Math 611 - Homework Solutions 3
Systems of Linear Equations

1. Matrix:

```
 1 1 1
 2 4 4
-2 2 4
Right Hand Side:
-1 -6 -10
U matrix
1 1 1
0 2 2
0 0 2
Answer:
x0 = 1
x1 = 0
x2 = -2
U is given above
L matrix
 1 0 0
 2 1 0
-2 2 1
```

2. Not assigned

3. Multiplying and nxn matrix by an n-vector entails n multiplications (and $n-1$ additions) to compute each entry in the n-vector answer. This gives n^2 multiplications and $n(n-1)$ additions.

   If the matrix is tridiagonal, there are 3 multiplications and 2 additions for each row except the first and last, which each have 2 multiplications and 1 addition. So for an nxn tridiagonal matrix we get $3n-2$ multiplications and $2n-2$ additions.

4. Not assigned.

5. Not assigned.

6. (a)
```
Matrix:
4 0 1
1 4 1
1 0 4
Right hand side:
5 3 -10
U matrix:
4.000000 0.000000 1.000000
0.000000 4.000000 0.750000
0.000000 0.000000 3.750000
Answer:
x0 = 2.000000
x1 = 1.000000
x2 = -3.000000
```

(b) Jacobi's method gives the method $\begin{pmatrix} x \\ y \\ z \end{pmatrix}^{(n+1)} = \frac{1}{4}\begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}^{(n)} + \frac{1}{4}\begin{pmatrix} 5 \\ 3 \\ -10 \end{pmatrix}$ So the matrix to analyze is: $\frac{1}{4}\begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$

(c) One step of Gauss-Seidel gives: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$

7. (a) Jacobi's method gives the method $\begin{pmatrix} x \\ y \end{pmatrix}^{(n+1)} = \frac{1}{4}\begin{pmatrix} 0 & -1/4 \\ -2/5 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}^{(n)}$.
Then $\begin{pmatrix} 3 \\ 11 \end{pmatrix} \to \begin{pmatrix} -2 \\ -1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(b) For Gauss-Seidel, we use $x^{(n+1)} = (3 - y^{(n)})/4$ and $y^{(n+1)} = (1 - 2x^{(n)})/5$. Thus, $\begin{pmatrix} 3 \\ 11 \end{pmatrix} \to \begin{pmatrix} -2 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$.

(c) Both methods should converge because the original matrix is diagonally dominant. Another way to see this is that the eigenvalues

of the Jacobi matrix $\frac{1}{4}\begin{pmatrix} 0 & -1/4 \\ -2/5 & 0 \end{pmatrix}$ are $\pm\frac{1}{\sqrt{10}}$. These are less than 1 in magnitude, so Jacobi's method converges.

For Gauss-Seidel, the matrix to analyze is $(L+D)^{-1}U$ or $\begin{pmatrix} 4 & 0 \\ 2 & 5 \end{pmatrix}^{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1/4 \\ 0 & -1/10 \end{pmatrix}$, which has eigenvalues 0 and $-1/10$. These are less than 1 in magnitude, so the Gauss-Seidel method converges.

(d) $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 5 \\ 7 \end{pmatrix} \to \begin{pmatrix} 5/7 \\ 1 \end{pmatrix} \to \begin{pmatrix} 27/7 \\ 45/7 \end{pmatrix} \to \begin{pmatrix} 3/5 \\ 1 \end{pmatrix}$

8. Not assigned.

9. Not assigned.

10. (a) The characteristic polynomial of A is $(9-\lambda)(-3-\lambda)+20 = \lambda^2 - 6\lambda - 7$. The characteristic polynomial of B is $-\lambda^3 + 12\lambda^2 - 39\lambda + 20 = 0$.

    (b) For A, $\lambda^2 - 6\lambda - 7 = (\lambda - 7)(\lambda + 1) \Rightarrow \lambda = -1, 7$. These are the eigenvalues of A.
    For B, $\lambda = 5, \frac{7\pm\sqrt{33}}{2}$.

    (c) $A*A-6A-7I = \begin{pmatrix} 61 & 12 \\ -60 & -11 \end{pmatrix} - \begin{pmatrix} 54 & 12 \\ -60 & -18 \end{pmatrix} - \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

11. Not assigned.

12. Not assigned.

13.
$$\begin{pmatrix} 2 & -1 & 1 & 7 \\ 3 & 0 & 2 & 9 \\ 1 & 2 & -1 & -6 \end{pmatrix} \to \begin{pmatrix} 3 & 0 & 2 & 9 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & -1 & -6 \end{pmatrix} \to \begin{pmatrix} 3 & 0 & 2 & 9 \\ 0 & -1 & \frac{-1}{3} & 1 \\ 0 & 2 & \frac{-5}{3} & -9 \end{pmatrix} \to$$
$$\begin{pmatrix} 3 & 0 & 2 & 9 \\ 0 & 2\frac{-5}{3} & -9 & \\ 0 & -1 & \frac{-1}{3} & 1 \end{pmatrix} \to \begin{pmatrix} 3 & 0 & 2 & 9 \\ 0 & 2 & \frac{-5}{3} & -9 \\ 0 & 0 & \frac{-7}{6} & \frac{-7}{2} \end{pmatrix}$$

