MATH 611 – Numerical Analysis – Fall 1990 – Final Exam

Instructor: B. Bukiet

Do all problems. Show all work.

Each EXTRA CREDIT problem is worth 10 points. Do not do them until you have finished the rest of the exam

Problem 1

(a). (5 points) Show that $f(x) = x^3 - x - 1$ has exactly one root on the interval [1,2].
(b). (5 points) Show that any root of $f(x) = x^3 - x - 1$ is a fixed point of the iteration scheme $x_{n+1} = (1 + x_n)^{1/3}$.
(c). (**EXTRA CREDIT**) Show the iteration scheme $x_{n+1} = (1 + x_n)^{1/3}$ converges to a solution of $x^3 - x - 1 = 0$ if the initial guess (x_0) is between 1 and 2.
(d). (5 points) Set up a quadratically convergent iteration scheme to find the zero of $f(x) = x^3 - x - 1$ on [1,2]. Let $x_0 = 1$. Find x_1 and x_2.
(e). (5 points) What is Aitken's method used for (in one line or less)? Would it be more useful in part (c) or part (d) of this problem?

Problem 2

Consider the function $f(x) = \sqrt[3]{1 + x}$
(a). (5 points) Find $f(-1), f(0), f(7), f'(x), f''(x)$.
(b). (5 points) Find the second degree Taylor polynomial approximation to $f(x)$ around $x = 0$. Use this to estimate $f(1)$. (c). (10 points) Use the information from part (a) to find a Lagrange polynomial of degree 1 to approximate $f(1)$. Give an upper bound to the error of this approximation.
(d). (**EXTRA CREDIT**) Find the highest degree Hermite polynomial possible using information only at $x = 0$ and $x = 7$.

Problem 3

Given a continuous smooth function $f(x)$ for which $f(0) = 8$, $f(1) = 5$, $f(2) = 3$, $f(3) = 2$, and $f(4) = 3$
(a). (5 points) Use a 3-point centered difference scheme to approximate $f''(2)$.
(b). (5 points) Use Richardson extrapolation to improve this result.
(c). (10 points) Use the (composite) Trapezoidal Rule and Simpson's Rule to approximate $\int_0^4 f(x)dx$.

Problem 4

(a). (15 points) Investigate the following multistep difference scheme for solution to differential equations of the form $y' = f(t, y)$ for consistency, stability and convergence.

$$w_{i+1} = \frac{3}{2}w_i - \frac{1}{2}w_{i-1} + \frac{1}{2}hf(t_i, w_i)$$

(b). (5 points) Write the third order differential equation $y''' + 2y'' - y' - 2y = e^t$ where $0 <= t <= 3$ and $y(0) = 1$, $y'(0) = 2$ and $y''(0) = 0$ as a system of first order differential equations. Find $y'''(0)$.
(c). (**EXTRA CREDIT**) Consider the modified Euler method

$$w_0 = \alpha w_{i+1} = w_i + \frac{h}{2}[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))]$$

Find the inequality to use in determining the step size such that the method will be stable for the differential equation $y' = -4y$. You DO NOT have to solve the inequality.

Problem 5

(a). (5 points) Set up the system of linear equations for finding a least squares line through the points $(-2,-1)$, $(-1,0)$, $(0,1)$, and $(7,2)$

(b). (5 points) Which of the following methods may be used to solve this system (DO NOT solve the system) (i) Gaussian Elimination, (ii) LU decomposition, (iii) Choleski's Method, (iv) LDL^T.

Consider the system of linear equations

$$\begin{array}{rcrcl} 9x & + & 2y & = & 7 \\ 8x & + & 4y & = & 4 \end{array} \tag{1}$$

(c). (5 points) Set up Jacobi iteration with initial guess $x = 0$, $y = 0$ and perform two steps of iteration.

(d). (5 points) Set up Gauss-Seidel iteration with initial guess $x = 0$, $y = 0$ and perform two steps of iteration.

(e). (**EXTRA CREDIT**) Determine whether Jacobi and Gauss-Seidel converge to the solution in this case and which converges faster. (Without doing more iterations or finding the exact solution).