Hwk #6 Solutions

Section 4.1 #7, 20, 24, 33

7. Suppose $S_n \to S$ (finite) and $\forall \varepsilon > 0$ $|S_n - t_n| < \varepsilon$ if n is large enough. Show that $\lim_{n\to\infty} t_n = S$.

Fix $\varepsilon > 0$, $S_n \to S \Rightarrow \exists N_1$ s.t. $n > N_1$ $|S_n - S| < \varepsilon/2$

From what is given $\exists N_2$ s.t. $n > N_2$ $|S_n - t_n| < \varepsilon/2$

$$\Rightarrow |t_n - S| = |t_n - S_n + S_n - S|$$
$$< |t_n - S_n| + |S_n - S|$$
$$< \varepsilon/2 + \varepsilon/2 \quad \text{if } n > \max(N_1, N_2).$$
$$= \varepsilon$$

$\Rightarrow t_n \to S$.

20. Prove: If $\lim_{n\to\infty} \dfrac{S_n - S}{S_n + S} = 0$, then $\lim_{n\to\infty} S_n = S$

Let $t_n = \dfrac{S_n - S}{S_n + S}$. Thus $t_n \to 0$ as $n \to \infty$

Solving for $S_n = \dfrac{(1 + t_n)S}{1 - t_n}$ we can take the limit

of this ratio since the indiviual limits of the numerator and denominator exist

$\Rightarrow S_n \to S$ as $n \to \infty$.

24. Find $\overline{S}$ and $\underline{S}$

a) $S_n = [(-1)^n + 1]n^2$

If n is even say $n = 2k$ then $S_{2k} = 2(2k)^2$ $\Rightarrow \overline{S} = \infty$

n odd say $n = 2k+1$ then $S_{2k+1} = 0$ $\underline{S} = 0$

b). $S_n = (1 - r^n)\sin\frac{n\pi}{2}$

The answer will clearly depend on the value of r, so lets consider this on a case by case basis.

If $r = 1$ then $S_n = 0 \Rightarrow \overline{S} = \underline{S} = 0$

If $|r| < 1$ then $1 - r^n \to 1$ and we only need to worry about $\sin\frac{n\pi}{2}$. This term takes on the values $-1, 0, 1$ depending on the value of n.

$\Rightarrow \overline{S} = 1$, $\underline{S} = -1$.

If $|r| > 1$ then $1 - r^n \to -\infty$ if $r > 1$

and oscillates towards $\pm\infty$ if $r < 1$

$\sin n\pi/2$ still takes on values $-1, 0, 1$

so $\overline{S} = +\infty$, $\underline{S} = -\infty$.

c) $S_n = \dfrac{r^{2n}}{1 + r^n}$ $r \neq -1$

If $|r| < 1$, then $S_n \to 0 \Rightarrow \overline{S} = \underline{S} = 0$

If $r > 1$ then $S_n \to +\infty \Rightarrow \overline{S} = \underline{S} = +\infty$

If $r < -1$ then ~~the~~ sequence oscillates and $\overline{S} = +\infty$ $\underline{S} = -\infty$.

(d) $s_n = n^2 - n = n(n-1) \to +\infty \Rightarrow \bar{s} = \underline{s} = +\infty$

(e) $s_n = (-1)^n t_n$ where $t_n \to t$.

$\Rightarrow \bar{s} = t, \ \underline{s} = -t$.

33. Let s_0 and s_1 be arbitrary and $s_{n+1} = \dfrac{s_n + s_{n-1}}{2} \quad n \geq 1$

Use Cauchy's convergence criterion to show s_n converges.

Thus we must show that $\forall \varepsilon > 0, \exists N$ s.t. if $n, m > N$ then $|s_n - s_m| < \varepsilon$.

First note that s_{n+1} is the average ~~between~~ of the two preceding values s_n and s_{n-1}. Therefore $\{s_n\}$ is bounded both above and below for all n.

Now fix $\varepsilon > 0$ and consider $m = n+k, \ k \geq 1$.

