Homework problems

Basics, Error

1. Consider the function $f(x) = \sin x$ near $x = 0$.
   (a) Write the Taylor polynomials $P_1(x)$, $P_3(x)$ and $P_5(x)$.
   (b) Graph $P_1(x)$, $P_3(x)$ and $P_5(x)$ as well as $f(x)$ over $[0, 2\pi]$.
   (c) Compute $P_1(0.1)$, $P_3(0.1)$ and $P_5(0.1)$ and find the absolute and relative error.
   (d) Compute $P_1(0.5)$, $P_3(0.5)$ and $P_5(0.5)$ and find the absolute and relative error.
   (e) Compute $P_1(1)$, $P_3(1)$ and $P_5(1)$ and find the absolute and relative error.
2. Consider the function $f(x) = \ln(1 + x)$ near $x = 0$.
   (a) Write the Taylor polynomials $P_1(x)$, $P_2(x)$ and $P_3(x)$.
   (b) Graph $P_1(x)$, $P_2(x)$ and $P_3(x)$ as well as $f(x)$ over $[0, 2]$.
   (c) Compute $P_1(0.1)$, $P_2(0.1)$ and $P_3(0.1)$ and find the absolute and relative error.
   (d) Compute $P_1(0.5)$, $P_2(0.5)$ and $P_3(0.5)$ and find the absolute and relative error.
   (e) Compute $P_1(1)$, $P_2(1)$ and $P_3(1)$ and find the absolute and relative error.
3. Construct a Taylor polynomial approximation to $f(x) = e^{-x}$ with $x_0 = 0$ that is accurate to within 10^{-3} on the interval [0,1]. Find a value M such that $|f(x_1) - f(x_2)| \le M|x_1 - x_2|$ for all $x_1,\ x_2$ on the interval. (Use the Mean Value Theorem).
4. Find the Taylor polynomials of order 2 and order 3 around $x_0 = 1$ for $f(x) = x^3 + x$.
5. Find intervals containing solutions to the following equations:
   (a) $x - 3^{-x} = 0$
   (b) $4x^2 - e^x = 0$
6. Perform the following calculations using 3 digit rounding, 3 digit chopping and exactly:
   (a) $12.3 + 0.0234$
   (b) $-0.0321 + 0.000136$
   (c) $12.3 - 0.0234$

(d) $-321 + 32.1$

(e) $132 * 0.987$

7. Evaluate the following polynomial for $x = 1.07$ using three-digit chopping after each operation: $2.75x^3 - 2.95x^2 + 3.16x - 4.67$. Find the absolute and relative errors of your results. Count the number of additions and multiplications needed.

   (a) Proceeding from left to right

   (b) Proceeding from right to left

   (c) Evaluting the polynomial in nested form $((2.75x - 2.95)x + 3.16)x - 4.67$

8. Write the polynomial $p(x) = 5x^6 + x^5 + 3x^4 + 3x^3 - x^2 + 1$ in nested form.

9. Suppose p^* must approximate p with relative error at most 10^{-3}. Find the largest interval in which p^* must lie for each value of p.

   $a.$ 150; $b.$ 1500;

10. Use 3-digit chopping (after each addition) for

$$\sum_{i=1}^{N} \frac{1}{i}$$

   (a) Find N such that N is the smallest integer for which your sum is not the exact solution (chopped to 3 digits).

   (b) What is the absolute error for this value of N?

   (c) What is the relative error for this value of N?

11. The number e can be calculated as

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

   Use three-digit chopping arithmetic to compute the following approximations to e. Also compute the relative and absolute errors. For the 'exact' value of e, use $e = 2.7182818$.

   (a) $S_1 = \sum_{n=0}^{5} \frac{1}{n!}$

   (b) $S_2 = \sum_{n=0}^{5} \frac{1}{(5-n)!}$

12. The first three terms of the Maclaurin series for the arctangent function are given by $P(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5$. Compute the absolute and relative error in the following approximations of π using $P(x)$ in place of the arctangent:

   $a.\ 4\left[\arctan(\frac{1}{2}) + \arctan(\frac{1}{3})\right]$ $b.\ 16\arctan(\frac{1}{5}) - 4\arctan(\frac{1}{239})$

13. Why will a naive construction be susceptible to significant rounding errors for certain values of x and explain how to avoid this error for:

   (a) $f(x) = (\sqrt{x+9} - 3)/x$

   (b) $f(x) = (1 - \cos x)/x$

14. Consider the difference schemes: $p_{n+1} = \frac{1}{4}p_n$ and $p_{n+2} = \frac{21}{4}p_{n+1} - \frac{5}{4}p_n$.

   (a) For each scheme, solve analytically given $p_0 = 1$ and $p_1 = 0.25001$.

   (b) Find the first 15 iterates of each method using single precision and explain why the answers differ.

15. Use 4 digit chopping to approximate $f'(1)$ where $f(x) = e^x$ using $\frac{f(x+h)-f(x)}{h}$ with $h = 1,\ 0.1,\ 0.05,\ 0.02,\ 0.01,\ 0.005,\ 0.001,\ 0.0001$.

16. Suppose two points (x_0, y_0) and (x_1, y_1) lie on a straight line with $y_1 \neq y_0$. Two formulas for the x-intercept are given by:

$$x = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0} \quad \text{and} \quad x = x_0 - \frac{(x_1 - x_0) y_0}{y_1 - y_0}$$

   (a) Show that both formulas are analytically correct.

   (b) Use the data $(x_0, y_0) = (1.31, 3.24)$ and $(x_1, y_1) = (1.93, 4.76)$ and 3 digit rounding to compute the x-intercept both ways. Which method is better and why?

17. Find the rates of convergence of the following sequences as $n \to \infty$.

$$a.\ \lim_{n\to\infty} \sin(\frac{1}{n}) \quad b.\ \lim_{n\to\infty} \sin(\frac{1}{n^2}) \quad c.\ \lim_{n\to\infty} [\ln(n+1) - \ln(n)]$$

18. Find the rates of convergence of the following sequences as $h \to 0$.

$$a.\ \lim_{h\to 0} \frac{\sin h - h\cos h}{h} \quad b.\ \lim_{h\to 0} \frac{1 - e^h}{h}$$

19. How many multiplications and additions are required to determine a sum of the form

$$\sum_{i=1}^{n} \sum_{j=1}^{i} a_i b_j$$

   Modify the above expression to an equivalent form that reduces the number of computations.

20. Using Taylor polynomials, we can compute e^{-x} as

$$1 - x + \frac{x^2}{2} - \frac{x^3}{6} + ...$$

or

$$\frac{1}{1+x+\frac{x^2}{2}+\frac{x^3}{6}+...}$$

Which of these should be more accurate and less susceptible to rounding error?

21. Construct a linear interpolating polynomial to the function $f(x) = 1/x$ using $x_0 = 1/2$ and $x_1 = 1$. Find an upper bound for the error of this approximation on this interval using the relevant theorem.

22. Using the Trapezoidal rule to compute $\int_0^1 x^3 dx$ with $h = \frac{1}{4}$ gives a computed value of $\frac{17}{64}$. (Check this). How small should h be taken such that the error in the computation is less than 10^{-3}? 10^{-6}?

Roots of Nonlinear equations

1. **TEST**

   (a) Find an interval $[a, b]$ with a and b integers and $b = a + 1$ such that $(25)^{1/3}$ lies in the interval. Let $a_0 = a$ and $b_0 = b$.

   (b) Use one step of the bisection method to narrow this interval. I.e., find a_1 and b_1.

   (c) Set up Newton's method such that the fixed point should be $(25)^{1/3}$ and no fractional power calculations need to be done.

   (d) Perform one iteration of Newton's method with initial guess a_0 from part (a).

   (e) For this problem in particular, show that Newton's method gives better than linear convergence.

2. **TEST** Consider the function $f(x) = 4x - 1 - \sin x$ on the interval $0 \leq x \leq 2$.

   (a) Perform 3 iterations of the Bisection Method on $f(x)$ using the endpoints of the interval as initial data. Show the new estimate x_n and $f(x_n)$ at each step.

   (b) Perform 2 iterations of Newton's Method on $f(x)$ with the initial guess $x_0 = 1$. Show x_n and $f(x_n)$ at each step.

   (c) What is the absolute difference between the final estimates of the root of $f(x)$ in parts (a) and (b)? If this is the error of the bisection method, how many more iterations of bisection are needed to find the root of $f(x)$ to within an accuracy of 10^{-4}?

3. **TEST** Consider the function $f(x) = e^x - x - 2$ on the interval $0 \leq x \leq 2$.

(a) Perform 3 iterations of the Bisection Method on $f(x)$ using the endpoints of the interval as initial data. Show the new estimate x_n and $f(x_n)$ at each step.

(b) Perform 2 iterations of Newton's Method on $f(x)$ with the initial guess $x_0 = 1$. Show x_n and $f(x_n)$ at each step.

(c) What is the absolute difference between the final estimates of the root of $f(x)$ in parts (a) and (b)? If this is the error of the bisection method, how many more iterations of bisection are needed to find the root of $f(x)$ to within an accuracy of 10^{-4}?

4. Approximate $\sqrt[3]{29}$ by letting $f(x) = \sqrt[3]{x}$ and linearizing around $x = 27$. Then use one step of Newton's method on the relevant polynomial with corresponding initial guess. Compare your results in the two computations and think about how they are related.

5. **TEST** Consider the function $f(x) = x^2 - \cos x$ on the interval $1/2 \leq x \leq 3/2$.

   (a) Perform 3 iterations of the Bisection Method on $f(x)$ using the endpoints of the interval as initial data. Show the new estimate x_n and $f(x_n)$ at each step.

   (b) Perform 3 iterations of Secant Method on $f(x)$ using the endpoints of the interval as initial data. Show x_n and $f(x_n)$ at each step.

   (c) Perform 2 iterations of Newton's Method on $f(x)$ with the initial guess $x_0 = 1$. Show x_n and $f(x_n)$ at each step.

   (d) How many iterations of bisection are needed to find the root of $f(x) = x^2 - \cos x$ with initial data $x_L = 0$ and $x_R = \pi$ to within an accuracy of 10^{-5}?

6. Let $f(x) = (x-1)^{10}$ with root $x^* = 1$ and $x_n = 1 + \frac{1}{n}$. Show that $|f(x_n)| < 10^{-3}$ whenever $n > 1$ but that $|x^* - x_n| < 10^{-3}$ requires that $n > 1000$.

7. **TEST** To approximate $\sqrt{2}$, we solve $f(x) = x^2 - 2$ for the positive root.

   (a) Show that there must be a root on $[1, 2]$.

   (b) Use the interval $[1, 2]$ and two steps of Bisection to approximate the root. Give an upper bound for the error after these two steps.

   (c) Use $x_0 = 1$ and $x_1 = 2$ and perform one step of Secant method to find x_2.

   (d) Use $x_0 = 1$ and perform one step of Newton's method.

8. **TEST** Let $f(x) = x^2 - \cos x$

   (a) Using only the terms x^2 and $\cos x$, find 3 fixed point representations that can be used to find the roots of $f(x)$.

(b) Which of these representations will converge to the positive root if the initial guess is $x_0 = 1$? Justify your answers *without* performing any iterations.

9. **TEST** Prove that Newton's method performed on $f(x)$ will converge quadratically to its root $x = r$ for $f'(r) \neq 0$.

10. Let A be a given positive constant and $g(x) = 2x - Ax^2$

    (a) Show that if fixed-point iteration converges to a nonzero limit, then the limit is $x^* = \frac{1}{A}$, so the reciprocal of a number can be found using only multiplications and subtractions.

    (b) Find the largest interval around $\frac{1}{A}$ where this fixed-point iteration scheme is guaranteed to converge, (by theorems we have learned) provided x_0 is in that interval.

11. **TEST**

    (a) Show that $x^3 - 36 = 0$ has exactly one root on the interval [3,4]. Call this root p.

