Math 340/611 - Homework Solutions 2
Roots of Nonlinear Equations

1. Not assigned
2. Not assigned
3. Not assigned.
4. $f(x+h) \approx f(x) \; + \; hf'(x)$.
   So $f(29) \approx f(27) \; + \; (29-27)f'(27)$ where $f(x) = x^{1/3}$.
   Thus, $f(29) \approx 3 + 2 * \frac{1}{3}27^{-2/3} \; = \; 3\frac{2}{27}$.

   Using Newton's method on $f(x) \; = \; x^3 - 29$ with starting guess $x_0 = 3$ gives: $x_1 \; = \; x_0 - \frac{f(x_0)}{f'(x_0)} \; = \; 3 - \frac{-2}{27} \; = \; 3\frac{2}{27}$.

   We get the same answer for both cases since Newton's method approximates the solution with a tangent line at $x = 3$ and linearizing does the same in the first case, although the curve is turned sideways. But bascially, we've done the same thing in both cases.

5. (a) Bisection

```
i      a      b     mid     f_of_a f_of_b     f_mid  err_bound
 0 0.500 1.500 1.0000 -0.62758 2.17926  0.45969 0.50000000
 1 0.500 1.000 0.7500 -0.62758 0.45969 -0.16918 0.25000000
 2 0.750 1.000 0.8750 -0.16918 0.45969  0.12462 0.12500000
 3 0.750 0.875 0.8125 -0.16918 0.12462814 -0.02752931 0.06250000
Sol is x = 0.824132 f = 1.71045e-09 iters = 29 err_bnd = 9.31323e-10
```

   (b) Secant

```
Initial interval is 0.500000 1.500000
i      x0       x1      f(x0)      f(x1)
 0 0.50000000 1.50000000 -0.62758256 2.17926280
 1 1.50000000 0.72359000 2.17926280 -0.22585120
 2 0.72359000 0.79649845 -0.22585120 -0.06480450
 3 0.79649845 0.82583650 -0.06480450 0.00406365
Solution is x = 0.824132 function = 0.000000 iterations = 6
```

   (c) Newton

```
Initial iterate is 1.000000
i     x      f(x)
0 1.000000 0.459698
1 0.838218 0.0338217
2 0.824241 0.0002610
3 0.824132 0.0000000
4 0.824132 0.000000
Solution is x = 0.824132 function = 0.000000 iterations = 4
```

(d) On $[0, \pi]$ the initial error (from $\pi/2$) is less than $\pi/2$. At each step this halves such that $E_n \le \frac{\pi/2}{2^n}$ so $E_n < 10^{-5} \Rightarrow n > \frac{\ln(\pi 10^5/2)}{\ln 2} \simeq$ 17.3 so 18 steps are needed.

6. $f(x_n) = (1 + \frac{1}{n} - 1)^3 = (\frac{1}{n})^3$. For $n > 10$, we have $(\frac{1}{n})^3 < \frac{1}{1000} = \frac{1}{10^3}$. Since $x^* = 1$, in order to get within 10^{-3} of x^*, we need $x_n < 1.001 = 1 + \frac{1}{10^3}$ or $n > 1000$.

7. (a) $f(x) = x^2 - 2$. So, $f(1) = -1$ and $f(2) = 2$ and $f(1) * f(2) < 0$. Since all polynomials are continuous, by the Intermediate Value Theorem, the exists at least one root on the interval [1,2].

(b) Bisection method

i	a	b	mid	f_of_a	f_of_b	f_mid	err_bnd
0	1.0000	2.0000	1.5000	-1.0000	2.0000	0.250000	0.5000
1	1.0000	1.5000	1.2500	-1.0000	0.2500	-0.437500	0.2500
2	1.2500	1.5000	1.3750	-0.4375	0.2500	-0.109375	0.1250

So an upper bound for the error after 2 steps is 0.125.

(c) Secant method $x_2 = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)} = 2 - 2\frac{2-1}{2--1} = \frac{4}{3}$

i	x0	x1	f(x0)	f(x1)
0	1.00000	2.00000	-1.00000	2.00000
1	2.00000	1.33333	2.00000	-0.22222

(d) Newton's method $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-1}{2} = \frac{3}{2}$

i	x	f(x)
0	1.000000	-1.000000
1	1.5000000	0.2500000

8. Not assigned.

9. Not assigned.

10. (a). If $x_* = 2x_* - Ax_*^2$ then $x_* = 0$ or $x_* = 1/A$.
    (b). We want to find an interval around $1/A$ in which $|g'(x)| < 1$. Thus, $|2 - 2Ax| < 1$, which leads to $\frac{1}{2A} < x < \frac{3}{2A}$. The scheme may converge on a larger interval, but we can guarantee at least this much.