This gives $x_2 = 3$; $2x_1 = -9+\frac{5}{3}3 \Rightarrow x_1 = -2$ and $3x_0 = 9-2*3 = 3 \Rightarrow x_0 = 1$.

```
Without pivoting:
The matrix is 3 by 3
Matrix:
2.000000 -1.000000 1.000000
3.000000 0.000000 2.000000
1.000000 2.000000 -1.000000
Right Hand Side:
7.000000 9.000000 -6.000000
U matrix:
2.000000 -1.000000 1.000000
0.000000 1.500000 0.500000
0.000000 0.000000 -2.333333
L matrix:
1.000000 0.000000 0.000000
1.500000 1.000000 0.000000
0.500000 1.666666 1.000000
Answer:
x0 = 1.000000
x1 = -2.000000
x2 = 3.000000
```

14. The matrix is 3 by 3

```
Matrix:
1.000000 2.000000 3.000000
1.000000 -1.000000 6.000000
2.000000 1.000000 0.000000
Right Hand Side:
4.000000 -1.000000 0.000000

    1.000000  2.000000  3.000000
U = 0.000000 -3.000000  3.000000
    0.000000  0.000000 -9.000000

    1.000000  0.000000  0.000000
L = 1.000000  1.000000  0.000000
    2.000000  1.000000  1.000000
```

```
Answer:
x0 = -1.000000
x1 =  2.000000
x2 =  0.333333
```

15. Not assigned.

16. Not assigned.

17.

$$\left(\begin{array}{cccc|cc} 3 & 2 & -1 & -4 & 10 & 2 \\ 1 & -1 & 3 & -1 & -4 & 3 \\ 2 & 1 & -3 & 0 & 16 & 1 \\ 0 & -1 & 8 & -5 & 3 & 3 \end{array}\right) \rightarrow \left(\begin{array}{cccc|cc} 3 & 2 & -1 & -4 & 10 & 2 \\ 0 & -5 & 10 & 1 & -22 & 7 \\ 0 & -1 & -7 & 8 & 28 & -1 \\ 0 & -1 & 8 & -5 & 3 & 3 \end{array}\right) \rightarrow$$

$$\left(\begin{array}{cccc|cc} 3 & 2 & -1 & -4 & 10 & 2 \\ 0 & -5 & 10 & 1 & -22 & 7 \\ 0 & 0 & -45 & 39 & 162 & -12 \\ 0 & 0 & 30 & -26 & 37 & 8 \end{array}\right) \rightarrow \left(\begin{array}{cccc|cc} 3 & 2 & -1 & -4 & 10 & 2 \\ 0 & -5 & 10 & 1 & -22 & 7 \\ 0 & 0 & -45 & 39 & 162 & -12 \\ 0 & 0 & 0 & 0 & 108 & 0 \end{array}\right)$$

Thus, for the first right hand side, we have a contradiction $0 = 108$. For the second right hand side, we have $0 = 0$ and have a one parameter family of solutions.

18. Not assigned.

19. (a)
```
The matrix is 5 by 5
Matrix:
-2.000000 1.000000 0.000000 0.000000 0.000000
1.000000 -2.000000 1.000000 0.000000 0.000000
0.000000 1.000000 -2.000000 1.000000 0.000000
0.000000 0.000000 1.000000 -2.000000 1.000000
0.000000 0.000000 0.000000 1.000000 -2.000000
Right Hand Side:
0.000000 0.000000 0.000000 0.000000 -6.000000
U matrix:
-2.000000 1.000000 0.000000 0.000000 0.000000
0.000000 -1.500000 1.000000 0.000000 0.000000
0.000000 0.000000 -1.333333 1.000000 0.000000
```

```
0.000000 0.000000 0.000000 -1.250000 1.000000
0.000000 0.000000 0.000000 0.000000 -1.200000
Answer:
x0 = 1.000000
x1 = 2.000000
x2 = 3.000000
x3 = 4.000000
x4 = 5.000000
```

(b) The number of operations is just 2 for each of $n-1$ columns in the elimination process since the right hand side is zero. For the back-substitution, there is one operation to find x_n and 2 to find each of the others. This gives a total of $2(n-1)+2(n-1)+1 = 4n-3$.

20. (a)

$$\begin{pmatrix} 1 & 2 & 0 \\ -2 & -3 & 2 \\ 0 & -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -3 \end{pmatrix}$$

(b)
```
    1.000000 2.000000 0.000000
U = 0.000000 1.000000 2.000000
    0.000000 0.000000 1.000000
Answer:
x0 = -7.000000
x1 =  3.000000
x2 = -1.000000
```

(c) Using the Gauss-Jordan Method gives the same result as in part (b).

(d)
```
    1  0  0        1  2  0
L =-2  1  0    U = 0  1  2
    0 -2  1        0  0  1
```

Then $Ly = b$ gives $y^T = [-1,\ 1,\ 1]$ and $Ux = y$ gives $x^T = [-7,\ 3,\ -1]$.

