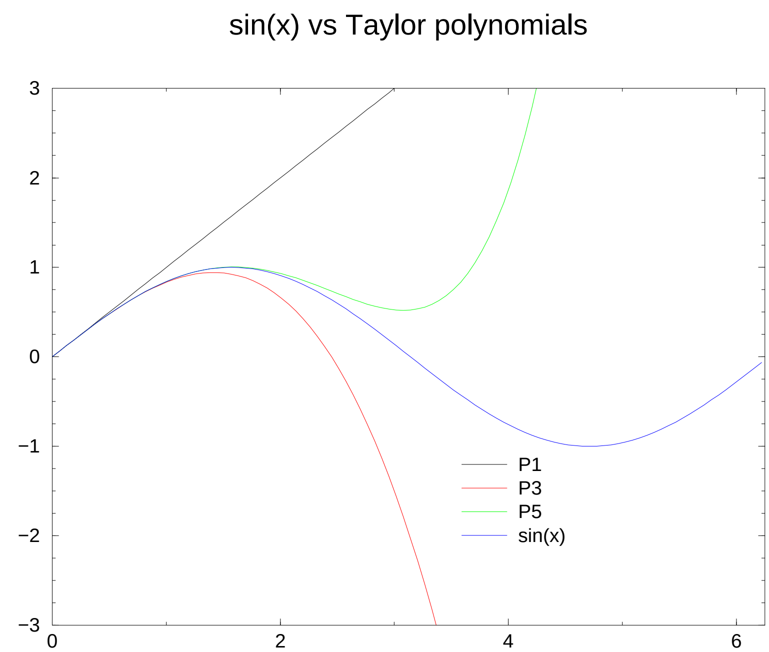

Figure 1: Problem 1

Math 340/611 - Homework Solutions 1
Basics, Error

1. (a). For $f(x) = \sin(x)$, we have

$$P_1(x) = x,$$

$$P_3(x) = x - \frac{x^3}{3!},$$

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

Table 1: Problem 1, parts c,d, and e

x	P_1			P_3			P_5		
	Val	Abs. Err.	Rel. Err.	Value	Abs. Err.	Rel. Err.	Value	Abs. Err.	Rel. Err.
0.1	0.1	.000167	.001669	.099833	.000000	.000001	.099833	.000000	.000000
0.5	0.5	.020574	.042915	.479167	.000259	.000540	.479427	.000002	.000003
1.0	1.0	.158529	.188395	.833333	.008138	.009671	.841667	.000196	.000233

(b). See Figure
(c). See Table
(d). See Table
(e). See Table

2. (a). For $f(x) = \ln(1+x)$, we have

$$\begin{aligned} P_1(x) &= x, \\ P_2(x) &= x - \frac{x^2}{2!}, \\ P_3(x) &= x - \frac{x^2}{2!} + 2\frac{x^3}{3!} \\ P_4(x) &= x - \frac{x^2}{2!} + 2\frac{x^3}{3!} - 6\frac{x^4}{4!} \end{aligned}$$

(b). See Figure
(c). See Table
(d). See Table
(e). See Table

3. $e^{-x} = 1 - x + \frac{x^2}{2} - \ldots$ with error term $\frac{1}{(n+1)!}x^{n+1}f^{(n+1)}(\xi)$. So for error to be less than 10^{-3} and knowing the nth derivative of $e^{-x} = (-1)^n e^{-x}$ gives the largest value of $f^{(n+1)}(\xi) = 1$ on [0,1]. We have error $\leq \frac{1}{(n+1)!} \leq 10^{-3}$. So $n = 6$ and the Taylor polynomial desired is: $1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720}$. Also, we have by the MVT $|f(x_1) - f(x_2| \leq 1|x_1 - x_2|$ since $f'(x) \leq 1$ on [0,1].