11. (a) Let $f(x) = x^3 - 36$. Then $f(3) = -9$ and $f(4) = 28$. Since $f(3) * f(4) < 0$ and $f(x)$ is a polynomial and is thus continuous, $f(x)$ has at least on root on [3,4] by the IVT.

    $f'(x) = 3x^2$, which is greater than 0 on the interval. So, $f(x)$ is monotonically increasing on [3,4], so there is at most one root on [3,4].

    Thus, there is exactly one root of $f(x)$ on [3,4].

    (b) The error after n steps is less than $\frac{1}{2}\frac{1}{2^n}L_0$. Solving $\frac{1}{2}\frac{1}{2^n}L_0 < 0.05$, where $L_0 = 1$, gives $n \geq -1 + \log_2 \frac{L_0}{\epsilon} = -1 + \frac{\ln \frac{L_0}{\epsilon}}{\ln 2} = -1 + \frac{\ln 20}{\ln 2} = 3.32$, since $\epsilon = 0.05$. Thus, 4 iterations are needed.

    (c) Fixed points, x of $x_{n+1} = 2\frac{x_n^3 + 18}{3x_n^2}$ satisfy $x = 2\frac{x^3 + 18}{3x^2}$. So, $3x^3 = 2x^3 + 36$ or $x^3 - 36 = 0$. So p is a fixed point of this iteration scheme.

    (d) If $x_0 = 3$, then $x_1 = 4$ and $x_2 = 2.25$. Note that if x is a fixed point of the iteration scheme $x_{n+1} = \frac{36}{x_n^2}$, then $x = \frac{36}{x^2}$. Then $x^3 - 36 = 0$, so p is a fixed point of this scheme. If we let $g(x) = \frac{36}{x^2}$, then $g'(x) = \frac{-72}{x^3}$. On the interval [3,4], $|g'(x)| \geq \frac{9}{8} > 1$ so the method does not converge to p.

    (e) If $x_0 = 3$, then $x_1 = 7/2$ and $x_2 = 631/196 \approx 3.219\ldots$. Note that if x is a fixed point of the iteration scheme $x_{n+1} = \frac{x_n^3 + 36}{2x_n^2}$, then $x = \frac{x^3 + 36}{2x^2}$. Then $2x^3 = x^3 + 36$, and $x^3 - 36 = 0$. So p is a fixed point of this scheme. If we let $g(x) = \frac{x^3 + 36}{2x^2}$, then $g'(x) = \frac{1}{2} - \frac{36}{x^3}$. On the interval [3,4], $-\frac{2}{3} \leq g'(x) \leq -\frac{1}{16}$. So, $|g'(x)| < 1$ and the method converges to p.

    (f) Newton's method has second order convergence (since p is not a multiple root). Thus, the scheme $x_{n+1} = x_n - \frac{x_n^3 - 36}{3x_n^2}$ is quadratically convergent to p. If $x_0 = 3$, then $x_1 = \frac{10}{3}$ and $x_2 = \frac{743}{225} \approx 3.302\overline{2}$.

(g) Aitken's method is useful for linearly convergent methods. Only the method of part (e) is both convergent and convergent linearly. So this is where Aitken's method would be useful.

We have the values $x_0 = 3,\ x_1 = 7/2$ and $x_2 = 631/196 \approx 3.129$ so $\tilde{x}_0 \ = \ x_0 - \frac{(x_1-x_0)^2}{x_2-2x_1+x_0} = 3.32026...$

The exact solution is $^3\sqrt{36} \ \approx 3.3019$.

12. Not assigned.

13. Take a look at $g(x)$ at $x = 4$ in each case. (The iterations you play with should confirm the results).

(a) $g'(x) \ = \ 2 - \frac{x}{2}$ and $g'(4) \ = \ 0$ so convergence is better than linear.

(b) $g'(x) \ = \ \frac{2x}{3}$ and $g'(4) \ = \ \frac{8}{3} > 1$ so the scheme diverges.

(c) $g'(x) \ = \ \frac{3}{2\sqrt{3x+4}}$ and $g'(4) \ = \ \frac{3}{8} < 1$ but no 0 so the scheme converges linearly.

(a) converges fastest, then (c) and (b) does not converge.