    (b) How many iterations of bisection would it take to find this root with error < 0.05? Explain. Let $x_L = 3$, $x_R = 4$ and $x_0 = 7/2$.

    (c) Is p a fixed point of the iteration scheme $x_{n+1} = 2\frac{x_n^3 + 18}{3x_n^2}$? Show reasoning.

    (d) If $x_0 = 3$, does x_n converge to p if $x_{n+1} = \frac{36}{x_n^2}$? Find x_1 and x_2.

    (e) If $x_0 = 3$, does x_n converge to p if $x_{n+1} = \frac{x_n^3 + 36}{2x_n^2}$? Find x_1 and x_2.

    (f) Set up a quadratically convergent iteration scheme to find p. Let $x_0 = 3$. Find x_1 and x_2.

    (g) Which of the above schemes would benefit most from using Aitken's method in conjunction with it? Use one step of Aitken's method on this scheme using x_0 x_1 and x_2 found earlier.

12. **TEST**

    (a) Show that $f(x) = x^3 - x - 1$ has exactly one root on the interval [1,2].

    (b) Show that any root of $f(x) = x^3 - x - 1$ is a fixed point of the iteration scheme $x_{n+1} = (1 + x_n)^{1/3}$.

    (c) Show the iteration scheme $x_{n+1} = (1 + x_n)^{1/3}$ converges to a solution of $x^3 - x - 1 = 0$ if the initial guess (x_0) is between 1 and 2.

    (d) Set up a quadratically convergent iteration scheme to find the zero of $f(x) = x^3 - x - 1$ on [1,2]. Let $x_0 = 1$. Find x_1 and x_2.

(e) What is Aitken's method used for (in one line or less)? Would it be more useful in part (c) or part (d) of this problem?

13. **TEST** Show $x = 4$ is a fixed point of the following iteration schemes:

    (a) $x_{n+1} = \frac{1}{4}(8x_n - x_n^2)$
    (b) $x_{n+1} = \frac{1}{3}(x_n^2 - 4)$
    (c) $x_{n+1} = \sqrt{3x_n + 4}$

    Compute a few iterates for each scheme (choose you own starting values, x_0). Then analyze theoretically (don't calculate iterates) to determine which of these methods should converge to the root at $x = 4$ given a starting guess close enough. Which method should converge fastest?

14. **TEST** Consider the Fixed Point Iteration Method with $g(x) = \frac{8}{4+x}$.

    (a) Show that the method has a unique fixed point x^* on the interval $1 \leq x \leq 2$.
    (b) Perform 3 iterations of the method starting with $x_0 = 1.5$ and give an approximation to the value of x^*.
    (c) Write down an equation which the fixed point x^* satisfies. Solve it exactly for x^* and compare the result with that of part (b).

15. If you borrow L dollars at an annual interest rate r for a period of m years, then the monthly payment M is given by $L = \frac{12M}{r}\left(1 - (1 + \frac{r}{12})^{-12m})\right)$. If the amount borrowed is \$150,000 and the payment you can afford is \$600 per month, find the maximum interest rate you can be afford for this 30 year mortgage. Use bisection.

16. Draw the graph of a single function f that satisfies all of the following:

    (a) f is defined and differentiable for all x.
    (b) There is a unique root at $x = \alpha > 0$.
    (c) Newton's method will converge for any $x_0 > \alpha$.
    (d) Newton's method will diverge for any $x_0 < 0$.

17. **TEST** Find a single polynomial whose roots are fixed points of the following iteration schemes:

    A. $x_{n+1} = \frac{2x_n^3 + 2x_n^2 + 2}{3x_n^2 + 4x_n - 1}$ **B.** $x_{n+1} = \frac{3x_n^3 + 5x_n^2 + 2x_n + 2}{4x_n^2 + 7x_n + 1}$.

    Which scheme is better for finding the root at $x = 1$?

18. Suppose p is a zero of multiplicity m of f, where f''' is continuous on an open interval containing p. Show that the folloiwng fixed-point method has $g'(p) = 0$:

$$g(x) \;=\; x \;-\; \frac{mf(x)}{f'(x)}$$

19. Consider the function

$$f(x) = (x-2)^2 - \ln x$$

on the interval $1 \leq x \leq 2$.

(a) Prove there is exactly root of this equation in the interval.

(b) Use the secant method to approximate a root of to 6 digits accuracy using the endpoints of the interval as initial data. Tabulate your data to show x_i, and $f(x_i)$ at each step.

(c) Use Newton's method with the initial guess $x_0 = 1.5$ to find a root of $f(x) = 0$ to 6 significant digits.

(d) How many iterates of the Bisection Method are needed to find an approximation to the root of $f(x) = 0$ in the interval to within an accuracy of 10^{-4}?

20. **TEST** Set up Newton's method for solving for the roots of

$$f(x) \;=\; x^4 - 3x^3 + 3x^2 - x \;=\; x(x-1)^3 \;=\; 0$$

DO NOT PERFORM ANY ITERATIONS.
Show why convergence to one root is super-linear (quadratic) while convergence to the other root is linear. Would convergence to the "linear" root be better or worse than bisection? Explain you answer.

21. **TEST** Consider the iteration scheme: $x_{n+1} \;=\; x_n^3 - 12x_n^2 + 54x_n - 105 + \frac{81}{x_n}$

(a) Perform 2 iterations of this scheme with $x_0 \;=\; 4$. (Find x_1 and x_2).

(b) Use the result of part(a) to guess the true solution (Hint: it's an integer) and verify you're correct.

(c) Use x_0, x_1, and x_2 to find the order of error. (A whole number - round to it).

(d) Use your understanding of analysis of error to show why this scheme should have a root of the order found in part (c) at the appropriate point.

22. If f is such that $|f''(x)| \leq 4$ for all x and $|f'(x)| \geq 2$ for all x and if the initial error in Newton's method is less than $\frac{1}{3}$, what is an upper bound on the error at each of the first 3 steps?

23. **TEST** Consider the iteration scheme $x_{n+1} \;=\; 2x_n - x_n^2$

(a) Find the fixed point(s).

(b) Analyze theoretically whether the method should converge to each point found in part(a) for a starting guess close enough. If it converges to the fixed point, find the order of convergence and the interval of starting values for which it will converge to this fixed point.

(c) Perform 2 steps of the method with initial guess $x_0 = 1.5$.

24. **TEST** We desire to compute $\sqrt{17}$.

(a) Find a simple polynomial equation with integer coefficients that has a solution $+\sqrt{17}$.

(b) Find an interval of length 1 that brackets the root where x_L and x_R are integers and show why it brackets a root.

(c) Perform 2 steps of bisection using this interval to reduce the interval length.

(d) How many steps of bisection are needed to guarantee your error is less than 0.0001?

(e) Use one step of secant method on the original interval (x_L,x_R) to get an update.

(f) Set up Newton's method for this problem. Use $x_0 = x_L$ (of part (b)) above to find x_1.

25. Consider the fixed point iteration scheme: $x_{n+1} = 1 + e^{-x_n}$. Show that this scheme converges for any $x_0 \in [1, 2]$. How many iterations does the theory predict it will take to achieve 10^{-5} accuracy?

26. **TEST**

(a) Find a "simple" polynomial with integer coefficients such that $(31)^{1/4}$ is a solution (a root) of the polynomial.

(b) Find an interval $[a, b]$ with a and b integers and $b = a + 1$ such that $(31)^{1/4}$ lies in the interval.

27. **TEST** Consider

$$x^3 + 2x - 4 = 0$$

(a) Show that $x^3 + 2x - 4 = 0$ has exactly one root on the interval [1,3].

(b) Use one step of secant method on the interval [1,3] to find an update for an approxiamtion of the root.

(c) Use one step of Newton's method for this problem. Use $x_0 = 2$.

28. **TEST** Show that there is exactly one solution to $2x^3 + x^2 + 5x - 4 = 0$ on $[0, 1]$. Then, use $x_0 = 0$ and $x_1 = 1$ and one step of the secant method to compute x_2.

29. **TEST** Suppose an iteration method for a root (with true solution $x^* = 2$) has the following iterates: $x_0 = 3$, $x_1 = 2.5$ and $x_2 = 2.0625$. Find the order of convergence.

30. **TEST**

    (a) Find the positive root of $x^2 - x = 0$.

    (b) Rank the iteration schemes that follow in order of which should converge best to this root (starting close enough to the root) to which method is worst. Do **NOT** perform any iterations!!

    i. $x_{n+1} = \sqrt{x_n}$

    ii. $x_{n+1} = \frac{x_n^2}{2x_n - 1}$

    iii. $x_{n+1} = x_n^2$

    (c) Use six steps of method (i.) along with Aitken's method to improve the approximations of the root, with initial guess $x_0 = 2$.

31. **TEST** Consider the equation $2x^2 = 3x + 1$ $(*)$.

    (a) Show there is at least one solution to the equation on the interval $[1, 2]$.

    (b) Show that any fixed point of the iteration scheme $x_{n+1} = \frac{3}{2} + \frac{1}{2x_n}$ is a solution of (*).

    (c) Prove that the iteration scheme in part b has exactly one fixed point in the interval $[1, 2]$ and that the scheme will converge to that root for any starting guess in $[1, 2]$.

    (d) Use $x_0 = 1$ and perform two iterations of the scheme in part (b).

    (e) Use Aitken's method to improve the last value computed in part d.

    (f) Set up Newton's method for (*) and perform one iteration with $x_0 = 1$.

32. The function $f(x) = e^{-x} + 4x^3 - 5$ has a root on the interval $[1, 2]$. Use bisection, secant, regula falsi and Newton's method (the latter with $x_0 = 1.5$) to approximate the root with error less than 10^{-6}.

33. **TEST**

    (a) Use 2 steps of the bisection method for finding a root of $f(x) = x^3 - 3$ on the interval [0,4]. ($a_0 = 0$ and $b_0 = 4$). What is the appropriate guess for the root after these two steps?

    (b) How many iterations of bisection would be needed to such that the error is less than 10^{-4}?

    (c) Set up Newton's method for the function of part (a) and perform 2 steps of Newton's method. Use b_0 of part (a) as your initial guess to find b_1 and b_2.

(d) Find the absolute and relative error of your value of b_2 in part (c).

34. The function $f(x) = e^{x-1} - 5x^3 + 5$ has a root near $x = 1$. Using Newton's method to approximate this root, does the number of "correct" digits double with each iteration?

35. The quadratic $f(x) = (x - 0.4)(x - 0.6) = x^2 - x + 0.24$ has zeroes at $x = 0.4$ and at $x = 0.6$. Why are the endpoints $[0, 1]$ not satisfactory to begin bisection?

36. Consider fixed point iteration with $g(x) = 1 + \frac{1}{4x}$.

    (a) Show that the method has a unique fixed point x^* on the interval $1 \leq x \leq 2$.

    (b) Use the method starting with $x_0 = 1.5$ and give an approximation to the value of x^*.

    (c) Write down an equation which the fixed point x^* satisfies. Solve it exactly for x^* and compare the result with that of part (b).

37. **TEST** Consider the iteration scheme

$$x_{n+1} = \frac{10}{x_n} + 3$$

    (a) Show $x_* = 5$ is a fixed point.

    (b) Show there is exactly one root on the interval [4,6].

    (c) Let $x_0 = 4$ and find x_1 and x_2.

    (d) Why should Aitken's method be able to improve the value of x_2? Use Aitken's method to improve x_2.

$$\tilde{x} = x_n - \frac{(x_{n+1} - x_n)^2}{x_{n+2} - 2x_{n+1} + x_n}$$

    (e) On what interval, i.e., for what values of x_0, should the iteration scheme converge to $x_* = 5$?

    (f) Find the other fixed point of the iteration scheme and explain why the scheme will not converge to that root for a guess close to it.

    (g) The iteration scheme

$$x_{n+1} = \frac{x^2 + 3x - 10}{6}$$

    has the exact same roots as the original scheme. Show that this scheme will converge to the second fixed point, but not to the one at $x_* = 5$.

