MATH 611 – FINAL EXAM – DEC 15, 1997

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Show all work.

Problem #1 (25 points)
a. Find an interval $[a,\, b]$ with a and b integers and $b = a + 1$ such that $(25)^{1/3}$ lies in the interval. Let $a_0 = a$ and $b_0 = b$.
b. Use one step of the bisection method to narrow this interval. I.e., find a_1 and b_1.
c. Set up Newton's method such that the fixed point should be $(25)^{1/3}$ and no fractional power calculations need to be done.
d. Perform one iteration of Newton's method with initial guess a_0 from part **a.**.
e. For this problem in particular, show that Newton's method gives better than linear convergence.

Problem #2 (10 points)
a. Find the Lagrange polynomial through the $(x,\, y)$ points (1,2), (2,5), (4,5) and (5,2).
b. From the polynomial found in part **a.**, approximate $y(3)$.

Problem #3 (15 points)
Using the smallest step size possible, approximate $\int_2^6 \frac{1}{1+x} dx$ using the Midpoint method, the Trapezoidal rule and Simpson's Rule using only the values of $f(x)$ at $x = 2, 3, 4, 5$ and 6.

Problem #4 (15 points)
Consider the ordinary differential equation $\frac{dy}{dt} = -t^2 + y^2$ with $y(0) = 1$. Use the Runge-Kutta method with $h = 2$ to approximate $y(2)$.

Problem #5 (15 points)
Write the ODE $y'' - y' + y = t^2$ with $y(2) = 2,\ y'(2) = 3$ and $h = 1$ as a system of first order ODEs. Use Euler's method to approximate $y(4)$.

Problem #6 (15 points)

Solve using Gaussian Elimination with back-substitution.
$$\begin{aligned} x + 2y + 3z &= -5 \\ 3x + 5y + z &= 2 \\ 2x + 6y &= 0 \end{aligned}$$

Problem #7 (10 points)

Perform one step of Jacobi's method method with starting guess $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ for the system of linear equations $\begin{aligned} 3x + y &= 4 \\ 2x + 4y &= -4 \end{aligned}$. Take the result and perform one step of the Gauss-Seidel method.

Problem #8 (10 points)
Use 2 iterations of the Power Method to approximate the eigenvalue of largest magnitude for $\begin{pmatrix} 8 & 5 \\ 5 & 8 \end{pmatrix}$ with initial guess: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Try to guess an eigenvalue and eigenvector from this?

Problem #9 (15 points)
Find the best (Least-Squares) line through the points $(0, 2)$ $(0, 8)$ $(1, -1)$ and $(3, 11)$.

DO JUST 1 of the FOLLOWING 3 PROBLEMS – (it is worth 20 points)

Problem #10 Given $f'(x_0) = \frac{1}{h}[f(x_0 + h) - f(x_0)] - \frac{h}{2}f''(x_0) + O(h^2)$.
Use the numerical derivative approximation $f'(x_0) \tilde{} \frac{1}{h}[f(x_0 + h) - f(x_0)]$ and step size of $2h$ combined with the idea of Richardson extrapolation to derive a 3 point formula for $f'(x_0)$ with improved (i.e., $O(h^2)$) error.

Problem #11 Find the condition on step size h such that the second order Taylor series method will be stable for the ODE $y' = -2y$.

Problem #12
Consider the multi-step method $w_{i+1} = \frac{3}{2}w_i + aw_{i-1} + bh[f(t_i, w_i) + 2f(t_{i-1}, w_{i-1})]$ where a and b are a constants. Find a and b such that the scheme is consistent. Is the scheme stable?

Useful info:
Euler's method: $w_{i+1} = w_i + h\ f(t_i, w_i)$
Taylor series method: $w_{i+1} = w_i + h\ f(t_i, w_i) + \frac{h^2}{2}\frac{d}{dt}f(t_i, w_i)|_{t_i, w_i}$
Runge-Kutta: $w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ where
$k_1 = hf(t_i, w_i)$;
$k_2 = hf(t_i + h/2, w_i + k_1/2)$;
$k_3 = hf(t_i + h/2, w_i + k_2/2)$;
$k_4 = hf(t_i + h, w_i + k_3)$;