$|s_m - s_n| = |s_{n+k} - s_n|$, we will systematically replace s_{n+k} with terms like $s_{n+k} - s_{n+k-1} + s_{n+k-1} - s_{n+k-2} + s_{n+k-2}$ etc.

But first observe the following

$$s_{n+1} - s_n = \frac{s_n + s_{n-1}}{2} - s_n = \frac{s_{n-1} - s_n}{2}$$

$$\Rightarrow |s_{n+1} - s_n| = \frac{|s_n - s_{n-1}|}{2}$$

Thus by induction

$$|S_{n+1} - S_n| = \frac{|S_1 - S_0|}{2^n}$$

Therefore

$$|S_m - S_n| = |(S_m - S_{m-1}) + (S_{m-1} - S_{m-2}) + \cdots + (S_{n+1} - S_n)|$$

$$\leq |S_m - S_{m-1}| + |S_{m-1} - S_{m-2}| + \cdots + |S_{n+1} - S_n|$$

$$\leq \frac{|S_1 - S_0|}{2^n} \underbrace{\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-n-1}}\right)}_{<2}$$

$$< \frac{|S_1 - S_0|}{2^{n-1}}$$

Thus by taking n sufficiently large, $\left|\frac{1}{2^{n-1}}\right| < \varepsilon$

$\Rightarrow$ by the Cauchy criterion $\{S_n\}$ converges.

Section 4.2, # 5, 18

5. Prove: If $s_n \to s$ and $\{s_n\}$ has a subsequence $\{s_{n_k}\}$ s.t. $(-1)^k s_{n_k} \geq 0$ then $s = 0$.

By theorem 4.2.2. every subsequence of $\{s_n\}$ must converge to s.

$\Rightarrow$ $s_{n_k} \to s$ as $n_k \to \infty$

~~and~~ $(-1)^k s_{n_k}$ must also converge since it is the product of a convergent sequence and a bounded one. where $(-1)^k s_{n_k} \to s$.

Suppose $s < 0$. Then $\forall \varepsilon > 0$, $\exists N$ s.t. $n_k > N \Rightarrow |(-1)^k s_{n_k} - s| < \varepsilon$

but this would imply $(-1)^k s_{n_k} < 0$ for n_k sufficiently large.

$\Rightarrow$ $\therefore$ $s \geq 0$

~~Suppose s > 0, then $|(-1)^k s_{n_k} - s| < \varepsilon \Rightarrow \exists_k (-1)^{n_k} s_{n_k}$ fo~~

Suppose $s > 0$, then for n sufficiently large

$0 < s - \varepsilon < s_n$ $\Rightarrow$ there cannot exist a

subsequence s_{n_k} s.t. $(-1)^k s_{n_k} \geq 0$ for n_k sufficiently large.

$\Rightarrow$ $s \leq 0$

$\Rightarrow$ $s = 0$

8. Suppose $\{S_n\}$ is bounded and all convergent subsequences of $\{S_n\}$ converge to the same limit. Then show that S_n is convergent. Give an example showing that the result need not hold if $\{S_n\}$ is unbounded.

First lets give the counter-example when $\{S_n\}$ is unbounded.

Let $S_n = \begin{cases} n & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$. Clearly S_n does not converge. But every convergent subsequence converges to 0.

Now suppose $|S_n| \leq M$ is bounded and that for any $\{n_k\}$

$S_{n_k} \to S$ as $n_k \to \infty$.

Let S' be the set of distinct numbers that occur in $\{S_n\}$ (see proof of theorem 4.2.5 on pg 192). If S' is finite, then some $\bar{x}$ occurs infinitely often in S_n and $S_n \to \bar{x} = S$.

If S' is infinite, then since it is bdd, the Bolzano-Weierstrass theorem states that S' must have at least one limit point $\bar{x}$. Since every infinite subsequence must also have a limit point and these, by assumption are all the same, $\bar{x} = S$ is unique.

$\Rightarrow S_n \to S$.