38. Plot $f(x) = x^2 - \cos x - x$ on $[0, 2]$ and find a root on this interval using Bisection, Regula Falsi and Newton's Method (with starting guess $x = 1$).

39. Find a root of $\tan x - c = 0$ for $c = 5$ and $c = 10$ with initial guess $x_0 = 1.3$ and $x_0 = 1.4$ using Newton's method. Explain the results.

40. The function $f(x) = e^x - 3x^2$ has 3 real roots. One rearrangement is

$$x = \pm\sqrt{\frac{e^x}{3}}$$

   (a) Show that convergence is to the root near -0.5 if we begin with $x_0 = 0$ and use the negative value.

   (b) Show that convergence is to the root near 1.0 if we begin with $x_0 = 0$ and use the positive value.

   (c) Show that this form does not converge to the third root, near $x = 4$, even though a starting value very close to this root is used.

   (d) Find a different rearrangement that will give convergence to the root near $x = 4$.

41. Use different fixed point iteration schemes to find the root near $x = 1$ for $x^{10} = e^x$.

42. Consider the polynomial

$$p(x) = 1 - x + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} + \ldots + \frac{x^n}{(n!)^2}$$

Find the smallest positive root for members of the sequence of polynomials: $p_1 = 1 - x$, $p_2 = 1 - x + \frac{x^2}{(2!)^2}, \ldots$. Do they appear to be converging?

43. Consider the polynomial

$$p(x) = (x-1)^2(x-3)^3 = x^5 - 11x^4 + 46x^3 - 90x^2 + 81x - 27$$

   (a) Can all the roots be obtained by bisection? Why or why not?

   (b) Use Newton's method with initial guess $x = 2$. Does the method converge?

   (c) Use Newton's method with initial guess $x = 2.9$. Discuss the order and rate of convergence?

   (d) Use Newton's method on $\frac{p(x)}{p'(x)}$ which has no multiple roots to obtain quadratic convergence.

   (e) Use Newton's method on $p(x)$ but using $x_{n+1} = x_n - m\frac{p(x_n)}{p'(x_n)}$ to obtain second order convergence.

44. Use the modified Newton's method $x_{n+1} = x_n - k\frac{f(x_n)}{f'(x_n)}$ to find the root $\alpha = 1$ for the function $f(x) = 1 - xe^{1-x}$. Is the quadratic convergence recovered?

45. Consider $f(x) = e^x \sin^2 x - x^2$, which has one root at $x = 0$ and another root near $x = 2.3$. Use Newton's method to approximate these roots. Use starting values of $x_0 = 1$ and $x_0 = 2$. Determine the order of convergence in each case.

46. When solving for several roots of a polynomial, we can obtain one root and then "deflate" the polynomial by synthetic division, but any error in early roots and round-off can build up with later roots. Investigate this phenomenon with

$$f(x) = x^3 - 4x^2 + 3x + 1$$

by deliberately using an imperfect root (error 1%) and see how this affects the accuracy of the other roots. The roots are 1.44504, 2.80193 and −0.246979. So give one root an error, use synthetic division to obtain a quadratic (ignoring any remainder) and use the quadratic formula to find the other 2 roots. The repeat starting with the other two roots.

47. Perform two iterations of Newton's method (for systems) with initial guess (1,1) for the system: $\begin{array}{rcl} 2x_1 - x_2 + \frac{1}{9}e^{-x_1} &=& -1 \\ -x_1 + 2x_2 + \frac{1}{9}e^{-x_2} &=& 1 \end{array}$

Rewrite the system as $Kx + \phi(x) = b$
where $K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$; $\phi(x) = \begin{pmatrix} \frac{1}{9}e^{-x_1} \\ \frac{1}{9}e^{-x_2} \end{pmatrix}$ and $b = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.
Perform 2 iterations of the scheme $x_{k+1} = \frac{1}{2}(b - \phi(x_k) - Kx_k + 2x_k)$ with $x_0 = (1,1)$.

Perform 2 iterations of the scheme $x_{k+1} = K^{-1}(b - \phi(x_k))$ with $x_0 = (1,1)$.

Systems of linear equations

1. **TEST** Consider the system of linear equations $\begin{array}{rrrcl} x & +y & +z &=& -1 \\ 2x & +4y & +4z &=& -6 \\ -2x & +2y & +4z &=& -10 \end{array}$

   (a) Use Gaussian Elimination with back substitution to solve the system.

   (b) Find the LU decomposition of the system and solve using L and U.

2. **TEST** Consider the system of linear equations:

$$\begin{array}{rrrcr} x_1 & +3x_2 & & = & -1 \\ 3x_1 & +4x_2 & +3x_3 & = & 11 \\ & 5x_2 & +2x_3 & = & 1 \end{array}$$

   (a) Write the system as $Ax = b$.

   (b) Solve the system using Gaussian Elimination with back-sustitution.

(c) Find the LU decomposition of A and solve using it.

(d) What is the determinant of A?

(e)

$$A^{-1} = \frac{1}{25}\begin{pmatrix} 7 & 6 & -9 \\ 6 & -2 & 3 \\ -15 & 5 & 5 \end{pmatrix}$$

What is the condition number of A? (Use the 1-norm).

3. What is the operation count for computing the matrix-vector product $\mathbf{Ax}$? What is the operation count if the matrix is tridiagonal?

4. **TEST** Consider the system of linear equations:

$$\begin{array}{rrrcr} x_1 & + 5x_2 & + 3x_3 & = & 4 \\ x_1 & - x_2 & + 6x_3 & = & 16 \\ 2x_1 & + x_2 & & = & 5 \end{array}$$

(a) Write the system as $Ax = b$.

(b) Solve the system using Gaussian Elimination with back-sustitution.

(c) What is the determinant of A?

(d) Perform one step of the Gauss-Seidel method on this system (put the equations in the most appropriate order), starting with initial guess $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

5. **TEST** Find the LU decomposition of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

Find the determinant of A using L and U.

6. **TEST** Consider the system of linear equations $\begin{array}{rrrcr} 4x & & + z & = & 5 \\ x & + 4y & + z & = & 3 \\ x & & + 4z & = & -10 \end{array}$

(a) Use Gaussian Elimination with back substitution to solve the system.

(b) Find the matrix that needs to be analyzed to determine whether Jacobi's iteration method will converge for this problem. (Do not analyze the relevant matrix, just find it).

(c) Perform one iteration of Gauss-Seidel, with starting guess $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

7. **TEST** Consider the system of linear equations

$$\begin{array}{ccccc} 4x & + & 1y & = & 3 \\ 2x & + & 5y & = & 1 \end{array}$$

(a) Set up Jacobi iteration with initial guess $x = 3$, $y = 11$ and perform two steps of Jacobi's method.

(b) Set up Gauss-Seidel iteration with initial guess $x = 3$, $y = 11$ and perform two steps of the Gauss-Seidel method.

(c) Explain why both methods should converge for this case.

(d) Perform 2 steps of the Power Method on the matrix

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix}$$

with initial guess $(1, 1)^T$ to approximate an eigenvalue and eigenvector of A.

8. **TEST** Solve using Gaussian Elimination with back-substitution.

$$\begin{array}{llll} x & + 2y & + 3z & = -5 \\ 3x & + 5y & + z & = 2 \\ 2x & + 6y & & = 0 \end{array}$$

9. Given the matrices

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -2 & 4 \\ 1 & 3 & -5 \\ 2 & 4 & -7 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & 0 & 2 \\ 4 & 2 & -1 \\ 2 & 3 & 1 \end{bmatrix}$$

(a) Show $AB = BA = I$

(b) Show $AI = IA = A$

(c) Show $AC \neq CA$ and $BC \neq CB$.

10. Let

$$A = \begin{bmatrix} 9 & 2 \\ -10 & -3 \end{bmatrix}, \quad \text{and } B = \begin{bmatrix} 6 & 1 & -1 \\ 1 & 4 & 1 \\ -1 & 3 & 2 \end{bmatrix}$$

(a) Find the characteristic polynomial of A and of B.

(b) Find the eigenvalues of A and of B.

(c) Show that if the characteristic polynomial of A is $p(\lambda)$, then $p(A)$ is a zero matrix.

11. Write the set of equations:

$$\begin{array}{rcl} 2x - 6y + z & = & 11 \\ -5x - y - 2z & = & -12 \\ x + 2y + 7z & = & 20 \end{array}$$

in matrix form and solve the set of equations.

12. **TEST** Consider the system of linear equations generated by

$$Ax = b, \quad A = \begin{bmatrix} -5 & 1 & -2 \\ 2 & -6 & 1 \\ 1 & 2 & 7 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

   (a) Perform 3 iterations of the Jacobi method with initial guess $(0,0,0)^T$ to find an approximation for the solution of the above matrix equation.

   (b) Perform 3 iterations of the Gauss-Seidel method with initial guess $(0,0,0)^T$ to approximate the solution.

   (c) Given the exact solution (-0.58824,0.05348,0.49733), calculate the error in parts (a) and (b) using the Euclidean vector norm.

13. Solve the following system of equations using Gaussian elimination with partial pivoting (row interchange) followed by back-substitution.

$$Ax = b, \quad A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ 9 \\ -6 \end{bmatrix}$$

   Find the LU decomposition of the matrix A above using Gaussian elimination without pivoting. Verify that $LU = A$.

14. Solve the following system of equations by hand using Gaussian elimination with no pivoting (row interchange) followed by back-substitution:

$$Ax = b \quad \text{where} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 6 \\ 2 & 1 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$$

   Save the coefficients used in the row-reduction process and construct the lower triangular matrix L and write down the LU decomposition of A.

15. **TEST** Find the LU decomposition of $\begin{pmatrix} 1 & 3 \\ 4 & -1 \end{pmatrix}$ and use it to solve

$$\begin{aligned} x + 3y &= 9 \\ 4x - y &= 10 \end{aligned}$$

16. The linear system

$$\begin{aligned} x + 2y &= 3 \\ 1.0001x + 2y &= 3.0001 \end{aligned}$$

   has solution $(1,1)^T$. Change the system slightly, to

$$\begin{aligned} x + 2y &= 3 \\ 0.9999x + 2y &= 3.0001 \end{aligned}$$

and compute the solution using 5 digit rounding. Compare the error to the estimate

$$\frac{||x - x^*||}{||x||} \le \frac{K(A)}{1 - K(A)(||\delta A||/||A||)} \frac{||\delta A||}{||A||}$$

Is the matrix ill-conditioned?

17. Show that the following system does not have a solution:

$$\begin{array}{rcrcrcrcr} 3x_1 & + & 2x_2 & - & x_3 & - & 4x_4 & = & 10 \\ x_1 & - & x_2 & + & 3x_3 & - & x_4 & = & -4 \\ 2x_1 & + & x_2 & - & 3x_3 & & & = & 16 \\ & - & x_2 & + & 8x_3 & - & 5x_4 & = & 3 \end{array}$$

If the right hand side is changed to $(2, 3, 1, 3)^T$, show there are an infinite number of solutions.

18. **TEST** Consider the following system of linear equations

$$\begin{array}{rrrl} 2x & -\ 3y & +\ z & = 16 \\ 4x & +\ y & -\ z & = 6 \\ -2x & -\ 2y & +\ 3z & = 10 \end{array}$$

(a) Use Gaussian Elimination with back substitution.

(b) Find the inverse of the matrix $\begin{bmatrix} 6 & 11 \\ 1 & 2 \end{bmatrix}$ using the Gauss-Jordan method.

19. **TEST**

(a) Use Gaussian Elimination with back substitution to solve

$$\begin{array}{rrrrrl} -2x_1 & +\ x_2 & & & & = 0 \\ x_1 & -2x_2 & +\ x_3 & & & = 0 \\ & x_2 & -2x_3 & +\ x_4 & & = 0 \\ & & x_3 & -2x_4 & +\ x_5 & = 0 \\ & & & x_4 & -\ 2x_5 & = -6 \end{array}$$

(b) Approximate the number of operations needed to solve the $n \times n$ system of the same form with right hand side all zeroes except for the last entry, which is $-1 - n$.

