Math 340 - Homework Solutions 4
Interpolation and Curve Fitting

1. The Lagrange Polynomial is:

$$P_2(x) = \frac{(x-3)(x-6)}{(2-3)(2-6)}0.69315 + \frac{(x-2)(x-6)}{(3-2)(3-6)}1.0986 + \frac{(x-2)(x-3)}{(6-2)(6-3)}1.7918$$

x	$P_2(x)$	$\ln(x)$	Abs. Error	Rel. Error
1	0.200540	0.000000	0.200540	N/A
2	0.693160	0.693147	0.000013	0.000019
3	1.098600	1.098612	0.000012	0.000011
4	1.416860	1.386294	0.030566	0.022049
5	1.647940	1.609438	0.038502	0.023923
6	1.791840	1.791759	0.000081	0.000045
7	1.848560	1.945910	0.097350	0.050028
8	1.818100	2.079442	0.261342	0.125679
9	1.700460	2.197225	0.496765	0.226087
10	1.495640	2.302585	0.806945	0.350452

2. Not assigned

3. (a) We make the chart

x	$f(x)$			
0	2	0	0	1
1	2	0	4	
3	2	12		
4	14			

So, the Lagrange Polynomial through the points is:
(from the top) $2 + 0x + 0x(x-1) + 1x(x-1)(x-3)$ or
(from the bottom) $14 + 12(x-4) + 4(x-4)(x-3) + 1(x-4)(x-3)(x-1)$.
They both equal $x^3 - 4x^2 + 3x + 2$.

(b) We make the chart

x	$f(x)$		
0	2	0	0
1	2	0	
3	2		

So, the Lagrange Polynomial through the points is:
(from the top) $2 + 0x + 0x(x-1)$ or

(from the bottom) $2 + 0(x-3) + 0(x-3)(x-1)$.
They both equal 2.

(c) We make the chart

x	$f(x)$		
1	2	0	4
3	2	12	
4	14		

So, the Lagrange Polynomial through the points is:
(from the top) $2 + 0(x-1) + 4(x-1)(x-3)$ or
(from the bottom) $14 + 12(x-4) + 4(x-4)(x-3)$.
They both equal $4x^2 - 16x + 14$.

(d) For part (a), $L(2) = 0$ with error term $\frac{(2-0)(2-1)(2-3)(2-4)f''''(\xi)}{4!}$.
For part (b), $L(2) = 2$ with error term $\frac{(2-0)(2-1)(2-3)f'''(\xi)}{3!}$.
For part (c), $L(2) = -2$ with error term $\frac{(2-1)(2-3)(2-4)f'''(\xi)}{3!}$.

(e) $f'(2) \approx \frac{f(4)-f(0)}{4-0} = 3$.

(f) Using $x=1$ and $x=3$, we get $f'(2) \approx \frac{f(3)-f(1)}{2-1} = 0$. Using Richardson extrapolation, we find a "better" value for $f'(2) \approx \frac{4(0)-1(3)}{3} \approx -1$.

4. (a) The Lagrange Polynomial is:

$$P_3(x) = \frac{(x-1)(x-3)(x-6)}{(-1-1)(-1-3)(-1-6)}1 + \frac{(x--1)(x-3)(x-6)}{(1--1)(1-3)(1-6)}3$$
$$+\frac{(x--1)(x-1)(x-6)}{(3--1)(3-1)(3-6)}(-3) + \frac{(x--1)(x-1)(x-3)}{(6--1)(6-1)(6-3)}(-27)$$

(b) Using a divided difference table, we find:

x	$f(x)$			
-1	1	1	-1	0
1	3	-3	-1	
3	-3	-8		
6	-27			

So the polynomial (cubic) is $1 + 1(x--1) - 1(x--1)(x-1)$, which is really a quadratic.

5. Not assigned.

6. With 6 equispaced points, the error term is:
$\frac{f^{(6)}(\xi)}{6!}(x-x_0)(x-x_1)...(x-x_5)$ where $x_{i+1} = x_i + h$. Thus, we need to

maximize $(x-x_0)(x-x_1)...(x-x_5)$. If we let θh be the (positive or negative) distance from the center of the interval ($x_2+h/2$ or $x_3-h/2$), to where the expression is maximized, this becomes, find the maximum of $(\theta h-\frac{5}{2}h)(\theta h-\frac{3}{2}h)(\theta h-\frac{1}{2}h)(\theta h+\frac{1}{2}h)(\theta h+\frac{3}{2}h)(\theta h+\frac{5}{2}h)$ since, for example x_0 is $\frac{5}{2}h$ to the left of the center. Factor out the h^6 and reordering terms, we let

$$F(\theta) = (\theta-\tfrac{5}{2})(\theta+\tfrac{5}{2})(\theta-\tfrac{3}{2})(\theta+\tfrac{3}{2})(\theta-\tfrac{1}{2})(\theta+\tfrac{1}{2})$$
$$F(\theta) = \theta^6-\tfrac{35}{4}\theta^4+\tfrac{259}{16}\theta^2-\tfrac{225}{64}$$
$$F'(\theta) = 6\theta^5-35\theta^3+\tfrac{259}{8}\theta.$$