(e) Solving for right hand sides of unit normal vectors gives:

$$A^{-1} = \begin{pmatrix} 13 & 6 & 4 \\ -6 & -3 & -2 \\ 4 & 2 & 1 \end{pmatrix}$$

(f) det(A)=det(U)=1

(g) The condition number of A is (using the 1-norm) $||A||\,||A^{-1}|| = 7*23 = 161$.

21. Using Gaussian Elimination on

```
2  5 -1
1  6  4
7 -4  2
```

gives

```
2.000000 5.000000 -1.000000
0.000000 3.500000  4.500000
0.000000 0.000000 33.142857
```

for a determinant of 2*3.5*33.1428.. = 232.
For the second matrix, we have:

```
 3.000000 -2.000000  8.000000  2.000000
-1.000000  3.000000  2.000000 -6.000000
-5.000000 -1.000000  3.000000 -9.000000
 2.000000  3.000000 -8.000000  1.000000
```

becomes

```
3.000000 -2.000000  8.000000   2.000000
0.000000  2.333333  4.666667  -5.333333
0.000000  0.000000 25.000000 -15.571429
0.000000  0.000000  0.000000  -4.131429
```

for a determinant of 3*2.333*25*-4.13 = 723.

22. Hilbert matrix question:

```
Double precision (3x3)
1.000000 0.500000 0.333333
0.500000 0.333333 0.250000
0.333333 0.250000 0.200000
Right Hand Side:
1.833333 1.083333 0.783333
U matrix
1.000000 0.500000 0.333333
0.000000 0.083333 0.083333
0.000000 0.000000 0.005556
Answer:
x0 = 1.000000
x1 = 1.000000
x2 = 1.000000
Numerical determinant  1/2160.

Double precision (5x5)
1.000000 0.500000 0.333333 0.250000 0.200000
0.500000 0.333333 0.250000 0.200000 0.166667
0.333333 0.250000 0.200000 0.166667 0.142857
0.250000 0.200000 0.166667 0.142857 0.125000
0.200000 0.166667 0.142857 0.125000 0.111111
Right Hand Side:
2.283333 1.450000 1.092857 0.884524 0.745635
U matrix
1.000000 0.500000 0.333333 0.250000 0.200000
0.000000 0.083333 0.083333 0.075000 0.066667
0.000000 0.000000 0.005556 0.008333 0.009524
0.000000 0.000000 0.000000 0.000357 0.000714
0.000000 0.000000 0.000000 0.000000 0.000023
Answer:
x0 = 1.000000
x1 = 1.000000
x2 = 1.000000
x3 = 1.000000
x4 = 1.000000
Numerical determinant  1/2.6*10^11.
```

```
Double precision (7x7)
1.000000 0.500000 0.333333 0.250000 0.200000 0.166667 0.142857
0.500000 0.333333 0.250000 0.200000 0.166667 0.142857 0.125000
0.333333 0.250000 0.200000 0.166667 0.142857 0.125000 0.111111
0.250000 0.200000 0.166667 0.142857 0.125000 0.111111 0.100000
0.200000 0.166667 0.142857 0.125000 0.111111 0.100000 0.090909
0.166667 0.142857 0.125000 0.111111 0.100000 0.090909 0.083333
0.142857 0.125000 0.111111 0.100000 0.090909 0.083333 0.076923
Right Hand Side:
2.592857 1.717857 1.328968 1.095635 0.936544 0.819877 0.730134
U matrix
1.000000 0.500000 0.333333 0.250000 0.200000 0.166667 0.142857
0.000000 0.083333 0.083333 0.075000 0.066667 0.059524 0.053571
0.000000 0.000000 0.005556 0.008333 0.009524 0.009921 0.009921
0.000000 0.000000 0.000000 0.000357 0.000714 0.000992 0.001190
0.000000 0.000000 0.000000 0.000000 0.000023 0.000057 0.000093
0.000000 0.000000 0.000000 0.000000 0.000000 0.000001 0.000004
0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000
Answer:
x0 = 1.000000
x1 = 1.000000
x2 = 1.000000
x3 = 1.000000
x4 = 1.000000
x5 = 1.000000
x6 = 1.000000
Numerical determinant less than  1/10^23.

8x8 comes out exact by 9x9 gives:
(Some digits removed to make it fit)
1.000 0.500 0.333 0.250 0.200 0.166 0.142 0.125 0.111
0.500 0.333 0.250 0.200 0.166 0.142 0.125 0.111 0.100
0.333 0.250 0.200 0.166 0.142 0.125 0.111 0.100 0.090
0.250 0.200 0.166 0.142 0.125 0.111 0.100 0.090 0.083
0.200 0.166 0.142 0.125 0.111 0.100 0.090 0.083 0.076
0.166 0.142 0.125 0.111 0.100 0.090 0.083 0.076 0.071
0.142 0.125 0.111 0.100 0.090 0.083 0.076 0.071 0.066
0.125 0.111 0.100 0.090 0.083 0.076 0.071 0.066 0.062
```

```
0.111 0.100 0.090 0.083 0.076 0.071 0.066 0.062 0.058
Right Hand Side:
2.828 1.928 1.519 1.269 1.096 0.968 0.868 0.787 0.721
U matrix
1.000 0.500 0.333 0.250 0.200 0.166 0.142 0.125 0.111
0.000 0.083 0.083 0.075 0.066 0.059 0.053 0.048 0.044
0.000 0.000 0.005 0.008 0.009 0.009 0.009 0.009 0.009