20. **TEST** Consider the system of linear equations:

$$\begin{array}{rrrcr} x_1 & +\ 2x_2 & & = & -1 \\ -2x_1 & -\ 3x_2 & +\ 2x_3 & = & 3 \\ & -\ 2x_2 & -\ 3x_3 & = & -3 \end{array}$$

(a) Write the system as $Ax = b$.

(b) Solve the system using Gaussian Elimination.

(c) Solve the system using the Gauss-Jordan method.

(d) Find the LU decomposition of A and solve using it.

(e) Compute A^{-1}.

(f) What is the determinant of A?

(g) What is the condition number of A? (Use the 1-norm).

21. Find the determinant of the following matrices using row operations to make them upper triangular:

$$\begin{bmatrix} 2 & 5 & -1 \\ 1 & 6 & 4 \\ 7 & -4 & 2 \end{bmatrix} \quad \begin{bmatrix} 3 & -2 & 8 & 2 \\ -1 & 3 & 2 & -6 \\ -5 & -1 & 3 & -9 \\ 2 & 3 & -8 & 1 \end{bmatrix}$$

22. Set up nxn Hilbert matrices (where $a_{ij} = \frac{1}{i+j-1}$) H_n to solve $H_n x = b$ where each row of b is just the sum of the entries of the row so that the correct answer is $[1, 1, \ldots, 1]^T$. Let n increase until the solution goes bad for single precision runs and then for double precision runs. For each of these matrices, find the (numerical) determinant. The fact that they are close to zero is a sign of the ill-conditioned nature of such systems.

23. (a) For a general $n \times n$ matrix, show that Gaussian elimination takes $O(n^3/3)$ operations.

(b) Show that back substitution takes $O(n^2/2)$ operations.

(c) Show that Gauss-Jordan takes $O(n^3/2)$ operations.

The following relations may be helpful.

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

24. Evaluate the norms $|| * ||_p$ where $p = 1, 2, \infty$ for

(a) $x = [2.15, \ -3.1\ 10.0, \ 2.2]$

(b) $y = [-4, \ -5\ 0, \ 3, \ 7]$

(c) $x + y$. Does the triangle inequality hold?

25. **TEST** Given

$$A = \begin{bmatrix} 2 & 4 \\ 19/10 & 41/10 \end{bmatrix} \text{ and } \mathrm{A}^{-1} = \begin{bmatrix} 41/6 & -20/3 \\ -19/6 & 10/3 \end{bmatrix}$$

(a) Evaluate the norms $||A||_p$ where $p = 1, \infty$.

(b) Find the condition number of A using the 1-norm.

26. Evaluate the norms $|| * ||_p$ where $p = 1, \infty$ for

(a)

$$A = \begin{bmatrix} -9 & 5 & -9 \\ -2 & 7 & 5 \\ 5 & 1 & 8 \end{bmatrix}$$

(b)

$$B = \begin{bmatrix} 8 & -2 & 1 \\ -2 & 2 & -1 \\ -2 & 4 & -3 \end{bmatrix}$$

(c) B^2, $A + B$, and AB. Does the triangle inequality hold?

27. Approximate the condition number for the Hilbert matrices H_1, H_2, H_3, H_4, and H_5 by computing their inverses and using the 1-norm or the infinity norm.

28. Consider the system $Ax = b$, where

$$A = \begin{bmatrix} 3.01 & 6.03 & 1.99 \\ 1.27 & 4.16 & -1.23 \\ 0.987 & -4.81 & 9.34 \end{bmatrix}, \quad and \ b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

.

(a) Solve the system using single precision arithmetic

(b) Solve the system using double precision arithmetic

(c) Change the 3,3 element of A from 9.34 to -9.34 and repeat parts (a) and (b).

(d) Are either of the 2 matrices ill-conditioned? What is your evidence of it?

(e) Find the condition number of the matrices (use the 1-norm). Find their inverses by solving $Ax = b$ for x, for b's that are standard normal unit vectors $(1, 0, 0)^T$ etc.

29. Let A be a matrix with all eigenvalues less than unity. Compute $(\mathrm{I} - \mathrm{A})^{-1}$ directly and using $(\mathrm{I} - \mathrm{A})^{-1} = \mathrm{I} + \mathrm{A} + \mathrm{A}^2 + \ldots$

30. Consider the system,

$$Ax = b, \quad A = \begin{bmatrix} 2 & -6 & 1 \\ -5 & 1 & -2 \\ 1 & 2 & 7 \end{bmatrix}, \quad b = \begin{bmatrix} 11 \\ -12 \\ 20 \end{bmatrix}$$

Solve using Jacobi and Gauss-Seidel with initial guess $[0, 0, 0]^T$ and compare the rates of convergence.

31. **TEST** Consider the linear system of equations

$$\begin{aligned} 3x \quad + y \quad &= 4 \\ x \quad + 4y \quad &= 5 \end{aligned}$$

(a) Start with initial guess $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and perform 2 steps of Jacobi's method.

(b) Start with initial guess $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and perform 2 steps of the Gauss-Seidel method.

(c) **Analyze** (i.e., analytically investigate) which of the two methods should work better for this system.

32. **TEST** Perform one step of Jacobi's method method with starting guess $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ for the system of linear equations

$$\begin{aligned} 3x \quad + y \quad &= 4 \\ 2x \quad + 4y \quad &= -4 \end{aligned}$$

Take the result and perform one step of the Gauss-Seidel method.

33. **TEST** Consider the system of linear equations

$$\begin{aligned} 9x + 2y &= 7 \\ 8x + 4y &= 4 \end{aligned}$$

(a) Set up Jacobi iteration with initial guess $x = 0, y = 0$ and perform two steps of iteration.

(b) Set up Gauss-Seidel iteration with initial guess $x = 0, y = 0$ and perform two steps of iteration.

(c) Determine whether Jacobi and Gauss-Seidel converge to the solution in this case and which converges faster (by analyzing the relevant matrices without doing more iterations or finding the exact solution).

34. **TEST** Given $\begin{bmatrix} 2 & 4 \\ 6 & 1 \end{bmatrix} x = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$, with initial guess $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(a) Find the first two iterates of a <u>converging</u> Jacobi iteration.

(b) Find the first two iterates of a <u>converging</u> Gauss-Seidel iteration.

35. **TEST**

(a) Consider the system: $\begin{aligned} 5x_1 \quad + x_2 \quad &= 7 \\ x_1 \quad + 5x_2 \quad &= 11 \end{aligned}$

(b) Set up the Jacobi method for solving the system and perform two iterations with starting guess $\begin{bmatrix} 26 \\ -23 \end{bmatrix}$

(c) Set up the Gauss-Seidel method for solving the system and perform two iterations with starting guess $\begin{bmatrix} 26 \\ -23 \end{bmatrix}$

(d) Which method works better for this system? Explain (show) carefully why.

36. **TEST** Consider the system of linear equations:

$$\begin{array}{rcrcr} 5x_1 & + & x_2 & = & 11 \\ x_1 & - & 20x_2 & = & -18 \end{array}$$

(a) Perform two steps of Jacobi iteration with starting guess $(12, 21)^T$.

(b) Perform two steps of Gauss-Seidel iteration with starting guess $(12, 21)^T$.

(c) Find the matrix needed to analyze convergence of Jacobi's method and discuss convergence.

(d) Find the matrix needed to analyze convergence for Gauss-Seidel and discuss convergence.

37. Consider heat conduction in a small wire carrying electrical current that is producing heat at a constant rate. The equation describing the temperature $y(x)$ along the wire ($0 \leq x \leq 1$cm) is

$$D\frac{\partial^2 y}{\partial x^2} = -S$$

with boundary conditions $y(0) = y(1) = 0^o$C, thermodiffusion coefficient $D = 0.01\text{cm}^2/\text{sec}$, and normalized source term $S = 1^o\text{C/sec}$.

If we discretize the domain into 20 equal sub-intervals, using $x_j = j/20$ for $j = 0$ to 20, we can approximate (1) at x_j to obtain

$$D\frac{y_{j-1} - 2y_j + y_{j+1}}{h^2} = -S$$

where y_j is the temperature at $x = x_j$ and $h = 0.05$ is the step size. If we apply the boundary conditions at x_0 and x_{20}, we are left with 19 equations for 19 unknown temperatures, y_1 to y_{19}. We can put these equations into the matrix form $A\mathbf{y} = \mathbf{b}$ where

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & \ldots & 0 \\ 1 & -2 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & -2 & 1 & 0 & \ldots & 0 \\ & & & \cdot & & & \\ & & & \cdot & & & \\ & & & \cdot & & & \\ 0 & 0 & \ldots & 0 & 1 & -2 & 1 \\ 0 & 0 & \ldots & 0 & 0 & 1 & -2 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_{18} \\ y_{19} \end{bmatrix}, \quad \text{and } \mathbf{b} = \begin{bmatrix} -0.25 \\ -0.25 \\ -0.25 \\ \cdot \\ \cdot \\ \cdot \\ -0.25 \\ -0.25 \end{bmatrix}$$

Solve the above steady-state problem by writing computer programs to implement Jacobi iteration and Gauss-Seidel iteration. Starting with an initial guess of the zero vector ($\mathbf{y} = \mathbf{0}$), plot $y_j^{(n)}$ vs. x_j, $j = 1, \ldots, 19$, for both methods for $n = 10$, 100, and 1000 (one plot per method). At what step does the maximum error (over y_1 to y_{19}) between successive estimates fall below 1% for the two methods?

$$E_{rel}^{max} = \max_{i=1,\ldots,19} \left| (y_i^{(n)} - y_i^{(n-1)})/y_i^{(n-1)} \right|$$

38. Consider the system of linear equations generated by

$$Ax = b, \quad A = \begin{bmatrix} 5 & 2 & 0 & 0 \\ -1 & 6 & 1 & 0 \\ 0 & 2 & 7 & 1 \\ 0 & 0 & -2 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 4 \end{bmatrix}$$

(a) Verify that the equation in part (a) is equivalent to

$$x = Tx + c, \quad T = \begin{bmatrix} 0 & -\frac{2}{5} & 0 & 0 \\ \frac{1}{6} & 0 & -\frac{1}{6} & 0 \\ 0 & -\frac{2}{7} & 0 & -\frac{1}{7} \\ 0 & 0 & \frac{1}{3} & 0 \end{bmatrix}, \quad c = \begin{bmatrix} \frac{1}{5} \\ \frac{1}{2} \\ -\frac{2}{7} \\ \frac{2}{3} \end{bmatrix}.$$

(b) Calculate $||T||_1$.

(c) Perform 4 iterations of the Jacobi method and find an approximation for the solution. What is your $||.||_1$- estimate of the error?

(d) How many Jacobi iterations do you have to perform so that the error between the n-th iterate x_n and the exact solution x is $||x_n - x||_1 \leq \epsilon$ with $\epsilon = 10^{-5}$.

39. Use Jacobi's Method and Gauss-Seidel to solve

$$\begin{bmatrix} -4 & 2 & 0 & . & . & . & 0 \\ 2 & -4 & 2 & 0 & . & . & 0 \\ 0 & 2 & -4 & 2 & 0 & . & 0 \\ 0 & 0 & . & . & . & 0 & 0 \\ 0 & . & 0 & . & . & . & 0 \\ 0 & . & . & 0 & 2 & -4 & 2 \\ 0 & . & . & . & 0 & 2 & -4 \end{bmatrix} x = \begin{bmatrix} 2 \\ 3 \\ . \\ . \\ . \\ . \\ 11 \end{bmatrix}.$$

40. **TEST** Perform two steps of the Power Method with initial guess $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ for the matrix $\begin{pmatrix} 5 & 0 & 5 \\ 0 & 2 & 0 \\ 5 & 0 & 5 \end{pmatrix}$. From the results, estimate an eigenvalue and an eigenvector.

41. **TEST** Use 2 iterations of the Power Method to approximate the eigenvalue of largest magnitude for $\begin{pmatrix} 8 & 5 \\ 5 & 8 \end{pmatrix}$ with initial guess: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Try to guess an eigenvalue and eigenvector from this?