Section 4.3 #4

4. a) Prove: If $\sum a_n$ converges then $\lim_{n \to \infty} (a_n + a_{n+1} + \cdots a_{n+r}) = 0$, $r \geq 0$

This is just a restatement of Cauchy's convergence Criterion for Series, Theorem 4.3.5 on pg 204.

which states that since $\sum a_n$ converges $\forall \varepsilon > 0$, $\exists N$ s.t. $|a_n + \cdots + a_m| < \varepsilon$ if $n, m \geq N$.

If we let $m = n + r$ then we obtain the result for $r \geq 0$.

b. No, the $\lim_{n \to \infty} (a_n + a_{n+1} \cdots + a_{n+r}) = 0$ <u>does not</u> imply $\sum a_n$ converges

Take $\sum \frac{1}{n}$ for example. Then $\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+r} \leq \frac{r+1}{n} \to 0$ as $n \to \infty$, but $\sum \frac{1}{n}$ does not converge.

Section 4.4. # 1a,d, 5(a,b), 9, 14

1 a) $F_n(x) = x^n(1-x^2)$

$$F_n(x) \to F(x) = \begin{cases} 0 & |x|<1 \quad \text{Since } x^n \to 0 \\ 0 & \text{if } x = \pm 1 \quad \text{Since } (1-x^2) = 0 \end{cases}$$

and diverges otherwise

$\Rightarrow F_n(x) \to 0$ on $|x| \le 1$.

d) $F_n(x) = \sin(1+\frac{1}{n})x$

Since $\sin x$ is a continuous function

$$\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \sin(1+\tfrac{1}{n})x = \sin\left(\lim_{n\to\infty} (1+\tfrac{1}{n})x\right)$$

$$= \sin x, \quad \forall x$$

5 a) $F_n(x) = x^n \sin nx$ on $(-1,1)$

Clearly if $x \in (-1,1)$ $x^n \to 0 \Rightarrow F_n(x) \to 0$ P.W

But at $x = \pm 1$ $F_n(x)$ does not converge.

Now let $I = [a,b] \subset (-1,1)$ be any closed subset of $(-1,1)$.

Then $|x^n \sin nx| \le |x^n|$

$< m^n$) let $m = \max\{|a|, |b|\}$

$< \varepsilon$ $m<1$

for n sufficiently large for any ε.

$\Rightarrow F_n(x) \to 0$ uniformly on $[a,b]$

5b. $F_n(x) = \dfrac{1}{1+x^{2n}}$ $\quad S = \{x : x \neq \pm 1\}$

$= (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

Then $F_n(x) \to \begin{cases} 0 & x < -1 \\ 1 & -1 < x < 1 \\ 0 & x > 1 \end{cases}$

By the way at $x = \pm 1$ $F_n(x) = \frac{1}{2}$ though technically F_n is not defined at these pts

Since each $F_n(x)$ is continuous, but $F(x)$ is not (if we include ± 1) the convergence cannot be uniform on S.

If we take any closed subset $T = [a,b] \subset S$, then each $F_n(x)$ is bounded and uniformly continuous on T. We can ... i-

Consider first $[a,b] \subset (-1, 1)$

Then $\left| \dfrac{1}{1+x^{2n}} - 1 \right| = \left| \dfrac{x^{2n}}{1+x^{2n}} \right| < |x^{2n}| < \varepsilon$ if n large enough since $|x| < 1$

next if $[a,b] \subset (1, \infty)$

$\left| \dfrac{1}{1+x^{2n}} - 0 \right| = \left| \dfrac{1}{1+x^{2n}} \right| < \dfrac{1}{x^{2n}} < \dfrac{1}{a^{2n}} < \varepsilon$ if n is large enough since $|a| > 1$

if $[a,b] \subset (-\infty, -1)$

$\left| \dfrac{1}{1+x^{2n}} - 0 \right| = \left| \dfrac{1}{1+x^{2n}} \right| < \dfrac{1}{x^{2n}} < \dfrac{1}{|b^n|} < \varepsilon$ if n is large enough since $|b| > 1$

~~$\Rightarrow$ In all cases the convergence is independent of x ...~~

In all cases, the convergence is independent of x and is thus uniform.