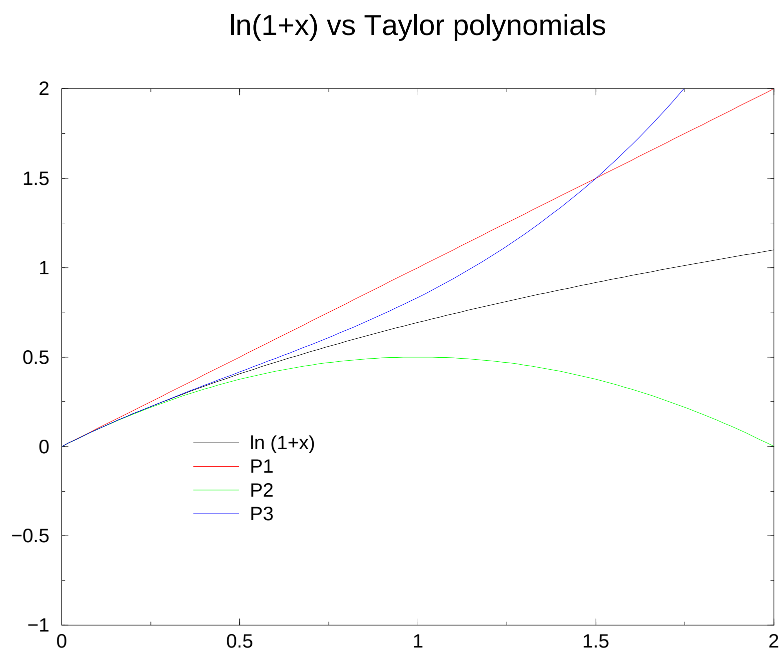

Figure 2: Problem 2

Table 2: Problem 2, parts c,d, and e

x	P_1			P_2			P_3		
	Val	Abs. Err.	Rel. Err.	Value	Abs. Err.	Rel. Err.	Value	Abs. Err.	Rel. Err.
0.1	0.1	.004690	.049208	.095000	.000310	.003253	.095333	.000023	.000241
0.5	0.5	.094535	.233152	.375000	.030465	.075136	.416667	.011202	.027628
1.0	1.0	.306853	.442695	.500000	.193147	.278652	.833333	.140186	.202246

4. $f(1) = 2$; $f'(x) = 3x^2 + 1$; $f'(1) = 4$; $f''(x) = 6x$; $f''(1) = 6$; $f'''(x) = 6$; $f'''(1) = 6$. So the order 2 polynomial is $p(x) \ = \ 2+4(x-1)+\frac{6(x-1)^2}{2}$ or $3x^2 - 2x + 1$ and the order 3 polynomial is $p(x) \ = \ 2 + 4(x-1) + \frac{6(x-1)^2}{2} + frac6(x-1)^3 6 \ = \ x^3 + x$

5. (a). If $f(x) \ = \ x - 3^{-x}$, then $f(0) \ = \ -3 < 0$ and $f(1) \ = \ \frac{2}{3} > 0 \ \Rightarrow \ \exists$ a root on [0,1].
(b). If $f(x) \ = \ 4x^2 \ - \ e^x$, then $f(0) \ = \ -1 < 0$ and $f(1) \ = \ 4 \ - \ e > 0 \ \Rightarrow \ \exists$ a root on [0,1].

6.

	a	b	c	d	e
Chopped	12.3	$-.0319$	12.2	-288	130
Rounded	12.3	$-.0320$	12.3	-289	130

7. (a). 2.75*1.07 = 2.94; 2.94*1.07=3.14; 3.14*1.07=3.35;
2.95*1.07=3.15; 3.15*1.07 = 3.37; 3.16*1.07=3.38;
So we get 3.35 - 3.37 + 3.38 - 4.67 = -1.31; (b). 1.07*1.07 = 1.14; 1.14*2.95 = 3.36;
1.07*1.07 = 1.14; 1.14*1.07 = 1.21; 1.21*2.75 = 3.32
So we get -4.67 + 3.38 - 3.36 + 3.32 = -1.33; (c). ((2.94-2.95)*1.07 + 3.16)*1.07 - 4.67
= (-0.0107 * 1.07 + 3.16)*1.07 - 4.67
= 3.14 * 1.07 - 4.67 = 3.36 - 4.67 = -1.31;
The true answer is -1.297...

8. $p(x) \ = \ 5x^6 + x^5 + 3x^4 + 3x^3 - x^2 + 1 \ = \ (((((5x + 1)x + 3)x + 3)x - 1)x + 0)x + 1$

9. (a). Error should less than 0.001*150 = 0.15 implies 149.85 $< p <$ 150.15;
(b). Error should less than 0.001*1500 = 1.5 implies 1498.5 $< p <$ 1501.5;

10. $N = 6$. The correct value chopped to 3 digits is 2.45 (which is exact), while the partial sums (chopped to 3 digits) are 1, 1.5, 1.83, 2.08, 2.28, 2.44
The absolute error is 0.01 and the relative error is 0.01/2.45 = 0.00408...

11. (a). $\displaystyle\sum_{n=0}^{5} \frac{1}{n!} \approx 2.70$ with absolute error 0.01828 and relative error 0.0067.
(b). $\displaystyle\sum_{n=0}^{5} \frac{1}{(5-n)!} \approx 2.71$ with absolute error 0.00828 and relative error 0.0030.

12. Not assigned.

13. a. $f(x) = \frac{\sqrt{x+9}-3}{x} \to \frac{0}{0}$ as $x \to 0$. So rationalize the numerator to get $\frac{\sqrt{x+9}-3}{x} \frac{\sqrt{x+9}+3}{\sqrt{x+9}+3} = \frac{1}{\sqrt{x+9}+3} \to \frac{1}{6}$.
b. $f(x) = \frac{1-\cos x}{x}$ goes to $\frac{0}{0}$ as $x \to 0$. Expand the numerator in Taylor series and factor out x to get $f(x) = \frac{1}{2}x + \ldots \to 0$ as $x \to 0$.