14. Not assigned.

15.
```
0.01 0.10
1
i      a        b       mid     f_of_a    f_of_b     f_mid       err_bound
 0 0.01000 0.10000 0.05500 36544.24 -81629.5 -44326.942 0.045000
 1 0.01000 0.05500 0.03250 36544.24 -44326.9 -12134.354 0.022500
 2 0.01000 0.03250 0.02125 36544.24 -12134.3   9616.236 0.011250
 3 0.02125 0.03250 0.02687  9616.23 -12134.3  -1829.261 0.005625
 4 0.02125 0.02687 0.02406  9616.23  -1829.2   3742.108 0.002812
 5 0.02406 0.02687 0.02546  3742.10  -1829.2    919.728 0.001406
 6 0.02546 0.02687 0.02617   919.72  -1829.2   -463.800 0.000703
 7 0.02546 0.02617 0.02582   919.72   -463.8    225.688 0.000351
 8 0.02582 0.02617 0.02599   225.68   -463.8   -119.623 0.000175
 9 0.02582 0.02599 0.02590   225.68   -119.6     52.890 0.000087
10 0.02590 0.02599 0.02595    52.89   -119.6    -33.401 0.000043
11 0.02590 0.02595 0.02593    52.89    -33.4      9.735 0.000021
12 0.02593 0.02595 0.02594     9.73    -33.4    -11.835 0.000010
13 0.02593 0.02594 0.02593     9.73    -11.8     -1.050 0.000005
```

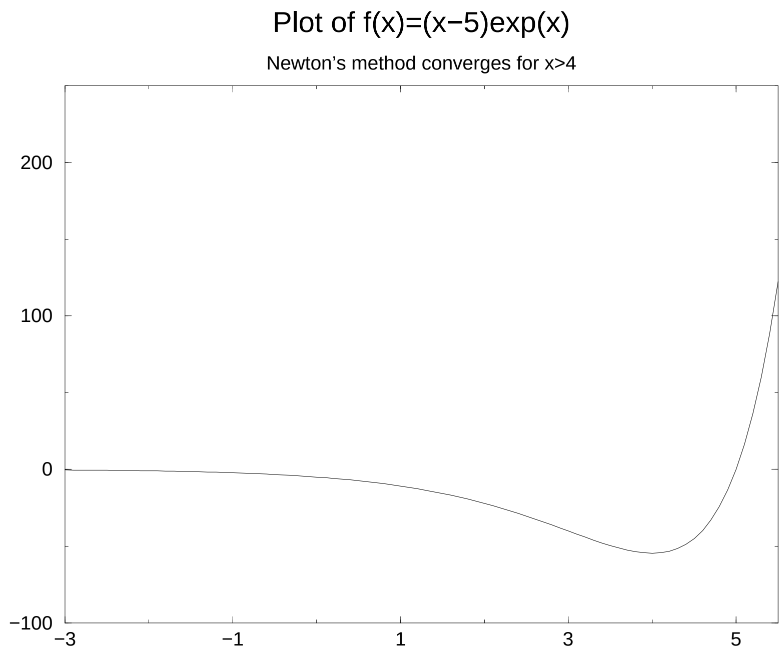

Figure 1: Problem 16 - $f(x) = (x-5)e^x$

```
14 0.02593 0.02593 0.02593      9.73      -1.0       4.342 0.000002
Sol is x = 0.025935 f = -0.001095 iters = 26 error_bound = 6.70552e-10
```

The maximum affordable interest rate is nearly 2.6%.

16. One such function is $f(x) = (x-5)e^x$. This has a unique root at $x = 5$, and is differentiable. Newton's method will converge for any $x > 4$ where the curve hits a minimum but will diverge whenever $x < 4$. See Figure 1.

17. Any fixed point of **A.** must satisfy $x = g(x) = \frac{2x^3 + 2x^2 + 2}{3x^2 + 4x - 1}$. So, $3x^3 + 4x^2 - x = 2x^3 + 2x^2 + 2$ and $x^3 + 2x^2 - x - 2 = 0$ and the polynomial is $x^3 + 2x^2 - x - 2$.

    Any fixed point of **B.** must satisfy $x = g(x) = \frac{3x^3 + 5x^2 + 2x + 2}{4x^2 + 7x + 1}$. So,

$4x^3 + 7x^2 + x = 3x^3 + 5x^2 + 2x + 2$ and $x^3 + 2x^2 - x - 2 = 0$ and the polynomial is $x^3 + 2x^2 - x - 2$.

To find which method is better for finding the root at $x = 1$, take the derivative, $g'(x)$ and plug in $x = 1$. The one with the smaller magnitude should converge more quickly for initial guesses "close enough" to the root. For part **A**, $g'(1) = 0$ so convergence is quadratic, while for part **B**, $g'(1) = \frac{1}{2}$ so convergence is linear with coefficient $\frac{1}{2}$. Thus, the method of part **A** is better.

18. Not assigned.

19. (a) Let $f(x) = (x - 2)^2 - \ln x$. Then $f(1) = 1$ and $f(2) = -\ln 2$. Since $f(1) * f(2) < 0$ and $f(x)$ is continuous on [1,2], $f(x)$ has at least on root on [1,2] by the IVT.

    $f'(x) = 2(x - 2) - \frac{1}{x}$, which is less than 0 on the interval. So, $f(x)$ is monotonically decreasing on [1,2], so there is at most one root on [1,2].

    Thus, there is exactly one root of $f(x)$ on [1,2].