9. Prove: If $\{F_n\}$ converges uniformly to F on S

then $\lim_{n\to\infty} \|F_n\|_S = \|F\|_S$

$F_n \xrightarrow{u} F$ means that $\|F_n - F\|_S \to 0$ as $n \to \infty$

Let's separately look at $\|F_n\|_S$ and $\|F\|_S$

for each n, $\|F_n\|_S = \sup_{x\in S} |F_n(x)| \doteq S_n$ define

Thus S_n is just a sequence of numbers.

$\|F\|_S = \sup_{x\in S} |F(x)| \doteq \bar{S}$ define

we are assume $F_n(x)$ and $F(x)$ are bounded

Thus we want to show that $|S_n - \bar{S}| \to 0$ as $n \to \infty$.

$\Rightarrow |S_n - \bar{S}| = |\,\|F_n\|_S - \|F\|_S\,|$ Reverse triangle Lemma 4.4.2

$< |\,\|F_n - F\|_S\,|$

$\to 0$ since $F_n \xrightarrow{u} F$.

14. a) $\lim_{n\to\infty} \int_1^4 \frac{n}{x} \sin \frac{x}{n}\, dx$

If we call $F_n(x) = \frac{n}{x} \sin \frac{x}{n}$, then we can show

$F_n(x) \xrightarrow{u} 1$. In which case $\lim_{n\to\infty} \int_1^4 \frac{n}{x} \sin\frac{x}{n}\, dx = \int_1^4 \lim_{n\to\infty} \frac{n}{x} \sin\frac{x}{n}\, dx$

$= \int_1^4 dx = 3$

To show $F_n(x) \xrightarrow{u} 1$, note

$\frac{n}{x} \sin \frac{x}{n} = \frac{n}{x}\left[\frac{x}{n} - f'''(c)\frac{x^3}{6n^3}\right]$ where $f(x) = \sin\frac{x}{n}$ and we have Taylor expanded.

$= \left[1 - \frac{mx^2}{6n^2}\right]$ Let $f'''(c) = m$

$\Rightarrow |F_n(x) - 1| = \left|1 - \frac{mx^2}{6n^2} - 1\right|$

$= \left|\frac{mx^2}{6n^2}\right|$

$< \frac{16m}{6n^2} < \varepsilon$ if n large enough independent of x.

$\Rightarrow F_n \xrightarrow{u} 1.$

b) $\lim_{n\to\infty} \int_0^2 \frac{dx}{1+x^{2n}} = \lim_{n\to\infty} \int_0^1 \frac{dx}{1+x^{2n}} + \int_1^2 \frac{dx}{1+x^{2n}}$ (4)

By 5-b, we know the convergence is uniform on any closed subset of $[0,1]$ and of $[1,2]$, Including the endpoints at $x=1$ does not change that $\lim_{n\to\infty} \int_0^1 \frac{dx}{1+x^{2n}} = \int_0^1 \lim_{n\to\infty} \frac{1}{1+x^{2n}} dx$

$= 1$

and $\lim_{n\to\infty} \int_1^2 \frac{dx}{1+x^{2n}} = \int_1^2 \lim_{n\to\infty} \frac{dx}{1+x^{2n}} = 0$

See also Theorem 4.4.10(b) on page 2.4.3

c. $\lim_{n\to\infty} \int_0^1 nxe^{-nx^2} dx = \lim_{n\to\infty} \frac{e^{-nx^2}}{2}\Big|_0^1 = 1/2$

d. $\lim_{n\to\infty} \int_0^1 \left(1+\frac{x}{n}\right)^n dx$

$F_n(x) = \left(1+\frac{x}{n}\right)^n \to e^x$ which is integrable on $[0,1]$

Further $\|F_n(x)\|_S = \left(1+\frac{1}{n}\right)^n \to e$ is bounded

Thus by Theorem 4.4.10,

$\lim_{n\to\infty} \int_0^1 \left(1+\frac{x}{n}\right)^n dx = \int_0^1 \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n dx = \int_0^1 e^x dx = e-1.$