14. (a). $p_{n+1} = \frac{1}{4}p_n$ leads to $r - \frac{1}{4} = 0$ so $r = \frac{1}{4}$ and $p_n = C(\frac{1}{4})^n$. Plugging in the initial condition gives $C = 1$ so $p_n = (\frac{1}{4})^n$.
$p_{n+2} = \frac{21}{4}p_{n+1} - \frac{5}{4}p_n$ leads to $r^2 - \frac{21}{4}r + \frac{5}{4} = 0$. So, $r = \frac{1}{4}$ or $r = 5$ yielding $p_n = C_1(\frac{1}{4})^n + C_2 5^n$. Plugging in the initial conditions gives $p_n = 0.999998\ (\frac{1}{4})^n + 0.000002\ 5^n$.

(b).

Iterate	First Method	Second Method
0	1.000000	1.000000
1	0.250010	0.250010
2	0.0625025	0.0625526
3	0.0156256	0.0158885
4	0.00390641	0.00522382
5	0.000976602	0.00756444
6	0.00024415	0.0331835
7	$6.10376e-05$	0.164758
8	$1.52594e-05$	0.8235
9	$3.81485e-06$	4.11743
10	$9.53713e-07$	20.5871
11	$2.38428e-07$	102.936
12	$5.9607e-08$	514.678
13	$1.49018e-08$	2573.39
14	$3.72544e-09$	12867
15	$9.3136e-10$	64334.8

The answers are not the same since the 5^n term grows very rapidly and eventually outweighs the solution we were looking for. A very small change in the initial data (e.g., from roundoff) leads to very large changes at later steps.

15. Of course, the exact value of $f'(x)$ is e^x and so $f'(1) = 2.71828...$

h	Approximation	
1.0000	4.671	$\frac{7.389-2.718}{1}$
0.1000	2.860	$\frac{3.004-2.718}{0.1}$
0.0500	2.780	$\frac{2.857-2.718}{0.05}$
0.0200	2.750	$\frac{2.773-2.718}{0.02}$
0.0100	2.700	$\frac{2.745-2.718}{0.01}$
0.0050	2.600	$\frac{2.731-2.718}{0.005}$
0.0010	3.000	$\frac{2.721-2.718}{0.001}$
0.0001	0.00	$\frac{2.718-2.718}{0.0001}$

16. Not assigned.

17. (a). Goes to zero like $\frac{1}{n}$
(b). Goes to zero like $\frac{1}{n^2}$
(c). The expression is equivalent to $\ln(1+\frac{1}{n})$ and, using $\ln(1+x) \approx x - \frac{1}{2}x^2 + ...$ the expression goes to zero like $\frac{1}{n}$

18. (a). $\lim_{h\to 0} \frac{\sin h - h\cos h}{h} = \frac{h - h^3/6 + ... - h(1-h^2/2 + ...)}{h} = \frac{h^3}{3h} + ... \to 0$ like h^2.
(b). $\lim_{h\to 0} \frac{1 - e^h}{h} = \frac{1-(1+h+h^2/2+...)}{h} \to -1$ like h.

19. $\sum_{i=1}^{n}\sum_{j=1}^{i} a_i b_j = \sum_{i=1}^{n} a_i(\sum_{j=1}^{i} b_j)$. The first has $\frac{n(n+1)}{2}$ multiplications and $\frac{n(n+1)}{2} - 1$ additions, $(a_1b_1 + a_2b_1 + a_2b_2 + a_3b_1 + a_3b_2 + a_3b_3 + ...)$ while the second has only n multiplications and $2(n-1)$ additions – assuming the partial sum of the bs is saved. $(a_1b_1 + a_2(b_1+b_2) + a_3(b_1 + b_2 + b_3) + ...)$

20. The first method can lead to problems due to subtractive cancellation since the terms alternate in sign. The second method is better since as you take more terms the denominator grows and so the quotient shrinks toward the correct value.

21. The linear approximation of $f(x) = \frac{1}{x}$ on [1/2,1] is: $p(x) = \frac{x-1/2}{1-1/2}1 + \frac{x-1}{1/2-1}2 = -2x+3$. Notice that this line goes through the points.
Error $\leq \frac{1}{8}M(1-1/2)^2 = \frac{1}{8}(16)\frac{1}{2^2} = 1/2$. The maximum value of the second derivative of $\frac{1}{x}$ on [1/2,1] is 16.

22. $\int_0^1 x^3 dx = \frac{1}{4}$. The Trapezoidal rule gives: $\frac{h}{2}[f(0) + 2f(\frac{1}{4}) + 2f(\frac{1}{2}) + 2f(\frac{3}{4}) + f(1)] = \frac{17}{64}$ so the error is $\frac{1}{64}$. Since the error is proportional to h^2, we have Error $\simeq Ch^2 \simeq \frac{1}{64}$ when $h = \frac{1}{4}$. Thus, $C \simeq \frac{1}{4}$. Then find h such that $\frac{1}{4}h^2 \leq 10^{-3}$ which gives $h < 0.0632$. For error $< 10^{-6}$ we have $h = 0.002$.