0.000 0.000 0.000 0.000 0.000 0.000 0.001 0.001 0.001
0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
Answer:
x0 = 1.000000
x1 = 1.000000
x2 = 1.000000
x3 = 1.000002
x4 = 0.999993
x5 = 1.000015
x6 = 0.999983
x7 = 1.000010
x8 = 0.999997
which is still quite good. 11x11 gives less than 1\% error but
finally the 12x12 computation gives significant errors (note x8).
(Some digits removed to make it fit)
1.000 0.500 0.333 0.250 0.200 0.166 0.142 0.125 0.111 0.100 0.090 0.083
0.500 0.333 0.250 0.200 0.166 0.142 0.125 0.111 0.100 0.090 0.083 0.076
0.333 0.250 0.200 0.166 0.142 0.125 0.111 0.100 0.090 0.083 0.076 0.071
0.250 0.200 0.166 0.142 0.125 0.111 0.100 0.090 0.083 0.076 0.071 0.066
0.200 0.166 0.142 0.125 0.111 0.100 0.090 0.083 0.076 0.071 0.066 0.062
0.166 0.142 0.125 0.111 0.100 0.090 0.083 0.076 0.071 0.066 0.062 0.058
0.142 0.125 0.111 0.100 0.090 0.083 0.076 0.071 0.066 0.062 0.058 0.055
0.125 0.111 0.100 0.090 0.083 0.076 0.071 0.066 0.062 0.058 0.055 0.052
0.111 0.100 0.090 0.083 0.076 0.071 0.066 0.062 0.058 0.055 0.052 0.050
0.100 0.090 0.083 0.076 0.071 0.066 0.062 0.058 0.055 0.052 0.050 0.047
0.090 0.083 0.076 0.071 0.066 0.062 0.058 0.055 0.052 0.050 0.047 0.045
0.083 0.076 0.071 0.066 0.062 0.058 0.055 0.052 0.050 0.047 0.045 0.043
```

```
Right Hand Side:
3.103 2.180 1.751 1.484 1.297 1.156 1.045 0.954 0.879 0.816 0.761 0.714
U matrix
1.000 0.500 0.333 0.250 0.200 0.166 0.142 0.125 0.111 0.100 0.090 0.083
0.000 0.083 0.083 0.075 0.066 0.059 0.053 0.048 0.044 0.040 0.037 0.035
0.000 0.000 0.005 0.008 0.009 0.009 0.009 0.009 0.009 0.009 0.008 0.008
0.000 0.000 0.000 0.000 0.000 0.000 0.001 0.001 0.001 0.001 0.001 0.001
0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
Answer:
x0 = 1.000000
x1 = 1.000003
x2 = 0.999916
x3 = 1.001125
x4 = 0.991859
x5 = 1.035385
x6 = 0.902315
x7 = 1.175419
x8 = 0.795768
x9 = 1.148663
x10 = 0.938525
x11 = 1.011022


For single precision:
2x2
1.000000 0.500000
0.500000 0.333333
Right Hand Side:
1.500000 0.833333
U matrix
1.000000 0.500000
```

```
0.000000 0.083333
Determinant = 1/12
Answer:
x0 = 1.000000
x1 = 1.000000
3x3 has some error
1.000000 0.500000 0.333333
0.500000 0.333333 0.250000
0.333333 0.250000 0.200000
Right Hand Side:
1.833333 1.083333 0.783333
U matrix
1.000000 0.500000 0.333333
0.000000 0.083333 0.083333
0.000000 0.000000 0.005556
Answer:
x0 = 0.999999
x1 = 1.000005
x2 = 0.999995
5x5 has about 1\% error
1.000000 0.500000 0.333333 0.250000 0.200000
0.500000 0.333333 0.250000 0.200000 0.166667
0.333333 0.250000 0.200000 0.166667 0.142857
0.250000 0.200000 0.166667 0.142857 0.125000
0.200000 0.166667 0.142857 0.125000 0.111111
Right Hand Side:
2.283334 1.450000 1.092857 0.884524 0.745635
U matrix
1.000000 0.500000 0.333333 0.250000 0.200000
0.000000 0.083333 0.083333 0.075000 0.066667
0.000000 0.000000 0.005556 0.008333 0.009524
0.000000 0.000000 0.000000 0.000357 0.000714
0.000000 0.000000 0.000000 0.000000 0.000023
Answer:
x0 = 1.000076
x1 = 0.998668
x2 = 1.005532
x3 = 0.991866
```

```
x4 = 1.003899

6x6 gives a poor result.
1.000000 0.500000 0.333333 0.250000 0.200000 0.166667
0.500000 0.333333 0.250000 0.200000 0.166667 0.142857
0.333333 0.250000 0.200000 0.166667 0.142857 0.125000
0.250000 0.200000 0.166667 0.142857 0.125000 0.111111
0.200000 0.166667 0.142857 0.125000 0.111111 0.100000
0.166667 0.142857 0.125000 0.111111 0.100000 0.090909
Right Hand Side:
2.450000 1.592857 1.217857 0.995635 0.845635 0.736544
U matrix
1.000000 0.500000 0.333333 0.250000 0.200000 0.166667
0.000000 0.083333 0.083333 0.075000 0.066667 0.059524
0.000000 0.000000 0.005556 0.008333 0.009524 0.009921
0.000000 0.000000 0.000000 0.000357 0.000714 0.000992
0.000000 0.000000 0.000000 0.000000 0.000023 0.000057
0.000000 0.000000 0.000000 0.000000 0.000000 0.000002
Numerical Determinant about 10^17
Answer:
x0 = 1.000391
x1 = 0.989231
x2 = 1.071208
x3 = 0.817770
x4 = 1.198709
x5 = 0.922440
```