    (b) Secant method

```
Initial interval is 1.000000 2.000000
i      x0           x1         f(x0)      f(x1)
 0 1.00000000 2.00000000  1.00000000 -0.69314718
 1 2.00000000 1.59061611 -0.69314718 -0.29652626
 2 1.59061611 1.28454785 -0.29652626  0.26146499
 3 1.28454785 1.42796611  0.26146499 -0.02902836
 4 1.42796611 1.41363464 -0.02902836 -0.00233981
 5 1.41363464 1.41237819 -0.00233981  0.00002446
 6 1.41237819 1.41239118  0.00002446 -0.00000002
 7 1.41239118 1.41239117 -0.00000002  0.00000000
Solution is x = 1.412391 function = 0.000000 iterations = 7
```

    (c) Newton's method

```
Initial iterate is 1.500000
i     x      f(x)
0  1.500000  -0.155465
1  1.4067209  0.0107186
2  1.4123700  0.0000400
```

```
3  1.4123912  0.0000000
4  1.412391   0.000000
Solution is x = 1.412391 function = 0.000000 iterations = 4
```

(d) The error after n steps is less than $\frac{1}{2}\frac{1}{2^n}L_0$. Solving $\frac{1}{2}\frac{1}{2^n}L_0 < 0.0001$, where $L_0 = 1$, gives $n \geq -1+\log_2 \frac{L_0}{\epsilon} = -1+\frac{\ln \frac{L_0}{\epsilon}}{\ln 2} = -1+\frac{\ln 10000}{\ln 2} \approx 12.29$, since $\epsilon = 0.0001$. Thus, 13 iterations are needed.

20. Not assigned.

21. Not assigned.

22. Newton's method error $E_{n+1} \leq \frac{1}{2}E_n^2\frac{f''(\xi_1)}{f'(\xi_2)}$. So the for information given, $E_0 < \frac{1}{3}$; $E_1 < \frac{1}{2}(\frac{1}{3})^2\frac{4}{2} = \frac{1}{9}$; $E_2 < \frac{1}{2}(\frac{1}{9})^2\frac{4}{2} = \frac{1}{81}$; $E_3 < \frac{1}{2}(\frac{1}{81})^2\frac{4}{2} = \frac{1}{6561}$;

23. Not assigned.

24. Not assigned.

25. $g(x) = 1 + e^{-x} \Rightarrow g(1) \simeq 0.632$ and $g(2) \simeq 0.865$. So on [1,2], $g \in [1,2]$ since $g'(x) = -e^{-x} < 0$ everywhere, i.e. $g(x)$ is decreasing. Also, $|g'(x)| < 1$ on [1,2] – it lies between 0.135 and 0.368 on [1,2]. So there is a unique fixed point on the interval and the iteration scheme will converge to the fixed point. Since $E_{n+1} \leq 0.368E_n$ and the initial error is less than 1, we need $0.368^n < 10^{-5}$ so $n > \frac{\ln 10^{-5}}{\ln 0.368} \simeq 11.6$ so 12 iterations are needed to be sure to be within 10^{-5} of the fixed point.

26. Not assigned.

27. Not assigned.

28. Not assigned.

29. $E_0 = 1$, $E_1 = 0.5$ and $E_2 = 0.0625$. Let $E_1 \approx CE_0^p$ and $E_2 \approx CE_1^p$ and divide to obtain $\dfrac{E_2}{E_1} \approx \left(\dfrac{E_1}{E_0}\right)^p \Rightarrow \dfrac{1}{8} = (\dfrac{1}{2})^p \Rightarrow p = 3$. So the method has order of convergence 3.

30. (a) $x^2 - x = 0 \Rightarrow x = 0$ or $x = 1$ so the positive root is $x = 1$.

(b) Find the value of $g'(1)$ for each case

i. $g'(x) = \frac{1}{2}x^{-1/2}$ and $g'(1) = \frac{1}{2}$. The method is linearly convergence with factor $\frac{1}{2}$.

ii. $g'(x) = \frac{(2x-1)(2x)-2x^2}{(2x-1)^2}$ and $g'(1) = 0$ so the method is quadratically convergent.

iii. $g'(x) = 2x$ and $g'(1) = 2 > 1$ so this method diverges and won't find the root at $x = 1$ for a nearby starting guess.

(c) Aitken's method (f0 is Aitken value)

```
Initial iterate is 2.000000
x0 = 2.000000 x1 = 1.414214 x2 = 1.189207 f0 = 1.048878
x1 = 1.414214 x2 = 1.189207 x3 = 1.090508 f0 = 1.013382
x2 = 1.189207 x3 = 1.090508 x4 = 1.044274 f0 = 1.003531
x3 = 1.090508 x4 = 1.044274 x5 = 1.021897 f0 = 1.000909
x4 = 1.044274 x5 = 1.021897 x6 = 1.010889 f0 = 1.000231
```

Notice how convergence is improved.

