Hwk #5 Solutions

Section 3.1: 2, 7, 9
3.2: 5

2. (a) Prove: If $\int_a^b f(x)\,dx$ exists, then $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|\sigma_1 - \sigma_2| < \varepsilon$ if σ_1 and σ_2 are Rieman sums of f over Partitions P_1 and P_2 of $[a,b]$ with norms less than δ.

Pf/ at this point we are only allowed to use Defn 3.1.1, namely f is Rieman integrable if $\exists L$ s.t. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|\sigma - L| < \varepsilon$ for any σ ~~for which~~ of f over $[a,b]$ with $\|P\| < \delta$.

Now let P_1 and P_2 be any partitions of $[a,b]$ s.t. $\|P_i\| < \delta$ and let σ_1 and σ_2 be the associated Rieman sums.

Then $|\sigma_1 - L| < \varepsilon/2$, $|\sigma_2 - L| < \varepsilon/2$ by Defn 3.1.1

$$\Rightarrow |\sigma_1 - \sigma_2| = |\sigma_1 - L + L - \sigma_2| \le |\sigma_1 - L| + |\sigma_2 - L| < \varepsilon.$$

(b) Suppose $\exists M > 0$ s.t. $\forall \delta > 0$, $\exists \sigma_1$ and σ_2 over P, $\|P\| < \delta$ s.t. $|\sigma_1 - \sigma_2| \ge M$. Show that f is not integrable.

Pf/ by contradiction suppose f is integrable. Then for any P_1 and P_2 ~~over~~ of $[a,b]$ with $\|P_i\| < \delta$, $|\sigma_1 - \sigma_2| < \varepsilon$. Choose $P_1 = P_2 = P$

Thus $\nexists M > 0$ s.t. $|\sigma_1 - \sigma_2| \ge M$ since ε is arbitrarily small.

7. Let f be bounded on $[a,b]$ and let P be a partition. Prove that the lower sum $s(P)$ of f over P is the infimum of the set of all Riemann sums of f over P

Pf/. let $P = \{x_0, \dots x_n\}$

$$s(P) = \sum_{j=1}^{n} m_j (x_j - x_{j-1}) \quad \text{where } m_j = \inf_{x_{j-1} \le x \le x_j} f(x)$$

An arbitrary Riemann sum of f over P is of the form

$$\sigma = \sum_{j=1}^{n} c_j (x_j - x_{j-1}) \qquad x_{j-1} \le c_j \le x_j$$

Note $f(c_j) \ge m_j \Rightarrow \sigma \ge s(P)$.

Thus the set of Riemann sums of f over P is bounded below.

Next let $\varepsilon > 0$ and choose $\bar{c}_j \in [x_{j-1}, x_j]$ s.t.

$$f(\bar{c}_j) < m_j + \frac{\varepsilon}{n(x_j - x_{j-1})}$$

Thus $$\bar{\sigma} = \sum_{j=1}^{n} f(\bar{c}_j)(x_j - x_{j-1}) < \sum_{j=1}^{n} \left[m_j + \frac{\varepsilon}{n(x_j - x_{j-1})} \right] (x_j - x_{j-1})$$

$$= s(P) + \varepsilon$$

That is for any $\varepsilon > 0$, we can find a Riemann sum that is greater than $s(P)$ but within ε of it

$\Rightarrow s(P) = \inf$ of all Riemann sums of f over P.

9. Find $\underline{\int_0^1} f(x)\,dx$ and $\overline{\int_0^1} f(x)\,dx$

a) $f(x) = \begin{cases} x & \in \mathbb{Q} \\ -x & \in \mathbb{Q}^c \end{cases}$

If P is a partition of $[0,1]$, then $M_j = \sup_{x_{j-1} \le x \le x_j} x = x_j$

$m_j = \inf_{x_{j-1} \le x \le x_j} (-x) = -x_j$

$$\Rightarrow S(P) = \sum_{j=1}^{n} x_j (x_j - x_{j-1})$$

we could explicitly calculate this sum or use that this is the ~~~~ upper sum for the function $g(x) = x$ on $[0,1]$ which by ~~problem~~ example 3.1.2 is $1/2$. $\Rightarrow \overline{\int_0^1} f(x)\,dx = 1/2$

$s(P) = \sum_{j=1}^{n} -x_j (x_j - x_{j-1})$, which is the lower sum for $g(x) = -x \Rightarrow \underline{\int_0^1} f(x)\,dx = -1/2$

b) $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ x & x \in \mathbb{Q}^c \end{cases}$

$\overline{\int_0^1} f(x)\,dx = 1 \qquad \underline{\int_0^1} f(x)\,dx = 1/2$.