23. Not assigned.

24. (a) 1 norm: $2.15 + 3.1 + 10.0 + 2.2 = 17.45$
    2 norm: $\sqrt{2.15^2 + 3.1^2 + 10.0^2 + 2.2^2} = \sqrt{119.0725} \approx 10.91$.
    ∞ norm: 10.

    (b) 1 norm: $4 + 5 + 0 + 3 + 7 = 19$
    2 norm: $\sqrt{16 + 25 + 0 + 9 + 49} = \sqrt{99} \approx 9.95$.
    ∞ norm: 7.

    (c) $x + y$ does not exist so we cannot check the triangle inequality.

25. (a) The 1-norm of A is 81/10 and the $\infty-$norm is 6.

(b) The condition number using the 1-norm is 81/10 * 10 = 81.

26. (a) 1 norm (maximum column sum): 23 ∞ norm (maximum row sum): 23

    (b) 1 norm (maximum column sum): 12 ∞ norm (maximum row sum): 11

    (c) Not assigned

27. The 1x1 Hilbert matrix inverse is: $\begin{pmatrix} 1 \end{pmatrix}$ so the condition number is 1*1=1.

    The 2x2 Hilbert matrix inverse is: $\begin{pmatrix} 4 & -6 \\ -6 & 12 \end{pmatrix}$ so the condition number is 3/2 * 18 =27.

    The 3x3 Hilbert matrix inverse is:

$$\begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix}$$

    So the condition number is 11/6 * 408 = 748.

    The 4x4 Hilbert matrix inverse is:

$$\begin{pmatrix} 16 & -120 & 240 & -140 \\ -120 & 1200 & -2700 & 1680 \\ 240 & -2700 & 6480 & -4200 \\ -140 & 1680 & -4200 & 2800 \end{pmatrix}$$

    So the condition number is 25/12 * 13620 = 28375.

    The 5x5 Hilbert matrix inverse is:

$$\begin{pmatrix} 25 & -300 & 1050 & -1400 & 630 \\ -300 & 4800 & -18900 & 26800 & -12600 \\ 1050 & -18900 & 79380 & -117600 & 56700 \\ -1400 & 26880 & -117600 & 179200 & -88200 \\ 630 & -12600 & 56700 & -88200 & 44100 \end{pmatrix}$$

    So the condition number is 137/60 * 413280 = 943646.

28. (a) Single precision

```
3.010000 6.030000 1.990000
1.270000 4.160000 -1.230000
0.987000 -4.810000 9.340000
Right Hand Side:
1.000000 1.000000 1.000000
3.010000 6.030000 1.990000
0.000000 1.615780 -2.069634
0.000000 0.000000 -0.006282
Answer:
x0 = 1592.355347
x1 = -631.814453
x2 = -493.541962
```

(b) Double precision

```
3.010000 6.030000 1.990000
1.270000 4.160000 -1.230000
0.987000 -4.810000 9.340000
Right Hand Side:
1.000000 1.000000 1.000000
3.010000 6.030000 1.990000
0.000000 1.615781 -2.069635
0.000000 0.000000 -0.006281
Answer:
x0 = 1592.599625
x1 = -631.911376
x2 = -493.617725
```

(c) Single precision

```
3.010000 6.030000 1.990000
1.270000 4.160000 -1.230000
0.987000 -4.810000 -9.340000
Right Hand Side:
1.000000 1.000000 1.000000
3.010000 6.030000 1.990000
0.000000 1.615780 -2.069634
0.000000 0.000000 -18.686283
Answer:
x0 = 0.150942
```

```
x1 = 0.145247
x2 = -0.165916

Double precision
3.010000 6.030000 1.990000
1.270000 4.160000 -1.230000
0.987000 -4.810000 -9.340000
Right Hand Side:
1.000000 1.000000 1.000000
3.010000 6.030000 1.990000
0.000000 1.615781 -2.069635
0.000000 0.000000 -18.686281
Answer:
x0 = 0.150942
x1 = 0.145247
x2 = -0.165916
```

(d) The first matrix (with +9.34) is ill-conditioned since single and double precision computations give different results.

(e) For the first matrix: $A^{-1} = \begin{pmatrix} -1078.272569 & 2157.065646 & 513.806548 \\ 428.054054 & -856.031178 & -203.934252 \\ 334.389189 & -668.793764 & -159.213150 \end{pmatrix}$ so the condition number is $||A||||A^{-1}|| = 15 * 3682 = 55230.$

For the second matrix: $A^{-1} = \begin{pmatrix} 0.492631 & -0.514392 & 0.172702 \\ -0.117162 & 0.330956 & -0.068547 \\ 0.112396 & -0.224797 & -0.053515 \end{pmatrix}$ so the condition number is $||A||||A^{-1}|| = 15 * 1.07 = 16.05$. This is much better conditioned.