$F'(\theta)=0 \Rightarrow \theta=0$ or $\theta^2=\frac{\frac{35}{6}\pm\sqrt{(\frac{35}{6})^2-\frac{259}{12}}}{2}=1.153,\ 4.68$. So, $\theta=\pm1.074,\ \pm2.163,\ 0$. There should be a local max or min between each consecutive pair of x_i. We can plug these values into F to find that the maximum (in magnitude) occurs when $\theta=\pm 2.163$, giving $F=16.9$. Thus, the error term is $\frac{f^{(6)}(\xi)}{6!}h^6 16.9=0.0235f^{(6)}(\xi)h^6$.

7. Not assigned.

8. Not assigned.

9. Not assigned.

10. Not assigned.

11. Not assigned.

12. Not assigned.

13. Not assigned.

14. We make the chart

x	$f(x)$				
0.5	−1.1518	−2.6494	1.0955	1.029	0.004
−0.2	0.7028	−2.4303	0.6840	1.027	
0.7	−1.4845	−2.2251	0.8894		
0.1	−0.14943	−2.8477			
0.0	0.13534				

So, a Lagrange polynomial here is:
$-1.1518-2.6494(x-0.5)+1.0955(x-0.5)(x+0.2)+$

$1.029(x-0.5)(x+0.2)(x-0.7)\ +$
$0.004(x-0.5)(x+0.2)(x-0.7)(x-0.1).$

15. Not assigned.

16. Not assigned.

17. $S(2) = 1 + 2a + 8b = 29.$
$S'(2) = a + 12b = 38.$
$S''(2) = 12b = 2c.$
These equations imply that $a = 2$; $b = 3$; $c = 18$ and, thus, $S(3) = 29+38+18-3 = 82.$

18. Using the setup of Gerald and Wheatley, we have
The system of equations for the cubic spline through these points is: (use eq. 3.15 of the text)

(The 1 and 3 in the denominator turns to 2s if use the new values of points 0,2,4,6).

$$\begin{aligned} 2S_0 + (2\cdot 2 + 2\cdot 3)S_1 + 3S_2 &= 6\left(\tfrac{53-(-4)}{3} - \tfrac{-4-(-2)}{2}\right) \\ 3S_1 + (2\cdot 3 + 2\cdot 1)S_2 + 1S_3 &= 6\left(\tfrac{112-(53)}{1} - \tfrac{53-(-4)}{3}\right) \end{aligned}$$

Since $S_0 = -6$ and $S_3 = 30$, we get

$$\begin{aligned} 10S_1 + 3S_2 &= 132 \\ 3S_1 + 8S_2 &= 210 \end{aligned}$$

We note that $S = 2c$ in my derivation. Then a, b and d can be obtained.

Using the setup of **Epperson**, we have

$$\begin{aligned} c_{-1} + 4c_0 + c_1 &= f(x_0) = -2 \\ c_0 + 4c_1 + c_2 &= f(x_1) = -4 \\ c_1 + 4c_2 + c_3 &= f(x_2) = 53 \\ c_2 + 4c_3 + c_4 &= f(x_3) = 112 \end{aligned}$$

where

$$f''(x_0) = c_{-1}B''_{-1}(x_0) + c_0B''_0(x_0) + c_1B''_1(x_0) = -6$$

$$f''(x_0) = c_{-1}\frac{6}{h^2} - c_0\frac{12}{h^2} + c_1\frac{6}{h^2} = -6$$

$$f''(x_3) = c_2\frac{6}{h^2} - c_3\frac{12}{h^2} + c_4\frac{6}{h^2} = 30$$

Using $h = 2$, we find: $c_{-1} = 2c_0 - c_1 - 4$ and $c_4 = 20 - c_2 + 2c_3$

This gives the system:
$$\begin{array}{rcrcrcrcr} 6c_0 & & & & & & & = & 2 \\ c_0 & + & 4c_1 & + & c_2 & & & = & -4 \\ & & c_1 & + & 4c_2 & + & c_3 & = & 53 \\ & & & & & & 6c_3 & = & 92 \end{array}$$