31. Not assigned.

32. Bisection

i	a	b	mid	f_of_a	f_of_b	f_mid	err_bnd
0	1.000000	2.000000	1.500000	-0.632120	27.135335	8.723130	0.5000000
1	1.000000	1.500000	1.250000	-0.632120	8.723130	3.099004	0.2500000
2	1.000000	1.250000	1.125000	-0.632120	3.099004	1.019964	0.1250000
3	1.000000	1.125000	1.062500	-0.632120	1.019964	0.143442	0.0625000
4	1.000000	1.062500	1.031250	-0.632120	0.143442	-0.256598	0.0312500
5	1.031250	1.062500	1.046875	-0.256598	0.143442	-0.059687	0.0156250
6	1.046875	1.062500	1.054687	-0.059687	0.143442	0.041094	0.0078125
7	1.046875	1.054687	1.050781	-0.059687	0.041094	-0.009491	0.0039063
8	1.050781	1.054687	1.052734	-0.009491	0.041094	0.015752	0.0019531
9	1.050781	1.052734	1.051757	-0.009491	0.015752	0.003117	0.0009766
10	1.050781	1.051757	1.051269	-0.009491	0.003117	-0.003190	0.0004883
11	1.051269	1.051757	1.051513	-0.003190	0.003117	-0.000036	0.0002441
12	1.051513	1.051757	1.051635	-0.000036	0.003117	0.001540	0.0001221
13	1.051513	1.051635	1.051574	-0.000036	0.001540	0.000751	0.0000610
14	1.051513	1.051574	1.051544	-0.000036	0.000751	0.000357	0.0000305
15	1.051513	1.051544	1.051528	-0.000036	0.000357	0.000160	0.0000153
16	1.051513	1.051528	1.051521	-0.000036	0.000160	0.000061	0.0000076

```
17 1.051513 1.051521 1.051517 -0.000036  0.000061  0.000012 0.0000038
18 1.051513 1.051517 1.051515 -0.000036  0.000012 -0.000012 0.0000019
19 1.051515 1.051517 1.051516 -0.000012  0.000012  0.000000 0.0000009
Solution is x = 1.051516 function = 0.000000 iterations = 19
```

Secant

```
Initial interval is 1.000000 2.000000
i      x0         x1         f(x0)       f(x1)
 0 1.00000000 2.00000000 -0.63212056 27.13533528
 1 2.00000000 1.02276480 27.13533528 -0.36095704
 2 1.02276480 1.03559344 -0.36095704 -0.20248005
 3 1.03559344 1.05198410 -0.20248005  0.00604338
 4 1.05198410 1.05150907  0.00604338 -0.00009626
 5 1.05150907 1.05151652 -0.00009626 -0.00000004
 6 1.05151652 1.05151652 -0.00000004  0.00000000
Solution is x = 1.051517 function = 0.000000 iterations = 6
```

Regula Falsi – Notice that convergence is from one side

```
Initial interval is 1 2
i     x0         x1         f(x0)        f(x1)
 0 1.00000000 2.00000000 -0.63212056 27.13533528
 1 1.02276480 2.00000000 -0.36095704 27.13533528
 2 1.03559344 2.00000000 -0.20248005 27.13533528
 3 1.04273640 2.00000000 -0.11244536 27.13533528
 4 1.04668681 2.00000000 -0.06209626 27.13533528
 5 1.04886338 2.00000000 -0.03418546 27.13533528
 6 1.05006013 2.00000000 -0.01878774 27.13533528
 7 1.05071739 2.00000000 -0.01031570 27.13533528
 8 1.05107813 2.00000000 -0.00566107 27.13533528
 9 1.05127605 2.00000000 -0.00310582 27.13533528
10 1.05138463 2.00000000 -0.00170367 27.13533528
11 1.05144418 2.00000000 -0.00093445 27.13533528
12 1.05147685 2.00000000 -0.00051252 27.13533528
13 1.05149476 2.00000000 -0.00028109 27.13533528
14 1.05150459 2.00000000 -0.00015416 27.13533528
15 1.05150997 2.00000000 -0.00008455 27.13533528
```

```
16 1.05151293 2.00000000 -0.00004637 27.13533528
17 1.05151455 2.00000000 -0.00002543 27.13533528
18 1.05151544 2.00000000 -0.00001395 27.13533528
19 1.05151593 2.00000000 -0.00000765 27.13533528
20 1.05151619 2.00000000 -0.00000420 27.13533528
21 1.05151634 2.00000000 -0.00000230 27.13533528
22 1.05151642 2.00000000 -0.00000126 27.13533528
Solution is x = 1.0515164 function = -0.000001 iterations = 22
```