29. Not assigned.

30. Neither Jacobi's method nor Gauss-Seidel converges for the equation written in this order. Swapping the first two equations makes the system diagonally dominant and we get:

```
Jacobi
  0.000000   0.000000   0.000000
  2.400000  -1.833333   2.857143
```

```
  0.890476  -0.557143   3.038095
  1.073333  -1.030159   2.889116
  1.038322  -0.994036   2.998141
  1.001937  -0.987536   2.992822
  1.005364  -1.000551   2.996162
  1.001425  -0.998852   2.999391
  1.000473  -0.999627   2.999468
  1.000287  -0.999931   2.999826
  1.000084  -0.999933   2.999939
  1.000038  -0.999982   2.999969
  1.000016  -0.999993   2.999990
  1.000006  -0.999996   2.999996
  1.000002  -0.999999   2.999998

Gauss-Seidel
  0.000000   0.000000   0.000000
  2.400000  -1.033333   2.809524
  1.069524  -1.008571   2.992517
  1.001279  -1.000821   3.000052
  0.999815  -1.000053   3.000042
  0.999973  -1.000002   3.000005
  0.999998  -1.000000   3.000000
```

So Gauss-Seidel converges in about half as many steps. Noting that the exact solution is $[1, \ -1, \ 3]^T$ and taking the ∞-norm of the error at each step gives:

```
Jacobi      Gauss-Seidel
3.000000    3.000000
1.400000    1.400000
0.442857    0.069524
0.110884    0.001279
0.038322    0.000185
0.012464    0.000027
0.003838    0.000002
```

Gauss-Seidel error reduces by about a factor of 7 at each step. Jacobi

error reduces by about a factor of 3.2 at each step. They are both linear with coefficients 0.31 for Jacobi and 0.14 for Gauss-Seidel.

31. (a) Jacobi's method gives $x = \frac{4-y}{3}$ and $y = \frac{5-x}{4}$. Starting with $(0,0)^T$ gives $(4/3, 5/4)^T$ and then $(11/12, 11/12)^T$.

    (b) Gauss-Seidel gives: $(4/3, 11/12)^T$ and $(37/36, 143/144)^T$.

    (c) For Jacobi: $D^{-1}(L+U) = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/3 \\ 1/4 & 0 \end{pmatrix}$
    with eigenvalues $\pm\frac{1}{\sqrt{12}}$

    For Gauss-Seidel $(L+D)^{-1}U = \begin{pmatrix} 1/3 & 0 \\ -1/12 & 1/4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/3 \\ 0 & -1/12 \end{pmatrix}$
    with eigenvalues 0 and $-1/12$, so the spectral radius is $1/12$ using Gauss-Seidel and thus this method is better than Jacobi's method for this matrix.

32. Not assigned.

33. Not assigned.

34. We must switch the order of the rows for Jacobi and Gauss-Seidel to converge here. We want the larger numbers on the diagonal. Thus we have