42. Compute the largest eigenvalue of a matrix using the Power method. Compute the matrix inverse and use the Power method on the inverse to compute the smallest eigenvalue of the original matrix.

43. **TEST** Use 2 iterations of the Power Method on $\begin{bmatrix} 5 & 0 & 5 \\ 0 & 1 & 0 \\ 5 & 0 & 5 \end{bmatrix}$ with initial guesses: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. Discuss the results.

44. Show that $A_1 = \begin{bmatrix} 1 & 0 \\ 1/4 & 1/2 \end{bmatrix}$ is not convergent, but $A_2 = \begin{bmatrix} 1/2 & 0 \\ 16 & 1/2 \end{bmatrix}$ is convergent.

45. **TEST** Use one iteration of Newton's method for systems with initial guess $(1/2, 1/2)^T$ on

$$\begin{aligned} x^2 + y^3 &= 1 \\ x^3 - y^2 &= -1/4 \end{aligned}$$

46. Use Newton's method for systems to find two solutions near the origin of

$$\begin{aligned} x^2 + x - y^2 &= 1 \\ y - \sin x^2 &= 0 \end{aligned}$$

47. **TEST** Perform one step of Newton's Method for systems with initial guess (-1,2) on:

$$\begin{aligned} x^2 + xy - y^2 &= 5 \\ x^3 + y &= 9 \end{aligned}$$

Interpolation and Curve Fitting

1. Given that $\ln(2) = 0.69315$, $\ln(3) = 1.0986$ and $\ln(6) = 1.7918$ interpolate using a Lagrange polynomial to approximate the natural logarithm of each integer from one to ten. Tabulate your results with the absolute and relative errors.

2. **TEST** Find the Lagrange polynomial through the $(x,\ y)$ points (1,2), (2,5), (4,5) and (5,2) and use it to approximate $y(3)$. Use a divided difference table to find the same polynomial.

3. **TEST** Consider a function $f(x)$ with the following values known:

x	0	1	3	4
$f(x)$	2	2	2	14

(a) Find the Lagrange Polynomial through all the points.

(b) Find the Lagrange Polynomial through $x = 0, 1$ and 3.

(c) Find the Lagrange Polynomial through $x = 1, 3$ and 4.

(d) Use the results of parts (a), (b) and (c) to approximate $f(2)$ and write an expression for the error terms. (Your answer may include $f^{(n)}(\xi)$).

(e) Approximate $f'(2)$ using $f(0)$ and $f(4)$.

(f) Use information in the table and Richardson extrapolation to improve your estimate of $f'(2)$.

4. **TEST** Consider a function $f(x)$ with the following values known:

x	-1	1	3	6
$f(x)$	1	3	-3	-27

(a) Set up the Lagrange Polynomial through all the points. (Do not multiply it out).

(b) Use a divided difference table to find the same polynomial as in part (a). (Do not multiply it out).

5. Construct the quadratic polynomial using Lagrange polynomials and divided differences for $y = \frac{1}{x}$ using the points $x = 1,\ 3/2,\ 2$ and find an upper bound for the error.

6. Show that the error in polynomial interpolation using 6 equally spaced points (quintic interpolation) satifies

$$|f - p_5| \leq 0.0235 h^6 |f^{(6)}$$

7. **TEST** Consider a function $f(x)$ with the following values known:

x	-1	0	2
$f(x)$	9	4	12

(a) Find the Lagrange Polynomial through the 3 points.

(b) Use the result of part (a) to approximate $f(1)$ and write an expression for the error term.

(c) Use the result of part (a) to approximate $f'(1)$.

(d) Use the result of part (a) to approximate $\int_0^2 f(x)\, dx$.

8. **TEST** Consider a function $f(x)$ with the following values known:

x	0	1	4
$f(x)$	2	1	3

(a) Find the Lagrange Polynomial (show set-up work) through the points.

(b) Use the results of part (a) to approximate $f(2)$, $f(3)$ and $f'(2)$.

9. **TEST** Find the Lagrange Polynomial through the points.

x	0	2	5	6
$f(x)$	-2	-4	53	112

and use the result to approximate $f(1)$.

10. **TEST** Consider the function $f(x) = \sqrt[3]{1 + x}$

(a) Find $f(-1), f(0), f(7), f'(x), f''(x)$.

(b) Find the second degree Taylor polynomial approximation to $f(x)$ around $x = 0$. Use this to estimate $f(1)$.

(c) Use the information from part (a) to find a Lagrange polynomial of degree 1 to approximate $f(1)$. Give an upper bound to the error of this approximation.

(d) Find the highest degree Hermite polynomial possible using information only at $x = 0$ and $x = 7$.

11. A car traveling along a straight road is clocked at a number of points. The data are given in the table, where time is in seconds, distance is in feet and speed is in feet per second:

Time	0	3	5	8	13
Distance	0	225	383	623	993
Speed	75	77	80	74	72

(a) Use a Hermite polynomial to predict the position of the car and its speed when $t = 10$.

(b) Use the derivative of the Hermite polynomial to determine whether the car ever exceeds a 55 mlies per hour speed. If so, what is the time that the car first exceeds this value?

12. **TEST** Given the data

x	0	1	2
$f(x)$	1/2	1/8	1/18

(a) Find the interpolating polynomial, $P_2(x)$, using divided differences?

(b) What is $P_2(3/2)$?

13. **TEST** For the data in problem 9 (this is the same as 9) Use the data in the previous problem to:

(a) Construct the Lagrange polynomial $P_2(x)$ that interpolates the function $f(x)$ at the nodes x_0, x_1, x_2.

(b) Find the value of $P_2(x)$ at $x = \frac{3}{2}$ and compare it with $f(x)$ at that point.

(c) Find the error bound for approximating $f(x)$ by $P_2(x)$ at $x = \frac{3}{2}$. Does it agree with the absolute error you found in part (b)?

14. Construct a divided difference table to compute a Lagrange polynomial using all of the following data:

x	0.5	−0.2	0.7	0.1	0.0
$f(x)$	−1.1518	0.7028	−1.4845	−0.14943	0.13534

15. **TEST** Let $f(x) = x^{5/2}$.

(a) Construct the second degree Lagrange polynomial $P_2(x)$ that interpolates $f(x)$ at the nodes $x_0 = 0$, $x_1 = 1$, $x_2 = 4$.

(b) Find the value of $P_2(x)$ at $x = 2$ and calculate the actual error at that point.

(c) Calculate the error estimate for approximating $f(x)$ by $P_2(x)$ at $x = 2$. Does it agree with the error in (b)?

16. Find the coefficient matrix and the right-hand side for fitting a cubic spline to the following data. (Use free boundary conditions).

x	0.15	0.27	0.76	0.89	1.07	2.11
$f(x)$	0.1680	0.2974	0.7175	0.7918	0.8698	0.9972

17. **TEST** $S(x)$ is a cubic spline. Find $S(3)$.

$$S(x) = \begin{array}{lll} 1 + ax + bx^3 & \text{on} & 0 \le x \le 2 \\ 29 + 38(x-2) + c(x-2)^2 - 3(x-2)^3 & \text{on} & 2 \le x \le 3 \end{array}$$

18. **TEST** Set up the system of equations for the cubic spline through the points:

x	0	2	5	6
$f(x)$	−2	−4	53	112

given $S_0 = g''(0) = -6$ and $S_3 = g''(6) = 30$. DO NOT SOLVE.

19. **TEST** Show $S(x)$ is a cubic spline. Is it natural or is it clamped?

$$S(x) = \begin{array}{lll} 1 + x + x^2 + x^3 & \text{on} & 0 \le x \le 1 \\ 4 + 6(x-1) + 4(x-1)^2 & \text{on} & 1 \le x \le 2 \end{array}$$

20. For what value of k is the following a cubic spline?

$$\begin{aligned} f_1(x) &= kx^2 + 3/2 & 0 \le x \le 1 \\ f_2(x) &= x^2 + x + 1/2 & 1 \le x \le 2 \end{aligned}$$

21. Is the following function a spline? Why or why not?

$$\begin{aligned} f_1(x) &= 0 & x < 0 \\ f_2(x) &= x^2 & 0 \le x \le 1 \\ f_3(x) &= -2x^2 + 6x + 3 & 1 \le x \le 2 \\ f_4(x) &= (x-3)^2 & 2 \le x \le 3 \\ f_5(x) &= 0 & x \ge 3 \end{aligned}$$

22. **TEST** Find the values of a, b, c, d, e and f such that the following functions define a cubic spline and find $f(0)$, $f(1)$, $f(2)$ and $f(3)$.

$$\begin{aligned} f_1(x) &= 2x^3 + 4x^2 - 7x + 5 & 0 \le x \le 1 \\ f_2(x) &= 3(x-1)^3 + a(x-1)^2 + b(x-1) + c & 1 \le x \le 2 \\ f_3(x) &= (x-2)^3 + d(x-2)^2 + e(x-2) + f & 2 \le x \le 3 \end{aligned}$$

23. **TEST** A natural cubic spline g on $[0, 2]$ is defined by:

$$\begin{aligned} g_0(x) &= 1 + 2x - x^3, & 0 \le x \le 1, \\ g_1(x) &= a + b(x-1) + c(x-1)^2 + d(x-1)^3, & 1 \le x \le 2. \end{aligned}$$

   (a) What conditions should g_0 and g_1 satisfy at $x = 1$?

   (b) What condition must $g_1(x)$ satisfy at $x = 2$?

   (c) Apply the conditions in (a) and (b) to find a, b, c, d.

24. If the data given are periodic and cover one period, the first and last points will have identical function values and slopes. Develop cubic spline relations (equations) for such periodic data.

25. Suppose that $f(x)$ is a polynomial of degree 3 on the interval $[a, b]$. Show that $f(x)$ is its own clamped cubic spline on this interval, but it cannot be its own free cubic spline.

26. Suppose the data $(x_i, f(x_i))$, $i = 1, 2, ..., n$ lie on a straight line. What can be said about the free and clamped cubic splines for f.

27. **TEST**

(a) Set up the system of linear equations for finding a least squares line through the points $(-2,-1)$, $(-1,0)$, $(0,1)$, and $(7,2)$

(b) Which of the following methods may be used to solve this system (DO NOT solve the system) (i) Gaussian Elimination, (ii) LU decomposition, (iii) Choleski's Method, (iv) LDL^T.

28. **TEST** Find the best (Least-Squares) line through the points $(-8,-9)$ $(-3,-4)$ $(-1,-2)$ and $(12,11)$.

29. Find the best "least squares" function of the form $z = ax + by + c$ for the data:

0	0	0.9573
0	1	2.0132
1	0	2.0385
1	1	1.9773
0.5	0.5	1.9936

29a. Compute the best (least squares) line through (R is a function of T)

R	765	826	873	942	1032
T	20.5	32.7	51.0	73.2	95.7

30. **TEST** Find the best (Least-Squares) line through the points $(0,2)$ $(0,8)$ $(1,-1)$ and $(3,11)$.

31. **TEST** Consider the following set of values for (x_i, Y_i): (1,1.9), (2,3.1), (4,4.8). Find the least-squares line, of the form $y = ax+b$, that interpolates this data. What is the squared error, $\sum_{i=1}^{3}(y_i - Y_i)^2$, for this interpolation?

32. **TEST** The data points $(0, e^{0.1})$, $(1, e^2)$ and $(3, e^3)$ should lie approximately on a curve of the form $y = ae^{bx}$. Set up a system of linear "least squares" equations that would be useful for approximating a and b. DO NOT SOLVE.

33. Consider the following overdetermined system of equations:

$$Ax = b, \quad A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

Find the best (least squares) solution.

34. Find the best (least squares) values for a and b by fitting the data below to $y = ae^{bx}$ where y is the Solubility. (Take logs to turn it into a linear least squares problem).