Newton

```
Initial iterate is 1.500000
i     x      f(x)
0  1.500000   8.723130
1  1.1742289  1.7852280
2  1.0642788  0.1669652
3  1.0516750  0.0020483
4  1.0515165  0.0000003
5  1.0515165  0.0000000
Solution is x = 1.051517 function = 0.000000 iterations = 6
```

33. Not assigned.

34. Using Newton's method with $x_0 = 2$ gives convergence in about in 6 steps

```
Initial iterate is 2.000000
i     x      f(x)
0  2.000000 -32.281718
1  1.4364394 -8.2722569
2  1.1551006 -1.5382325
3  1.0734798 -0.1089235
4  1.0667599 -0.0007014
5  1.0667161  0.0000000
6  1.066716   0.000000
Solution is x = 1.066716 function = 0.000000 iterations = 6
```

The number of correct digits is 0,1,1,2,5,$\geq$ 7 so yes, the number of correct digits approximately doubes with each iteration.

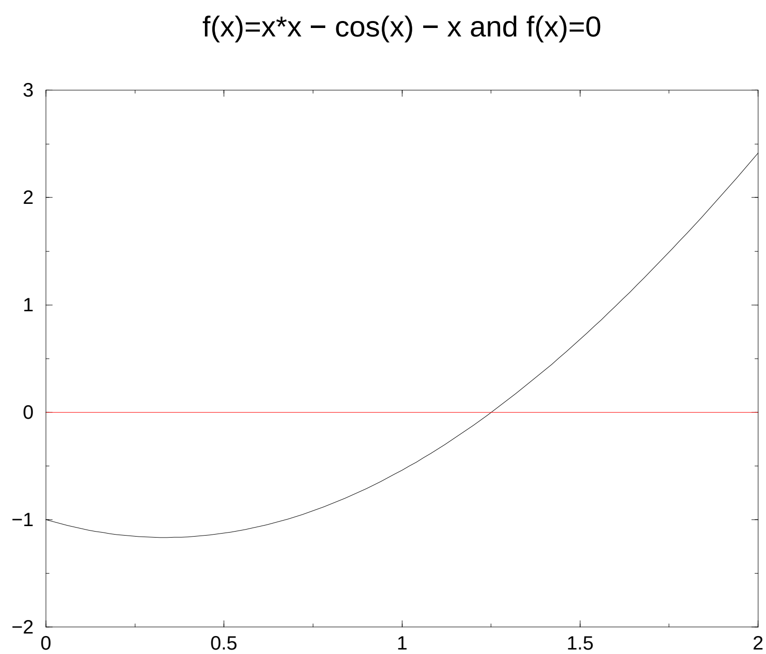

Figure 2: Problem 38 - $f(x) = x^2 - cos(x) - x$

35. $f(0) = f(1) = 0.24 > 0$ so we do not have $f(a) * f(b) < 0$ and are not guaranteed to bracket the root. Thus, this is not a good interval on which to start bisection.