    (a) $(0,0)^T \to (7/6,\ 3/2)^T \to (11/12,\ 11/12)^T$

    (b) $(0,0)^T \to (7/6,\ 11/12)^T \to (73/72,\ 143/144)^T$

35. Not assigned.

36. Not assigned.

37. Not assigned.

38. Not assigned.

39. Here are the results with the first 10 steps and then every 10th step. The stopping crtierion was that the sum of the absolute values of the iterates changed by less than 0.0001. Jacobi took roughly 300 steps and Gauss-Seidel only about 150.

```
Jacobi
 0.00  0.00  0.00  0.00  0.00 0.00 0.00 0.00 0.00 0.00                0
-0.50 -0.75 -1.00 -1.25 -1.50 -1.75 -2.00 -2.25 -2.50 -2.75
-0.88 -1.50 -2.00 -2.50 -3.00 -3.50 -4.00 -4.50 -5.00 -4.00
-1.25 -2.19 -3.00 -3.75 -4.50 -5.25 -6.00 -6.75 -6.75 -5.25
-1.59 -2.88 -3.97 -5.00 -6.00 -7.00 -8.00 -8.62 -8.50 -6.12
-1.94 -3.53 -4.94 -6.23 -7.50 -8.75 -9.81 -10.50 -9.88 -7.00
-2.27 -4.19 -5.88 -7.47 -8.99 -10.41 -11.62 -12.09 -11.25 -7.69
-2.59 -4.82 -6.83 -8.69 -10.44 -12.06 -13.25 -13.69 -12.39 -8.38
-2.91 -5.46 -7.76 -9.88 -11.87 -13.59 -14.87 -15.07 -13.53 -8.95
-3.23 -6.08 -8.67 -11.06 -13.24 -15.12 -16.33 -16.45 -14.51 -9.52
-3.54 -6.70 -9.57 -12.21 -14.59 -16.54 -17.79 -17.67 -15.48 -10.00    10
-6.34 -12.16 -17.40 -21.94 -25.59 -27.98 -28.70 -27.06 -22.36 -13.65
-8.38 -16.09 -22.94 -28.67 -32.99 -35.46 -35.64 -32.88 -26.54 -15.85
-9.77 -18.76 -26.67 -33.17 -37.90 -40.39 -40.17 -36.66 -29.25 -17.26
-10.69 -20.53 -29.15 -36.16 -41.15 -43.65 -43.16 -39.15 -31.03 -18.19
-11.30 -21.70 -30.79 -38.14 -43.30 -45.80 -45.14 -40.79 -32.21 -18.80
-11.71 -22.48 -31.88 -39.45 -44.72 -47.22 -46.45 -41.88 -32.98 -19.21
-11.98 -23.00 -32.60 -40.31 -45.66 -48.16 -47.31 -42.60 -33.50 -19.48
-12.15 -23.34 -33.07 -40.88 -46.29 -48.78 -47.88 -43.07 -33.84 -19.65
-12.27 -23.56 -33.39 -41.26 -46.70 -49.20 -48.26 -43.39 -34.06 -19.77 100
-12.35 -23.71 -33.59 -41.51 -46.97 -49.47 -48.51 -43.59 -34.21 -19.85
-12.40 -23.81 -33.73 -41.68 -47.15 -49.65 -48.68 -43.73 -34.31 -19.90
-12.43 -23.87 -33.82 -41.79 -47.27 -49.77 -48.79 -43.82 -34.37 -19.93
-12.46 -23.92 -33.88 -41.86 -47.35 -49.85 -48.86 -43.88 -34.42 -19.96
-12.47 -23.94 -33.92 -41.91 -47.40 -49.90 -48.91 -43.92 -34.44 -19.97
-12.48 -23.96 -33.95 -41.94 -47.43 -49.93 -48.94 -43.95 -34.46 -19.98
-12.49 -23.98 -33.97 -41.96 -47.46 -49.96 -48.96 -43.97 -34.48 -19.99
-12.49 -23.98 -33.98 -41.97 -47.47 -49.97 -48.97 -43.98 -34.48 -19.99
-12.49 -23.99 -33.99 -41.98 -47.48 -49.98 -48.98 -43.99 -34.49 -19.99
-12.50 -23.99 -33.99 -41.99 -47.49 -49.99 -48.99 -43.99 -34.49 -20.00 200
-12.50 -24.00 -33.99 -41.99 -47.49 -49.99 -48.99 -43.99 -34.50 -20.00
-12.50 -24.00 -34.00 -41.99 -47.49 -49.99 -48.99 -44.00 -34.50 -20.00
-12.50 -24.00 -34.00 -42.00 -47.50 -50.00 -49.00 -44.00 -34.50 -20.00
-12.50 -24.00 -34.00 -42.00 -47.50 -50.00 -49.00 -44.00 -34.50 -20.00
-12.50 -24.00 -34.00 -42.00 -47.50 -50.00 -49.00 -44.00 -34.50 -20.00
-12.50 -24.00 -34.00 -42.00 -47.50 -50.00 -49.00 -44.00 -34.50 -20.00
-12.50 -24.00 -34.00 -42.00 -47.50 -50.00 -49.00 -44.00 -34.50 -20.00
```

```
-12.50 -24.00 -34.00 -42.00 -47.50 -50.00 -49.00 -44.00 -34.50 -20.00 280

Gauss-Seidel
0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00                      0
-0.50 -1.00 -1.50 -2.00 -2.50 -3.00 -3.50 -4.00 -4.50 -5.00
-1.00 -2.00 -3.00 -4.00 -5.00 -6.00 -7.00 -8.00 -9.00 -7.25
-1.50 -3.00 -4.50 -6.00 -7.50 -9.00 -10.50 -12.00 -12.12 -8.81
-2.00 -4.00 -6.00 -8.00 -10.00 -12.00 -14.00 -15.31 -14.56 -10.03
-2.50 -5.00 -7.50 -10.00 -12.50 -15.00 -17.16 -18.11 -16.57 -11.04
-3.00 -6.00 -9.00 -12.00 -15.00 -17.83 -19.97 -20.52 -18.28 -11.89
-3.50 -7.00 -10.50 -14.00 -17.41 -20.44 -22.48 -22.63 -19.76 -12.63
-4.00 -8.00 -12.00 -15.96 -19.70 -22.84 -24.73 -24.50 -21.06 -13.28
-4.50 -9.00 -13.48 -17.84 -21.84 -25.04 -26.77 -26.16 -22.22 -13.86
-5.00 -9.99 -14.91 -19.63 -23.83 -27.05 -28.61 -27.67 -23.26 -14.38  10
-9.02 -17.59 -25.39 -32.04 -37.09 -40.01 -40.18 -36.97 -29.67 -17.59
-10.97 -21.19 -30.22 -37.64 -42.95 -45.63 -45.15 -40.93 -32.39 -18.95
-11.83 -22.77 -32.35 -40.09 -45.51 -48.09 -47.32 -42.66 -33.58 -19.54
-12.21 -23.46 -33.28 -41.17 -46.63 -49.16 -48.26 -43.41 -34.10 -19.80
-12.37 -23.76 -33.68 -41.64 -47.12 -49.63 -48.68 -43.74 -34.32 -19.91
-12.44 -23.90 -33.86 -41.84 -47.33 -49.84 -48.86 -43.89 -34.42 -19.96
-12.48 -23.95 -33.94 -41.93 -47.43 -49.93 -48.94 -43.95 -34.47 -19.98
-12.49 -23.98 -33.97 -41.97 -47.47 -49.97 -48.97 -43.98 -34.49 -19.99
-12.50 -23.99 -33.99 -41.99 -47.49 -49.99 -48.99 -43.99 -34.49 -20.00 100
-12.50 -24.00 -33.99 -41.99 -47.49 -49.99 -48.99 -44.00 -34.50 -20.00
-12.50 -24.00 -34.00 -42.00 -47.50 -50.00 -49.00 -44.00 -34.50 -20.00
-12.50 -24.00 -34.00 -42.00 -47.50 -50.00 -49.00 -44.00 -34.50 -20.00
-12.50 -24.00 -34.00 -42.00 -47.50 -50.00 -49.00 -44.00 -34.50 -20.00
-12.50 -24.00 -34.00 -42.00 -47.50 -50.00 -49.00 -44.00 -34.50 -20.00
```