T^o	77	100	185	239	285
Solubility	2.4	3.4	7.0	11.1	19.6

35. It is suspected (from theoretical considerations) that the rate of flow from a fire hose is proportional to some power of the pressure at the nozzle. (Flow = constant Pressure^{power}). Find the least squares values of the constant and the power given the following data.

Flow (gal/min)	94	118	147	180	230
Pressure (psi)	10	16	25	40	60

Numerical Differentiation and Integration

1. Use the Lagrange polynomial through $f(x_0 - h)$, $f(x_0)$ and $f(x_0 + h)$ to construct an approximation for $f''(x_0)$.

2. **TEST** What does the difference scheme approximate and give its error order? $\frac{1}{2h}\left[f(x+3h) + f(x-h) - 2f(x)\right]$.

3. **TEST** Given $f'(x_0) = \frac{1}{h}\left[f(x_0+h) - f(x_0)\right] - \frac{h}{2}f''(x_0) + O(h^2)$. Use the numerical derivative approximation $f'(x_0) \approx \frac{1}{h}\left[f(x_0+h) - f(x_0)\right]$ and step size of $2h$ combined with the idea of Richardson extrapolation to derive a 3 point formula for $f'(x_0)$ with improved (i.e., $O(h^2)$) error.

4. **TEST** Derive a (the best possible) difference formula for $f''(x_0)$ through $f(x_0)$, $f(x_0 - h)$ and $f(x_0 + 2h)$ and find the leading error term.

5. **TEST** Derive a (the best possible) difference formula for $f''(x_0)$ through $f(x_0)$, $f(x_0 + h)$ and $f(x_0 + 3h)$ and find the leading error term.

6. Derive a three-point centered difference scheme for $f'(x_1)$ with the unevenly spaced nodes x_0, x_1, x_2 be defined as: $x_0 = x_1 - h$, $x_2 = x_1 + 2h$

   (a) Construct the interpolation polynomial $P_2(x)$ in terms of the values of the function $f(x)$ at these nodes.

   (b) Differentiate the interpolation polynomial $P_2(x)$ with respect to x and evaluate it at $x = x_1$. Write down both the difference formula and the remainder term. What order of accuracy is this scheme?

   (c) Now let $f(x) = \ln x$, $x_1 = 1$, and take $h = 0.1$. Calculate the approximation of $f'(1)$ and compare it with the exact value. What is the actual error? Calculate the error from the remainder term obtained in (b) and compare it to the actual error. Do they agree?

7. Analyze the round-off errors to obtain the optimal step size for the formula $f'(x_0) = \frac{f(x_0+h)-f(x_0)}{h} - \frac{h}{2}f''(\xi_0)$.

8. **TEST** Given $f'(x) = \frac{1}{3h}\left[f(x+2h) - f(x-h)\right] + O(h)$. If $g(x) = 2x^3 + x^2 + 1$, use the formula given to approximate $g'(-1)$ using $h = 2$. Do the same for $h = 1$. Improve the result using Richardson extrapolation.

9. Show the approximation

$$f'(x) = \frac{8f(x+h) - 8f(x-h) - f(x+2h) + f(x+2h)}{12h}$$

in $O(h^4)$.

10. **TEST** Using the smallest step size possible, approximate $\int_2^6 \frac{1}{1+x} dx$ using the Midpoint Rule, the Trapezoidal rule and Simpson's Rule using only the values of $f(x)$ at $x = 2, 3, 4, 5$ and 6.

11. The length of a curve $y = g(x)$ is given by $\int_a^b \sqrt{1 + (g'(x))^2} dx$. Use Trapezoidal rule and Simpson's rule to compute the length of one arch of the sine curve.

12. Let $f(x) = |x|$; Use the trapezoidal rule with one interval and the smallest Simpson's iteration possible (3 points) and compare the result with the exact value for $\int_{-1}^{1} |x| dx$. Comment on the error and the theoretical value of the error.

13. Use the Midpoint Rule with $h = \frac{1}{4}$ to approximate $\int_0^1 x(1-x^2)dx$. How small does h have to be to get the error less than 10^{-3}? 10^{-6}? and compare to the theoretical value.

14. Use the Midpoint Rule with $h = 2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$ to approximate $\int_1^3 \ln x dx$. Confirm the approximations are converging at the correct rate. How small does h have to be theoretically to get the error less than 10^{-3}? 10^{-6}? Use Romberg Integration on this problem assuming that the error has only even power terms.

15. If the following data comes from a midpoint rule computation, is it converging as it should?

Number of Points	4	8	16	32	64
Values	-0.91595	-0.95732	-0.97850	-0.98921	-0.99459

16. **TEST** Approximate

$$\int_0^4 \frac{1}{1+x} dx$$

using the Trapezoidal rule (with $h = 1$) and Simpson's Rule (with $h = 1$). Find a theoretical error bound in each case.

17. Apply the Trapezoidal rule to the integral $\int_0^1 \sqrt{x} dx$ using $h = 1/2,\ 1/4,\ 1/8,\ \ldots$. Do you get the expected rate of convergence? Explain.

18. **TEST** Given

x	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$f(x)$	440	0	-162	-160	-84	0	50	48	0	-64	-90	0

(a) Approximate $\int_0^6 f(x)dx$ using Trapezoidal rule with $h = 2$.

(b) Approximate $\int_0^6 f(x)dx$ using Trapezoidal rule with $h = 6$.

(c) Improve the results of parts (a) and (b) using Romberg's (Richardson's) Method.

(d) Approximate $\int_0^6 f(x)dx$ using Simpson's rule using the largest step size possible with the data given. What is the (global) error order for Simpson's rule?

(e) Find the Lagrange Polynomial through $f(-5)$, $f(-2)$, $f(3)$ and $f(4)$ by setting it up by the definition. (Do not multiply it out).

(f) Find the Lagrange Polynomial through $f(-5)$, $f(-2)$, $f(3)$ and $f(4)$ using a divided difference table.

(g) Find the least squares line through $f(-5)$, $f(-2)$, $f(3)$ and $f(4)$.

19. Write a computer program that uses the composite Simpson's rule to find the value of the following integrals. Tabulate the results using $n = 2, 4, 8, 16, \ldots, 16384$ intervals using double precision. Compare these with the best approximations available (that is, the ones obtained at the largest value of n) and determine the number of intervals n needed for the numerical solution to be within an absolute error of 10^{-6}.

(a)
$$I_1 = \int_{-2}^{2} e^{-2x} dx.$$

(b)
$$I_2(t) = \int_{-\pi}^{\pi} \frac{250x^2}{\cosh^2 [500(x-t)]} dx$$

for values $t = 0, 0.5, 1.0, 2.0$

(c) Find the error bound as a function of n for part (a). Does it agree with your approximation to the exact value of the integral?

20. **TEST** Given a continuous smooth function $f(x)$ for which $f(0) = 8$, $f(1) = 5$, $f(2) = 3$, $f(3) = 2$, and $f(4) = 3$

(a) Use a 3-point centered difference scheme to approximate $f''(2)$.

(b) Use Richardson extrapolation to improve this result.

(c) Use the (composite) Trapezoidal Rule and Simpson's Rule to approximate $\int_0^4 f(x)dx$.

21. **TEST** Let $f(x) = e^{-x^2}$ and consider the integral

$$I = \int_0^1 f(x)dx.$$

(a) Use the composite Trapezoidal rule to approximate the integral I using $h = 0.25$.

(b) Use the composite Simpson's 1/3 rule to approximate the integral I using $h = 0.25$. Use this result to estimate the absolute error in part (a).

(c) Calculate the bound on the absolute error for the Trapezoidal rule in part (a) and compare it to the error estimated in part (b).

22. **TEST** Determine whether you need to subdivide the interval in

$$\int_0^{0.2} e^{-10x^2}\,dx,$$

if you wish to achieve the accuracy $\epsilon = 10^{-5}$ in the adaptive quadrature method (Using Simpson's Rule).

23. **TEST** Let $f(x) = \frac{1}{\sqrt{x}}$.

(a) Use the composite trapezoidal rule with five subintervals to approximate $\int_0^1 f(x)dx$. Compare the result with the exact value.

(b) Calculate the step size for the composite trapezoidal method you need to approximate the value of $\int_0^1 f(x)dx$ to within $\epsilon = 10^{-5}$?

(c) Use the forward difference formula with step $h = 0.2$ to approximate $f'(x)$ at $x = 1$. Calculate the absolute error.

(d) Calculate the error bound for part (c). Does it agree with the actual error you found in part (c)?

24. **TEST** The following data give approximations, I, to an integral $\int_a^b f(x)dx$ for a scheme with error terms $E = K_1 h + K_2 h^3 + K_3 h^5 + \ldots$.

$$I(h) = 2.3965, \quad I(h/3) = 2.9263, \quad I(h/9) = 2.9795$$

Construct an extrapolation table to obtain a better approximation.

25. **TEST** Consider the points

x	1	2	3	4	5
$f(x)$	0.01	0.69	1.10	1.39	1.61

(a) Approximate $f'(3)$ using centered difference with $h = 2$.

(b) Approximate $f'(3)$ using centered difference with $h = 1$.

(c) Use Richardson extrapolation to improve the results in parts (a) and (b).

(d) Approximate $f''(3)$ using centered difference with $h = 1$.

(e) Set up the finest possible Trapezoidal rule approximation of $\int_1^5 f(x)\,dx$. DON'T ADD UP.

(f) Set up the coarsest possible Trapezoidal rule approximation of $\int_1^5 f(x)\,dx$. DON'T ADD UP.

(g) Set up the finest possible Simpson's rule approximation of $\int_1^5 f(x)\,dx$. DON'T ADD UP.

(h) Set up the coarsest possible Simpson's rule approximation of $\int_1^5 f(x)\,dx$. DON'T ADD UP.

26. **TEST**

(a) What does the expression $\frac{4f(x+h)-3f(x)-f(x-2h)}{6h}$ approximate at x?

(b) What is the leading order of the error in part a (I.e., h, h^2, etc.)?

(c) Use $f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$ on the data

x	-3	-2	-1	0	1	2	3
$f(x)$	25	17	11	7	5	5	7

to approximate $f'(0)$ using $h = 3$ and $h = 1$.

(d) Use Richardson extrapolation to improve the result in part (c).

27. The following data give approximations to the integral $\int_0^\pi \sin x dx$ for a scheme with error terms $E = K_1h^2 + K_2h^4 + K_3h^6 + \ldots$. $I(h) = 1.570796$, $I(h/2) = 1.896119$, $I(h/4) = 1.974232$, $I(h/8) = 1.993570$. Construct an extrapolation table to obtain better approximations.

28. Suppose that $I(h)$ is an approximation to an integral M for every $h > 0$ and that $M = I(h) + K_1h + K_2h^2 + K_3h^3 + \ldots$. Use the values $I(h)$, $I(h/3)$ and $I(h/9)$ to produce an $O(h^3)$ approximation to M.

29. The Trapezoidal rule applied to $\int_0^2 f(x)dx$ gives the value 4 and Simpson's rule gives 2. What is $f(1)$?

30. Derive Simpson's 3/8 rule (i.e. using $f(x_0)$, $f(x_1)$, $f(x_2)$ and $f(x_3)$) and compute the error term by matching with exact integrals on $[0, a]$ for as many terms as possible.

31. Consider the data in the table:

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
$f(x)$	1.543	1.669	1.811	1.971	2.151	2.352	2.577	2.828	3.107

(a) Approximate $f'(1.4)$ using forward (non-centered) differences using $h = 0.1,\ 0.2,$ and 0.4.

(b) Use Richardson extrapolation to improve the results.

(c) Approximate $f'(1.4)$ using centered differences using $h = 0.1,\ 0.2,$ and 0.4.

(d) Use Richardson extrapolation to improve the results.

(e) Use the Trapezoidal Rule to approximate $\int_1^{1.8} f(x)dx$ using $h = 0.1,\ 0.2,\ 0.4$ and 0.8.

(f) Use Romberg's method to improve the results.