36. Not assigned.

37. Must do.

38. See Figure 2 for plot.

```
Bisection

i      a        b       mid     f_of_a  f_of_b    f_mid  err_bound
 0 0.00000 2.00000 1.00000 -1.00000 2.41614 -0.54030 1.00000000
```

```
 1 1.00000 2.00000 1.50000 -0.54030 2.41614  0.67926 0.50000000
 2 1.00000 1.50000 1.25000 -0.54030 0.67926 -0.00282 0.25000000
 3 1.25000 1.50000 1.37500 -0.00282 0.67926  0.32107 0.12500000
 4 1.25000 1.37500 1.31250 -0.00282 0.32107  0.15472 0.06250000
 5 1.25000 1.31250 1.28125 -0.00282 0.15472  0.07483 0.03125000
 6 1.25000 1.28125 1.26562 -0.00282 0.07483  0.03572 0.01562500
 7 1.25000 1.26562 1.25781 -0.00282 0.03572  0.01638 0.00781250
 8 1.25000 1.25781 1.25390 -0.00282 0.01638  0.00676 0.00390625
 9 1.25000 1.25390 1.25195 -0.00282 0.00676  0.00196 0.00195312
10 1.25000 1.25195 1.25097 -0.00282 0.00196 -0.00042 0.00097656
11 1.25097 1.25195 1.25146 -0.00042 0.00196  0.00076 0.00048828
12 1.25097 1.25146 1.25122 -0.00042 0.00076  0.00016 0.00024414
13 1.25097 1.25122 1.25109 -0.00042 0.00016 -0.00013 0.00012207
14 1.25109 1.25122 1.25115 -0.00013 0.00016  0.00001 0.00006104
15 1.25109 1.25115 1.25112 -0.00013 0.00001 -0.00005 0.00003052
16 1.25112 1.25115 1.25114 -0.00005 0.00001 -0.00001 0.00001526
17 1.25114 1.25115 1.25115 -0.00001 0.00001  0.00000 0.00000763
18 1.25114 1.25115 1.25114 -0.00001 0.00000 -0.00000 0.00000381
19 1.25114 1.25115 1.25115 -0.00000 0.00000 -0.00000 0.00000191
20 1.25115 1.25115 1.25115 -0.00000 0.00000 -0.00000 0.00000095
Solution is x = 1.25115 iterations = 20 error_bound = 9.5e-07


Regula Falsi (Notice x_1 never changes)
Initial interval is 0 2
i     x0        x1       f(x0)     f(x1)
 0 0.000000 2.000000 -1.00000000 2.41614684
 1 0.585454 2.000000 -1.07615824 2.41614684
 2 1.021348 2.000000 -0.50041160 2.41614684
 3 1.189261 2.000000 -0.14726302 2.41614684
 4 1.235837 2.000000 -0.03727402 2.41614684
 5 1.247446 2.000000 -0.00906721 2.41614684
 6 1.250260 2.000000 -0.00218424 2.41614684
 7 1.250937 2.000000 -0.00052493 2.41614684
 8 1.251100 2.000000 -0.00012608 2.41614684
 9 1.251139 2.000000 -0.00003028 2.41614684
10 1.251148 2.000000 -0.00000727 2.41614684
11 1.251151 2.000000 -0.00000175 2.41614684
```

```
12 1.251152 2.000000 -0.00000042 2.41614684
Solution is x = 1.25115 function = 0.000000 iterations = 12
```

```
Secant Method
Initial interval is 0.000000 2.000000
i     x0      x1     f(x0)     f(x1)
 0 0.000000 2.000000 -1.00000000  2.41614684
 1 2.000000 0.585454  2.41614684 -1.07615824
 2 0.585454 1.021348 -1.07615824 -0.50041160
 3 1.021348 1.400207 -0.50041160  0.39060951
 4 1.400207 1.234121  0.39060951 -0.04141587
 5 1.234121 1.250043 -0.04141587 -0.00271641
 6 1.250043 1.251160 -0.00271641  0.00002209
 7 1.251160 1.251151  0.00002209 -0.00000001
 8 1.251151 1.251151 -0.00000001  0.00000000
Solution is x = 1.25115 function = 0.000000 iterations = 8
```

```
Newton's method
Initial iterate is 1.000000
i     x      f(x)
0  1.000000  -0.540302
1  1.2934080  0.1056515
2  1.2519531  0.0019652
3  1.2511521  0.0000007
4  1.2511518  0.0000000
Solution is x = 1.251152 function = 0.000000 iterations = 4
```

39. For $c = 5$:

```
Initial iterate is 1.300000
i     x      f(x)
0 1.300000  -1.397898
1 1.4000274  0.7988334
2 1.3769574  0.0941473
3 1.3734640  0.0016458
4 1.3734008  0.0000005
```

```
5 1.3734008  0.0000000
6 1.373401   0.000000
Solution is x = 1.373401 function = 0.000000 iterations = 6

Initial iterate is 1.400000
i     x      f(x)
0  1.400000   0.797884
1  1.3769501  0.0939496
2  1.3734638  0.0016390
3  1.3734008  0.0000005
4  1.3734008  0.0000000
5  1.373401   0.000000
Solution is x = 1.373401 function = 0.000000 iterations = 5
```

For $c = 10$

```
Initial iterate is 1.300000
i     x      f(x)
0    1.300000   -6.397898
1    1.7578055 -15.2848477
2    2.2861516 -11.1509023
3    7.0831132  -8.9705099
4   11.4380471 -12.1105730
5   13.6583299  -8.0737134
6   15.3722831 -10.3488839
7   24.5981898 -10.5920473
8   32.4411299  -8.3527107
9   34.6903748  -9.8663571
10  44.3836068  -9.5756621
11  52.4981440 -11.2835356
12  56.7601612  -9.7852957
13  66.1142528  -9.8582549
14  75.7783399  -9.6004525
15  84.0571740 -10.9616056
16  89.7524458 -14.5345446
17  90.4265243 -10.8081037
18  96.9648773 -10.4519749
19 105.6438919 -12.3616789
```

```
20 107.5232728  -9.1419958
21 112.7888800 -10.3186254
22 122.1564859 -10.3828415
23 131.2120719 -10.9036346
24 137.2144400 -11.6123133
25 140.4404821 -11.3442012
26 144.4820562 -10.0312160
27 154.5035069  -9.3654085
28 161.1801773  -8.5748239
29 164.0091000  -9.2456457
30 169.9016102  -9.7386769
31 179.0177485 -10.0530826
32 189.0425835  -9.3909817
33 195.8927979  -7.9649930
34 197.4420295 -10.5184617
35 205.7320994  13.6717003
36 205.7077443   4.9985688
37 205.6856225   1.2448515
38 205.6758549   0.1231653
39 205.6746647   0.0014663
40 205.6746502   0.0000002
41 205.674650    0.000000
Solution is x = 205.674650 function = 0.000000 iterations = 41

Initial iterate is 1.400000
i     x      f(x)
0  1.400000  -4.202116
1  1.5213942 10.2255823
2  1.4964583  3.4272749
3  1.4775535  0.6935908
4  1.4715408  0.0418950
5  1.4711294  0.0001724
6  1.4711277  0.0000000
7  1.471128   0.000000
Solution is x = 1.471128 function = 0.000000 iterations = 7
```