40. Not assigned.

41. The Power Method gives $(1,\ 2)^T \to (18,\ 21)^T \to (6/7,\ 1) \to (83/7,\ 86/7)^T$
So it looks like an eigenvector of $(1,\ 1)^T$ and an eigenvalue around 12
(exact eigenvalues are 13 and -3).

42. Not assigned.

43. Not assigned.

44. We must find the eigenvalues of A_1 and A_2. The eigenvalues of A_1 are 1 and 1/2 so A_1 is not convergent (since the spectral radius is not strictly less than 1). In particular, the vector $[1,\ 0]^T$ just stays the same no when multiplied by A_1.

    The eigenvalues of A_2 are both 1/2 and thus are strictly less than 1 in magnitude. Thus A_2 is convergent (since the spectral radius is strictly less than 1).

45. $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - J^{-1} \begin{pmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{pmatrix}$ where $f(x,y) = x^2 + y^3 - 1$ and $g(x,y) = x^3 - y^2 + \frac{1}{4}$ and $J = \begin{pmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{pmatrix} = \begin{pmatrix} 2x & 3y^2 \\ 3x^2 & -2y \end{pmatrix}$

    For this problem, we have

$$J^{-1} = \frac{1}{-4xy - 9x^2y^2} \begin{pmatrix} -2y & -3y^2 \\ -3x^2 & 2x \end{pmatrix}$$

    So,

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} - \frac{1}{-1 - \frac{9}{16}} \begin{pmatrix} -1 & \frac{-3}{4} \\ \frac{-3}{4} & 1 \end{pmatrix} \begin{pmatrix} \frac{-5}{8} \\ \frac{1}{8} \end{pmatrix} = \begin{pmatrix} \frac{21}{25} \\ \frac{22}{25} \end{pmatrix}$$

46. Using Newton's method for systems, starting at (1,1) we find

```
i     x       f(x,y)        y        g(x,y)
0 1.000000  0.000000 1.000000  0.158529
0 0.622006 -0.178599 0.433009  0.055697
1 0.726607  0.006741 0.497818 -0.005952
2 0.725929 -0.000026 0.502919  0.000000
3 0.725951  0.000000 0.502947  0.000000
5 0.725951  0.000000 0.502947  0.000000
Solution is x = 0.725951 y = 0.502947 f,g = 0.0000 0.0000 iter = 5
```

    Starting at (-1,0) we find

```
Initial iterates are -1.000000 0.000000
i     x       f(x,y)        y        g(x,y)
0 -1.000000 -1.000000 0.000000 -0.841471
```

```
0 -2.000000 -2.694375 1.922076  2.678878
1 -1.417382 -0.995924 0.766495 -0.139032
2 -1.746058  0.042345 0.510210  0.417468
3 -1.668521 -0.015852 0.362343  0.012287
4 -1.670030 -0.000287 0.345337  0.000007
5 -1.670093  0.000000 0.345134  0.000000
6 -1.670093  0.000000 0.345134  0.000000
8 -1.670093  0.000000 0.345134  0.000000
Solution is x = -1.670093 y = 0.345134 f,g = 0.0000 0.0000 iter = 8
```

47. $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - J^{-1} \begin{pmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{pmatrix}$ where $f(x, y) = x^2 + xy - y^2 - 5$ and $g(x, y) = x^3 + y - 9$ and $J = \begin{pmatrix} f_x(x, y) & f_y(x, y) \\ g_x(x, y) & g_y(x, y) \end{pmatrix} = \begin{pmatrix} 2x + y & x - 2y \\ 3x^2 & 1 \end{pmatrix}$

For this problem, we have

$$J^{-1} = \frac{1}{2x + y - 3x^3 + 6x^2y} \begin{pmatrix} 1 & 2y - x \\ -3x^2 & 2x + y \end{pmatrix}$$

So,

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \frac{1}{15} \begin{pmatrix} 1 & 5 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} -5 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$