(g) Use Simpson's Rule to approximate $\int_1^{1.8} f(x)dx$ using $h = 0.1,\ 0.2,$ and 0.4.

(h) Use Romberg's method to improve the results.

32. Consider the data in the table:

x	$f(x)$
1.6	4.953
1.8	6.050
2.0	7.389
2.2	9.025
2.4	11.023
2.6	13.464
2.8	16.445
3.0	20.086

(a) Approximate $\int_{1.8}^{2.8} f(x)dx$ using the Trapezoidal rule ($h = 0.2$).

(b) Compute the exact error, noting that $f(x) = e^x$.

(c) Compute the error bound for the Trapezoidal Rule for this case.

(d) If we did not know the true function, we would have to approxiamte the maximum second derivative of $f(x)$ using the data. Compute the error bound in this manner.

(e) If we want the computation to be correct to 5 decimal places (error ≤ 0.000005) how small should the step size h be?

33. **TEST**

(a) Approximate $\int_{-3}^{3} f(x)dx$ for the data:

x	-3	-2	-1	0	1	2	3
$f(x)$	25	17	11	7	5	5	7

using the Trapezoidal rule with step sizes $h = 1$ and 2.

(b) Use Romberg integration to improve the result.

(c) Approximate $\int_{-3}^{3} f(x)dx$ for the data using Simpson's rule with two different step sizes.

(d) Compare the results of parts (b) and (c) and explain.

34. **TEST** Consider a function $f(x)$ with the following values known:

x	1	5/4	3/2	7/4	2
f(x)	11	7	5	4	6

.

(a) Use the Trapezoidal Rule with 2 trapezoids to approximate $\int_1^2 f(x)\,dx$.

(b) Use the Trapezoidal Rule with 4 trapezoids to approximate $\int_1^2 f(x)\,dx$.

(c) Use parts (a) and (b) and Romberg Integration (Richardson Extrapolation) to obtain a “better” approximation of $\int_1^2 f(x)\,dx$.

(d) Use Simpson’s Rule (with the smallest step size possible with the given data) to approximate of $\int_1^2 f(x)\,dx$.

(e) Use the Midpoint Rule (Rectangle Rule) with the smallest step size possible with the given data to approximate of $\int_1^2 f(x)\,dx$.

35. Use Trapezoidal Rule and Simpson’s Rule to approximate the following integrals, refining the step size until you believe the answer is reasonable. What is the step size in each case? (These integrals might fool automatic step size finding algorithms).

$$(a).\ \int_0^1 x^{0.001}dx \quad (b).\ \int_0^1 \frac{dx}{1+(230x-30)^2}$$

36. Find the degree of precision of the quadrature formula

$$\int_{-1}^{1} f(x)dx = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$$

37. The quadrature formula $\int_{-1}^1 f(x)dx = c_0 f(-1) + c_1 f(0) + c_2 f(1)$ is exact for all polynomials of degree less than or equal to 2. Determine c_0, c_1 and c_2.

38. The quadrature formula $\int_0^1 f(x)dx = c_0 f(0) + c_1 f(x_1)$ has the highest possible degree of precision. Determine c_0, c_1 and x_1.

39. The quadrature formula $\int_0^1 f(x)dx = \frac{1}{2} f(x_0) + c_1 f(x_1)$ has the highest possible degree of precision. Determine x_0, c_1 and x_1.

40. Use the transformation $t = \frac{1}{x}$ and Simpson’s rule to approximate

$$a.\ \int_1^\infty \frac{1}{1+x^4} \quad \text{and} \quad b.\ \int_0^\infty \frac{1}{1+x^4}$$

within 10^{-6}. You may want to break the second integral into two parts.

Differential Equations IVPs and BVPs

1. **TEST** Consider the ordinary differential equation $\frac{dy}{dt} = -t^2 + y^2$ with $y(0) = 1$. Use the Runge-Kutta method with $h = 2$ to approximate $y(2)$.

2. **TEST** Consider the ordinary differential equation $\frac{dy}{dt} = (1-t)y$ with $y(0) = 3$.

   (a) Use Euler's method with $h = 2$ to approximate $y(2)$ and $y(4)$.

   (b) Use Fourth Order Runge Kutta method with $h = 2$ to approximate $y(2)$.

   (c) Use part (a) at $t = 2$ to initialize Adams-Bashforth

$$w_{n+1} = w_n + \frac{h}{2}[3f_n - f_{n-1}]$$

   and Adams-Moulton $w_{n+1} = w_n + \frac{h}{12}[5f_{n+1} + 8f_n - f_{n-1}]$ to approximate $y(4)$. (I.e., all 3 – part (a), Adams-Bashforth and Adams-Moulton are all used to obtain one answer).

3. **TEST** Consider the ordinary differential equation

$$\frac{dy}{dt} = -t^2 y \quad \text{with} \quad y(2) = 2$$

   (a) Use Euler's method with $h = 1$ to approximate $y(3)$.

   (b) Use the second order Taylor series method with $h = 1$ to approximate $y(3)$.

   (c) Use Fourth Order Runge Kutta method with $h = 1$ to approximate $y(3)$.

   (d) Use the Adams-Bashforth Two-Step method with $h = 1$ and the results of part (a) to approximate $y(4)$.

   (e) Use the Adams-Moulton Two-Step method to update your result in part (d).

4. **TEST** Consider the ordinary differential equation $\frac{dy}{dt} = ty - 1$ with $y(0) = 3$.

   (a) Use Euler's method with $h = 2$ to approximate $y(2)$.

   (b) Use Modified Euler's method with $h = 2$ to approximate $y(2)$.

   (c) Use Fourth Order Runge Kutta method with $h = 2$ to approximate $y(2)$.

   (d) Use Taylor Series Method up to the x^3 term to approximate $y(2)$.

   (e) Use the result of part (b) along with the predictor-corrector method to approximate $y(4)$.

5. **TEST** Consider the ordinary differential equation $\frac{dy}{dt} = 3\frac{y}{t}$ with $y(1) = 2$ and (when useful below) $y(-1) = -2$.

   (a) Use Euler's method with $h = 1$ to approximate $y(3)$.

(b) Use the 2nd order Taylor series method with $h = 2$ to approximate $y(3)$.

(c) Use the 4th-order Runge-Kutta method with $h = 2$ to approximate $y(3)$.

(d) Use the Adams-Bashforth Two-Step method to approximate $y(3)$.

(e) Use the Adams-Moulton Two-Step method to approximate $y(3)$.

6. **TEST** Consider the initial-value problem

$$\frac{dy}{dt} = \frac{t}{y}, \quad y(0) = 1, \quad 0 \leq t \leq 1$$

(a) Find the approximate solution to the above problem using Euler's Method with $h = 0.25$.

(b) Given that the exact value is $y(1) = 1.41421356$, find the absolute error in part (a) and determine the h needed to get an absolute error of 10^{-7}.

7. **TEST** Consider the ordinary differential equation $\frac{dy}{dt} = 2ty$ with $y(1) = 2$.

(a) Use Euler's method with $h = 0.5$ to approximate $y(2)$.

(b) Use the second order Taylor series method with $h = 1$ to approximate $y(2)$.

(c) Use the Runge-Kutta method with $h = 1$ to approximate $y(2)$.

(d) Use the Adams-Bashforth Two-Step method with $h = 1$ and the results of part (a) to approximate $y(3)$.

(e) Use the Adams-Moulton Two-Step method to update your result in part (e).

(f) Compare the stability of the Adams-Bashforth and Adams-Moulton Two Step methods for the ODE $\frac{dy}{dt} = -4y$ with $h = 1$.

8. Consider the differential equation $\frac{dy}{dx} = x + y + xy$ with $y(0) = 1$.

(a) Use the Taylor series method with terms through x^3 to approximate $y(0.1)$ and $y(0.5)$.

(b) Use Euler's method to approximate $y(0.1)$ and $y(0.5)$. Use step size 0.1.

(c) Use Improved Euler's method to approximate $y(0.1)$ and $y(0.5)$. Use step size 0.1.

(d) Use the Fourth Order Runge-Kutta method to approximate $y(0.1)$ and $y(0.5)$. Use step size 0.1.

9. Consider the differential equation $\frac{dy}{dx} = x + y$ with $y(0) = 1$. (The analytical solution is $y(x) = 2e^x - x - 1$.)

(a) Use the Taylor series method with terms through x^3 to approximate $y(0.1)$ and $y(0.5)$.

(b) Use Euler's method to approximate $y(0.1)$ and $y(0.5)$. Use step size 0.1.

(c) Use Improved Euler's method to approximate $y(0.1)$ and $y(0.5)$. Use step size 0.1.

(d) Use the Fourth Order Runge-Kutta method to approximate $y(0.1)$ and $y(0.5)$. Use step size 0.1.

10. For the differential equation $\frac{dy}{dx} = (\frac{y}{2})(1 - \frac{y}{5})$ with $y(0) = 1$: (The analytical solution is $y(x) = \frac{5}{1 + 4e^{-x/2}}$.)

    (a) Use the Taylor series method with terms through x^3 to approximate $y(0.1)$ and $y(0.5)$.

    (b) Use Euler's method to approximate $y(0.1)$ and $y(0.5)$. Use step size 0.1.

    (c) Use Improved Euler's method to approximate $y(0.1)$ and $y(0.5)$. Use step size 0.1.

    (d) Use the Fourth Order Runge-Kutta method to approximate $y(0.1)$ and $y(0.5)$. Use step size 0.1.

11. For the differential equation $\frac{dy}{dx} = \frac{x}{y}$ with $y(0) = 1$:

    (a) Use Euler's method to approximate $y(1)$. Use step size 0.1 and 0.2.

    (b) Extrapolate to improve the results of part (a) assuming that error proportional to step size. Compare to the analytical result ($y^2 = 1 + x^2$).

    (c) Repeat parts (a) and (b) with modified Euler noting that error is proportional to step size squared.

12. For the differential equation $\frac{dy}{dt} = y - t^2$ with $y(0) = 1$ with $y(0.2) = 1.2186$, $y(0.4) = 1.4682$ and $y(0.6) = 1.7379$.

    (a) Use the Adams-Bashforth four-step method to compute the solution through $t = 1.2$

    (b) Use the Adams-Bashforth four-step method in combination with the Adams-Moulton method of the same order (3-step) to compute the solution through $t = 1.2$

13. Show that the Midpoint method, the Modified (Improved) Euler method, and Heun's Method give the same approximations for the IVP $y' = -y + t + 1$, on $0 \leq t \leq 1$, where $y(0) = 1$ for any choice of h. Why is this true?

14. Consider the initial-value problem:

$$y' = xy^2, \quad 0 \le x \le 2, \quad y(0) = \frac{2}{5}.$$

(a) Write a computer program to solve the ODE using Euler's method and Improved-Euler's method. Use your program to compute approximate solution $y(x)$ up to $x = 2$ for step sizes $h = \frac{1}{2^n}$, with $n = 1, 2, \ldots, 10$. Plot the approximate solutions for each n.

(b) Use the 4th-order Runge-Kutta method to compute approximate solutions to the ODE for the same step sizes as in Part 1. Given the exact solution, $y = \frac{2}{5-x^2}$, make log-log plots of the absolute error in $y(2)$ vs. h for the three methods. What do these plots tell you?

(c) Using 4th-order Runge Kutta method with $n = 4$, i.e., $h = 0.0625$ to compute the solution to the ODE up to $x = 3$. What happens and why?

15. Use the Adams-Moulton method (two-step) for the linear ODE $\frac{dy}{dt} = (1 + 3t^2)y$ with $y(0) = 1$ and $y(0.1) = 1.10628$ (The exact solution is e^{t+t^3}) to compute an approximation of $y(1)$. Notice that since the ODE is linear, the corrector method can be applied on its own. That is, Adams-Moulton reduces to an explicit method.

16. **TEST** Write the ODE $y'' - y' + y = t^2$ with $y(2) = 2$, $y'(2) = 3$ and $h = 1$ as a system of first order ODEs. Use Euler's method to approximate $y(4)$.