The starting guess of $x_0 = 1.3$ was not good for $\tan(x) - 10 = 0$ and convergence was to a far away root. The starting guess of 1.4 was much better. Note that there is a root every π radians for this function.

40. Not assigned.

41. Not assigned.

42. Not assigned.

43. Not assigned.

44. By noting that $f(1) = f'(1) = 0$ and $f''(1) = 1$ we see there is a double root. Thus, we use $k = 2$ in the modified Newton method to get (with initial iterate $x_0 = 2$).

```
Initial iterate is 2.000000
i     x      f(x)
0  2.000000 0.264241
1  0.5634363  0.1281495
2  0.9428404  0.0016972
3  0.9989263  0.0000006
4  0.9999996  0.0000000
5  1.0000000  0.0000000
Solution is x = 1.000000 function = 0.000000 iterations = 5
```

With the usual Newton's method, we have linear convergence with factor 1/2, as in the theory for a double root. See the first several steps below.

```
Initial iterate is 2.000000
i     x      f(x)
0   2.000000   0.264241
1   1.2817182  0.0329612
2   1.1266449  0.0073735
3   1.0605625  0.0017615
4   1.0296606  0.0004313
5   1.0146826  0.0001067
6   1.0073052  0.0000266
7   1.0036437  0.0000066
8   1.0018196  0.0000017
```

9	1.0009093	0.0000004
10	1.0004545	0.0000001
11	1.0002272	0.0000000
12	1.0001136	0.0000000
13	1.0000568	0.0000000
14	1.0000284	0.0000000

45. Starting at $x = 1$, convergence is to the root at $x = 0$ but convergence is slow since there is a triple root there. Note that the convergence is linear, with coefficient 2/3. For example, $\frac{x_{11}}{x_{10}} = 0.66695$. Convergence rate used exact root of 0.

Initial iterate is 1.000000

i	x	f(x)	conv_rate
0	1.000000	0.924743	
1	0.6141228	0.2365017	
2	0.4040985	0.0682853	0.858
3	0.2692718	0.0201336	0.970
4	0.1799758	0.0059703	0.993
5	0.1203292	0.0017728	0.999
6	0.0804131	0.0005264	1.001
7	0.0537065	0.0001562	1.001
8	0.0358513	0.0000463	1.001
9	0.0239227	0.0000137	
10	0.0159585	0.0000041	
11	0.0106436	0.0000012	
12	0.0070978	0.0000004	
13	0.0047328	0.0000001	
14	0.0031556	0.0000000	
15	0.0021039	0.0000000	
16	0.0014027	0.0000000	
17	0.0009352	0.0000000	
18	0.0006235	0.0000000	
19	0.0004156	0.0000000	
20	0.0002771	0.0000000	
21	0.0001847	0.0000000	
22	0.0001232	0.0000000	
23	0.0000821	0.0000000	

```
24  0.0000547  0.0000000
25  0.0000365  0.0000000
26  0.0000243  0.0000000
27  0.0000162  0.0000000
28  0.0000108  0.0000000
```

If instead, we used Newton's method knowing the root was a triple root and used $x_{n+1} = x_n - 3\frac{f(x_n)}{f'(x_n)}$ convergence seems quadratic but we really don't have enough iterates.

```
Initial iterate is 1.000000
i     x          f(x)    conv_rate
0  1.000000   0.924743
1  0.6141228  0.2365017
2 -0.0159499 -0.0000040  7.5
3  0.0000146  0.0000000  2.0
4  0.0000000  0.0000000
Solution is x = 0.000000 function = 0.000000 iterations = 4
```

For the root near $x = 2$, convergence is quadratic. (Convergence rate took iterate 5 as exact).

```
Initial iterate is 2.000000
i     x          f(x)     conv_rate
0  2.000000   2.109433
1  2.6057022 -3.2593537
2  2.3655241 -0.3702460   -2.30
3  2.3290587 -0.0118781    1.71
4  2.3278082 -0.0000142    1.98
5  2.3278067  0.0000000
Solution is x = 2.327807 function = 0.000000 iterations = 5
```

46. Not assigned.

47. Must do.