Section 3.2

5 Suppose f is integrable and g is bounded on $[a,b]$ and g differs from f only on a set H with the following property: For each $\varepsilon > 0$, H can be covered by a finite number of closed subintervals of $[a,b]$, the sum of whose lengths is less than ε. Show ~~f~~ g is integrable on $[a,b]$ and that $\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$.

We need to find an appropriate Riemann sum for g by decomposing H into a part that can be swapped for a Riemann sum for f and into a part that takes care of the set H.

We know that f is integrable. Thus for fixed $\varepsilon > 0$, $\exists P$ s.t. $\|P\| < \delta$ and $|\sigma - L| < \varepsilon$ where $L = \int_a^b f(x)\,dx$ and σ is a Riemann sum of f over P.

$$g(x) = \begin{cases} f(x) & x \notin H \\ \neq f(x) & x \in H \end{cases}$$

Let I_j be a finite set of closed intervals $j = 1, \ldots k$

s.t. $H \subset \bigcup_{j=1}^{k} I_j$

and $\sum_{j=1}^{k} L(I_j) < \varepsilon$ where $L(I_j)$ is the length of I_j

Let us define the intervals $I_j = [y_{j_1}, y_{j_2}]$ $j = 1, \dots, k$

Note that each $L(I_j) < \varepsilon$.

Let $P = \{x_0, x_1, \dots x_n\}$ be a partition of $[a,b]$ s.t. $\|P\| < \delta$

and all of the points $\{y_{1_1}, y_{1_2}, y_{2_1}, y_{2_2}, \dots y_{k_1}, y_{k_2}\}$ are contained in the partition s.t. any pair y_{j_1}, y_{j_2} are consecutive points in the partition. Thus P can be rewritten as two sets of points $P_1 = \{$ partition points that don't contain $y_{j_1}, y_{j_2}\}$

$P_2 = \{$ partition points y_{j_1}, y_{j_2} $j = 1, 2, \dots k\}$

Notice that on any subintervals defined by points in P_1, $f(x) = g(x)$ and for any subintervals ~~out~~ defined by points in P_2 $g(x)$ is bounded

Thus for $P = P_1 \cup P_2$, consider $\|P\| < \delta$

$$\left| \sum_P g(c_j)(x_j - x_{j-1}) - L \right| = \left| \sum_{P_1} g(c_j)(x_j - x_{j-1}) + \sum_{P_2} g(c_j)(x_j - x_{j-1}) - L \right|$$

$$= \left| \sum_{P_1} f(c_j)(x_j - x_{j-1}) + \sum_{P_2} g(c_j)(x_j - x_{j-1}) - L \right|$$

This notation means the sum over the partition P

$$= \left| \sum_{P_1} f(c_j)(x_j - x_{j-1}) + \sum_{P_2} f(c_j)(x_j - x_{j-1}) - L \right.$$
$$\left. - \sum_{P_2} f(c_j)(x_j - x_{j-1}) + \sum_{P_2} g(c_j)(x_j - x_{j-1}) \right|$$

$$= \left| \sum_{P} f(c_j)(x_j - x_{j-1}) - L + \sum_{P_2} (g(c_j) - f(c_j))(x_j - x_{j-1}) \right|$$

$$\leq \left| \sum_{P} f(c_j)(x_j - x_{j-1}) - L \right| + (M_g + M_f) \sum_{P_2} |x_j - x_{j-1}|$$

$$= \left| \sum_{P} f(c_j)(x_j - x_{j-1}) - L \right| + (M_g + M_f) \sum_{j=1}^{k} L(I_j)$$

If $\|P\| < \delta$

$$< \varepsilon + (M_g + M_f)\varepsilon$$

$$= (1 + M_g + M_f)\varepsilon \implies g \text{ is integrable and } \int_a^b g(x)\,dx = L.$$

In the above argument $M_g = \sup_{a \le x \le b} g(x)$ and $M_f = \sup_{a \le x \le b} f(x)$

M_g exits by the assumption that $g(x)$ is bounded.

M_f exits since $f(x)$ is integrable.