17. **TEST** Set up the first-order system of ODEs (the initial value problem) for the ODE initial value problem $y''' + 3y''y - 6(y')^2 + 2y = 3t$ with $y(0) = 1$; $y'(0) = 2$; $y''(0) = 3$.

18. **TEST** Investigate the following multistep difference scheme for solution to differential equations of the form $y' = f(t, y)$ for consistency, stability and convergence.

$$w_{i+1} = \frac{3}{2}w_i - \frac{1}{2}w_{i-1} + \frac{1}{2}hf(t_i, w_i)$$

19. **TEST** Find a and b such that the method for solving ODEs given by:

$$w_{n+1} = aw_{n-1} + bh[f(t_{n-1}, w_{n-1}) + 4f(t_n, w_n) + f(t_{n+1}, w_{n+1})]$$

is consistent. Is it stable? For extra credit – find the local truncation error.

20. **TEST** Consider the multi-step scheme

$$w_{i+1} = \frac{5}{4}w_i - aw_{i-1} + h[bf(t_i, w_i) + cf(t_{i-1}, w_{i-1})].$$

Find a, b and c to obtain the highest order scheme possible. Is the method stable? Find the inequality that must be satisfied for the method to be stable for the ODE $y' = -y$ for a step size h.

21. **TEST** Consider the Gear solver for solving ordinary differential equations:

$$w_{i+1} = \frac{18}{11}w_i - \frac{9}{11}w_{i-1} + \frac{2}{11}w_{i-2} + ahf(t_{i+1}, w_{i+1})$$

where a is a constant.

   (a) Find the characteristic polynomial to study the stability of this method in general.

   (b) Find a such that the method is consistent.

   (c) Noting that one solution to the polynomial you hopefully found in part (a) is 1, determine whether the method is stable.

22. Use the approximation $y'(t_n) = \dfrac{y(t_{n+1}) - y(t_n)}{2h}$ to derive a numerical method for ODEs. What is the residual? What is the truncation error? Is the method consistent?

23. Use the approximation $y'(t_{n-1}) = \dfrac{-y(t_{n+1}) + 4y(t_n) - 3y(t_{n-1})}{2h}$ to derive a numerical method for ODEs. What is the residual? What is the truncation error? Is the method consistent?

24. Show that the method $y_{n+1} = 4y_n - 3y_{n-1} - 2hf(t_{n-1}, y_{n-1})$ is unstable.

25. The IVP $y' = e^y$, on $0 \le t \le 0.20$ with $y(0) = 1$ has solution $y(t) = 1 - \ln(1 - et)$. Apply the three step Adam-Moulton method. You will get the nonlinear equation:

$$w_{i+1} = w_i + \frac{h}{24}\left[9e^{w_{i+1}} + 19e^{w_i} - 5e^{w_{i-1}} + e^{w_{i-2}}\right]$$

   (a) With $h = 0.01$, obtain w_{i+1} by functional iteration for $i = 1, 2, ..., 19$ using exact starting values $w_0,\ w_1,\ w_2$. At each step, use w_i as your first guess.

   (b) Solve each time using Newton's method with w_i as your first guess. Does Newton's method speed convergence over functional iteration?

26. **TEST**

   (a) Consider the one-step method for solving ordinary differential equations:

$$w_{i+1} = w_i + \frac{h}{4}\left[f(t_i, w_i) + 3f(t_i + \frac{2h}{3}, w_i + \frac{2h}{3}f(t_i, w_i))\right]$$

   What inequality must be satisfied by the step size h such that the method will be stable for the ODE $y' = -y$? (Do not solve the inequality).

(b) Consider the multi-step method

$$w_{i+1} = \frac{1}{2}(w_i + w_{i-1}) + \frac{h}{4}[f(t_i, w_i) + 5f(t_{i-1}, w_{i-1})]$$

Is this method consistent? Is this method stable?

27. Derive Simpson's method for ODEs by applying Simpson's rule to the integral in:

$$y(t_{i+1}) - y(t_{i-1}) = \int_{t_{i-1}}^{t_{i+1}} f(t, y(t))dt$$

28. Show that the fourth order Runge-Kutta method, when applied to the ODE $y' = \lambda y$, can be written in the form

$$w_{i+1} = \left(1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3 + \frac{1}{24}(h\lambda)^4\right) w_i$$

29. Discuss consistency, stability and convergence for the Implicit Trapezoidal method $w_{i+1} = w_i + \frac{h}{2}[f(t_{i+1}, w_{i+1}) + f(t_i, w_i)]$ for $y' = -\lambda y$ with $\lambda > 0$.

30. (a). Discuss A-stability for the Backwards Euler method $w_{i+1} = w_i + hf(t_{i+1}, w_{i+1})$ for $y' = -\lambda y$ with $\lambda > 0$.

    (b). Discuss A-stability for the 2 Step BDF method $w_{i+1} = \frac{4}{3}w_i - \frac{1}{3}w_{i-1} + \frac{2}{3}hf(t_{i+1}, w_{i+1})$ for $y' = -\lambda y$ with $\lambda > 0$.

31. **TEST** Is the multi-step scheme $w_{i+1} = \frac{1}{3}(2w_i + w_{i-1}) + \frac{4h}{3}f(t_i, w_i)$ for $y' = f(t, y)$ stable?

32. Investigate the conditions for stability with respect to step size for the Midpoint method and for Heun's method. (Just set it up. Do not solve).

33. **TEST** Find the condition on step size h such that the second order Taylor series method will be stable for the ODE $y' = -2y$.

34. **TEST** Consider the multi-step method

$$w_{i+1} = \frac{3}{2}w_i + aw_{i-1} + bh[f(t_i, w_i) + 2f(t_{i-1}, w_{i-1})]$$

where a and b are a constants. Find a and b such that the scheme is consistent. Is the scheme stable?

35. Find the inequality that must be satisfied for the Taylor series method of order n to be stable.

36. **TEST** Consider the modified Euler method

$$w_0 = \alpha w_{i+1} = w_i + \frac{h}{2}[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))]$$

Find the inequality to use in determining the step size such that the method will be stable for the differential equation $y' = -4y$. You DO NOT have to solve the inequality.

37. **TEST** Write the third order differential equation $y''' + 2y'' - y' - 2y = e^t$ where $0 <= t <= 3$ and $y(0) = 1$, $y'(0) = 2$ and $y''(0) = 0$ as a system of first order differential equations. Find $y'''(0)$.

38. **TEST** Set up (but **DO NOT SOLVE**) a system of first-order linear equations for solving the ODE Boundary Value Problem $y'' + y' + 3xy = 9x^2$ on $0 \le x \le 1$ with $y(0) = 1$ and $y(1) = 2$. Use grid spacing $h = \frac{1}{3}$.

39. **TEST** Consider the boundary-value problem

$$y'' - xy' + 3y = 11x, \quad y(1) = 1.5, \quad y(2) = 15$$

   (a) Convert the above second-order differential equation into a system of two first-order equations.

   (b) Consider solving the given boundary-value problem using the equations in part (a) and the Linear Shooting Method. If for the 4th-order Runge-Kutta method with $h = 0.1$, choosing $y'(1) = 1$ gives $y(2) = 9.48535$ and choosing $y'(1) = 2$ gives $y(2) = 10.71083$, what is the appropriate value of $y'(1)$ to use to solve this problem?

40. **TEST** Consider the boundary-value problem

$$y'' + xy' - x^2y = 2x^2, \quad y(0) = 1, \quad y(1) = -1$$

   (a) Write down the central-difference approximations to $y'(x_i)$ and $y''(x_i)$ for any x_i and h.

   (b) Using $h = 0.25$ and the approximations in part (a), write down the equations needed to solve the given problem by the Finite-Difference Method.

41. Consider the boundary-value problem:

$$y'' = y' + 2y + \cos x, \quad 0 \le x \le \frac{\pi}{2}, \quad y(0) = -0.3, \quad y\left(\frac{\pi}{2}\right) = -0.1 \quad (1)$$

   (a) Obtain the exact solution to (1) by assuming the form $y(x) = A\sin x + B\cos x$ and applying the given boundary conditions to find the constants A and B.

   (b) Write a computer program that uses the Modified Euler Method and Linear Shooting to approximate the solution to (1) for step sizes $h = \pi/12$, $h = \pi/24$, and $h = \pi/48$. For each h compare the numerical approximations of $y(x)$ to the exact solution by plotting the functions and computing the mean-square error in y over the numerical solution points:

$$\text{mean square error} = \sqrt{\frac{1}{n}\sum_{i=1}^{n-1}(y_i - y(x_i))^2}$$

(c) Now use your program to approximate the solution $y(x) = e^{-10x}$ to the boundary-value problem:

$$y'' = 100y, \quad 0 \le x \le 1, \quad y(0) = 1, \quad y(1) = e^{-10} \qquad (2)$$

Use step sizes $h = 0.1$ and $h = 0.01$ and compare the numerical approximations to the exact solution.

42. **TEST** Is the multi-step scheme $y_{i+1} = \frac{1}{3}(2y_i + y_{i-1}) + \frac{4h}{3} f(t_i, y_i)$ for $y' = f(t, y)$ stable?

Miscellaneous

1. **TEST** Short answer questions (match the term on the left with the most appropriate term on the right): Each answer is used exactly once.

a. Partial Pivoting	A. Accelerates convergence of sequence of linear iterates
b. Cubic splines	B. Used for solution to a nonlinear equation
c. Predictor – Corrector method	C. Improves stability in solving systems of linear equations
d. Aitken's Method	D. A second order Runge – Kutta method
e. Heun's method	E. Order n^3 operations in general
f. Muller's method	F. Multi – step ODE solving methods
g. Gaussian Elimination	G. Interpolates piecewise with f, f' and f'' continuous

2. **TEST** Short answer questions (match the term on the left all appropriate terms on the right): Answers may be used more than once. Assume any functions implied are continuous.

a. Bisection	A. Always finds a root if bracket it initially
b. Secant	B. Convergence order $\approx$ 1.62
c. Regula Falsi	C. Iterates may oscillate or diverge
d. Newton	D. Can usually apply Aitken's method on it when it converges
e. Fixed Point Iteration	E. Quadratic (second order) convergence (usually)
	F. Linear convergence in most or all cases (when it converges)
	G. Need to compute derivative
	H. Approximates function as a line
	I. Need two initial iterates

Useful Formulas for exams

Euler's method: $w_{i+1} = w_i + h\, f(t_i, w_i)$

Taylor series method: $w_{i+1} = w_i + h\, f(t_i, w_i) + \frac{h^2}{2} \frac{d}{dt} f(t_i, w_i)|_{t_i, w_i}$

Runge-Kutta: $w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ where

$k_1 = hf(t_i, w_i)$;

$k_2 = hf(t_i + h/2, w_i + k_1/2)$;

$k_3 = hf(t_i + h/2, w_i + k_2/2)$;

$k_4 = hf(t_i + h, w_i + k_3);$

Steps	Adams Bashforth Implicit	Truncation error
2	$w_{n+1} = w_n + \frac{h}{2}[3f_n - f_{n-1}]$	$\frac{5}{12}y'''h^2$
3	$w_{n+1} = w_n + \frac{h}{12}[23f_n - 16f_{n-1} + 5f_{n-2}]$	$\frac{3}{8}y^{(4)}h^3$
4	$w_{n+1} = w_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}]$	$\frac{251}{720}y^{(5)}h^4$

Steps	Adams Moulton Implicit	Truncation error
2	$w_{n+1} = w_n + \frac{h}{12}[5f_{n+1} + 8f_n - f_{n-1}]$	$\frac{-1}{24}y^{(4)}h^3$
3	$w_{n+1} = w_n + \frac{h}{24}[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}]$	$\frac{-19}{720}y^{(5)}h^4$
4	$w_{n+1} = w_n + \frac{h}{720}[251f_{n+1} + 646f_n - 264f_{n-1} + 106f_{n-2} - 19f_{n-3}]$	$\frac{-3}{160}y^{(6)}h^5$