Math 481 Hwk #2 Solutions

Section 5.2 # 1a, 1b, 11

Section 5.3 # 8, 14

5.2

#1a Prove $\lim_{(x,y)\to(1,2,1)} 3x+4y+z-2 = 10$

Fix $\varepsilon>0$ and consider $f(x,y,z)=3x+4y+z-2$

$\Rightarrow |f((x,y,z)) - 10| = |3x+4y+z-2-10|$

$= |3(x-1) + 4(y-2) + (z-1)|$

$= |(3,4,1)\cdot(x-1, y-2, z-1)|$

$\leq \sqrt{9+16+1}\cdot|\mathbf{X}-\mathbf{X}_0|$ $\quad \swarrow \quad \mathbf{X} = (x,y,z)$, $\mathbf{X}_0 = (1,2,1)$

$\leq \varepsilon$ if $|\mathbf{X}-\mathbf{X}_0| < \delta = \frac{\varepsilon}{\sqrt{26}}$

1b. $\lim_{(x,y)\to(1,1)} \frac{x^3-y^3}{x-y} = 3$

Let $f(x,y) = \frac{x^3-y^3}{x-y}$

Fix $\varepsilon>0$, then

$|f(x,y)-3| = \left|\frac{x^3-y^3}{x-y} - 3\right|$

$= |x^2+xy+y^2-3|$

$= |(x+2)(x-1) + (x+y+1)(y-1)|$

$= |(x+2, x+y+1)\cdot(x-1, y-1)|$

$\leq \sqrt{(x+2)^2+(x+y+1)^2}\,|\underline{X}-\underline{X}_0|$ $\quad \underline{X}=(x,y)$
$\underline{X}_0=(1,1).$

Now assume $|\underline{X}-\underline{X}_0|<1$

$\Rightarrow |x-1|<1 \Rightarrow \quad x<2 \Rightarrow |x+2|<4$
$|y-1|<1 \qquad y<2 \qquad |x+y+1|<5$

$\Rightarrow |f(x,y)-3| \leq \sqrt{4^2+5^2}\,|\underline{X}-\underline{X}_0|$
$= \sqrt{41}\,|\underline{X}-\underline{X}_0|$
$< \varepsilon \quad$, if $|\underline{X}-\underline{X}_0|<\delta=\dfrac{\varepsilon}{\sqrt{41}}$ This choice depends on the assumption $|\underline{X}-\underline{X}_0|<1$

$\Rightarrow$ if we choose $\delta=\min\left(\dfrac{\varepsilon}{\sqrt{41}},1\right)$
the result holds

11. Give an example of a function f on $\mathbb{R}^2$ s.t. f is not continuous at $(0,0)$ but $f(0,y)$ is a continuous function of y for all $y\in(-\infty,\infty)$ and $f(x,0)$ is continuous on $x\in(-\infty,\infty)$.

There are many such functions. Consider

$f(x,y)=\dfrac{xy}{x^2+y^2}$. Then if $x=y$ $\lim\limits_{(x,y)\to(0,0)} f(x,y)=\dfrac{x^2}{2x^2}=\tfrac{1}{2}$

but $f(x,0)=\dfrac{0}{x^2}=0 \to 0$ as $(x,y)\to(0,0)$
and is continuous $\forall x$

$f(0,y)=\dfrac{0}{y^2}=0 \to 0$ as $(x,y)\to(0,0)$
and is continuous $\forall y$.

8. Find a function $f(x,y)$ s.t. f_{xy} exists $\forall (x,y)$ but f_y exists nowhere.

Again there are many such functions

Consider $f(x,y) = \begin{cases} 1 & y \in \mathbb{Q} \\ 0 & y \in \mathbb{Q}^c \end{cases}$

This function is independent of $x \Rightarrow f_x = 0$
$\Rightarrow f_{xy} = 0$

But f_y DNE as we can easily prove from techniques learned last semester.

14. Show $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^6 + 2y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ has a directional derivative in ~~on~~ the direction of an arbitrary unit vector $\vec{u}$ at $(0,0)$ but f is not cont at $(0,0)$.

First let's show f is not cont at $(0,0)$ using the path $y = x^4$

Then $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ y=x^4}} \frac{x^2 y}{x^6+2y^2} = \lim_{x\to 0} \frac{x^6}{x^6+2x^8} = \lim_{x\to 0}\frac{1}{1+2x^2} = 1 \neq 0 = f(0,0)$

$\Rightarrow f$ is not cont at $(0,0)$.

But $\dfrac{\partial f}{\partial \Phi} = \lim\limits_{t \to 0} \dfrac{f(\bar{x} + t\phi) - f(\bar{x})}{t}$ exists for all ϕ.

Specifically if we take $h(t) = f(x + t\phi)$, then

$h'(0) = \dfrac{\partial f}{\partial \phi}$ if it exists. Let $\phi = (\phi_1, \phi_2)$

Now $h(t) = \begin{cases} \dfrac{(x + t\phi_1)^2 (y + t\phi_2)}{(x + t\phi_1)^{6} + 2(y + t\phi_2)} & t \neq 0 \\ 0 & t = 0 \end{cases}$

We are interested in $h'(0)$ at $(x, y) = 0$

$\Rightarrow h(t)\Big|_{(x,y)=(0,0)} = \dfrac{t^2\phi_1^2 \, t\phi_2}{t^6\phi_1^6 + 2t^2\phi_2^2} = \dfrac{t\phi_1^2 \phi_2}{t^4\phi_1^6 + 2\phi_2^2}$ (*)

$\Rightarrow h'(t)\Big|_{t=0} = \dfrac{\phi_1^2\phi_2}{t^4\phi_1^6 + 2\phi_2^2} - \dfrac{t\phi_1^2\phi_2 \cdot 4t^3\phi_1^6}{(t^4\phi_1^6 + 2\phi_2^2)^2}\Bigg|_{t=0}$

$= \dfrac{\phi_1^2\phi_2}{2\phi_2^2} = \dfrac{\phi_1^2}{2\phi_2}$ ← This holds if $\phi_2 \neq 0$

If $\phi_2 = 0$ consider $\Phi = (1, 0)$ then from (*) $h = 0 \Rightarrow h'(0) = 0$.

Prove: If S is connected and f is continuous on S, then $f(S)$ is connected.

Pf/. Assume not, that $f(S)$ is not connected. Then $\exists$ $A, B \subset \mathbb{R}$ s.t. A, B open, $A \cap B = \emptyset$ $\bar{A} \cap B = A \cap \bar{B} = \emptyset$ and $A \cup B = f(S)$.

~~A and B must be intervals since these are the only~~
Let's assume that A and B are the intervals
$A = [x_1, x_2]$ $B = [x_3, x_4]$ s.t. $x_2 < x_3$.
Let $c \in (x_2, x_3)$. Then by the IVT $\exists$ $y \in S$ s.t. $f(y) = c$ provided S is a region, that is an open connected set that contains some, all or none of its boundary. But $f(y) = c$ contradicts $f(S)$ being disconnected $\Rightarrow$ either S is not a region or f is not continuous $\rightarrow \Leftarrow$.
Either way we have proved the result.