The result is:
$$\begin{array}{rcr} c_0 & = & 1/3 \\ c_1 & = & -11/3 \\ c_2 & = & 31/3 \\ c_3 & = & 46/3 \end{array}$$

So the spline is: $\frac{1}{3}[B(\frac{x}{2}) - 11B(\frac{x-2}{2}) + 31B(\frac{x-4}{2}) + 46B(\frac{x-6}{2})]$

19. Not assigned.

20. Not assigned.

21. Not assigned.

22. $f_1(x) \Rightarrow f(0) = 5.$
$f_1(x) \Rightarrow f(1) = 4 = f_2(1) \Rightarrow c = 4.$
$f_1'(x) = 6x^2 + 8x - 7 \Rightarrow f_1'(1) = 7.$
$f_1''(x) = 12x + 8 \Rightarrow f_1''(1) = 20.$
$f_2'(x) = 9(x-1)^2 + 2a(x-1) + b \Rightarrow f_1'(1) = 7.$
$f_2'(1) = b = f_1'(1) = 7$. So $b = 7$
$f_2''(x) = 18(x-1) + 2a$
$f_2''(1) = 2a = f_1''(1) = 20 \Rightarrow a = 10.$
Thus, $f_2(x) = 3(x-1)^3 + 10(x-1)^2 + 7(x-1) + 4.$
This implies, $f_2(2) = 24,\ f_2'(2) = 36$ and $f_2''(2) = 38.$
Since, $f_3(x) = (x-2)^3 + d(x-2)^2 + e(x-2) + f$, we obtain
$f_3(2) = f = 24, f_3'(2) = e = 36$ and $f_3''(2) = 2d = 38 \Rightarrow d = 19.$
So, $f_3(x) = (x-2)^3 + 19(x-2)^2 + 36(x-2) + 24.$

23. Not assigned.

24. Not assigned.

25. The cubic spline $P(x)$ must have $P(a) = f(a),\ P(b) = f(b),\ P'(a) = f'(a)$ and $P'(b) = f'(b)$. Only one cubic can match these 4 conditions so the cubic spline and the cubic itself must be the same curve. For the free cubic spline case, a cubic with "free" end conditions has (for

$P(x) = a_1(x-a)^3 + b_1(x-a)^2 + c_1(x-a) + d_1)$ $P''(x) = 6a_1(x-a)+2b_1$ with $P''(a) = P''(b) = 0 \Rightarrow a_1 = b_1 = 0$. So $P(x)$ is not a cubic, but just a line.

26. Not assigned.

27. (a) We construct the table:

x	y	x^2	xy
-2	-1	4	2
-1	0	1	0
0	1	0	0
7	2	49	14
4	2	54	16

This leads to the equations for $y = a + bx$ of: $\begin{pmatrix} 4 & 4 \\ 4 & 54 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 16 \end{pmatrix}$.

(b) All four methods could be used for this system of equations.

28. We construct the table:

x	y	x^2	xy
-8	-9	64	72
-3	-4	9	12
-1	-2	1	2
12	11	144	132
0	-4	218	218

This leads to the equations for $y = a + bx$ of: $\begin{pmatrix} 4 & 0 \\ 0 & 218 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -4 \\ 218 \end{pmatrix}$.

Thus, $a = -1$ and $b = 1$ and $y = -1 + x$ is the best (in this case, it is the exact) line.

29. We set up the problem as:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0.5 & 0.5 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0.9573 \\ 2.0132 \\ 2.0385 \\ 1.9936 \\ 1.9773 \end{pmatrix}$$

Using $A^TAx = A^Tb \Rightarrow \begin{pmatrix} 9/4 & 5/4 & 5/2 \\ 5/4 & 9/4 & 5/2 \\ 5/2 & 5/2 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 5.0126 \\ 4.9873 \\ 8.9799 \end{pmatrix}$

This leads to, $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0.522650 \\ 0.497350 \\ 1.285980 \end{pmatrix}$. So the "best" plane through the points is $z = 0.522650x + 0.497350y + 1.285980$.

30. We construct the table:

x	y	x^2	xy
0	2	0	0
0	8	0	0
1	-1	1	-1
3	11	9	33
4	20	10	32

This leads to the equations for $y = a + bx$ of: $\begin{pmatrix} 4 & 4 \\ 4 & 10 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 20 \\ 32 \end{pmatrix}$ which gives $y = 3 + 2x$.

31. Not assigned.

32. $y = ae^{bx} \Rightarrow \ln y = \ln(a) + bx$
Let $Y = \ln y$ and $A = \ln(a)$ (so $a = e^A$). We then construct the table:

x	y	x^2	xy
0	0.1	0	0
1	2.0	1	2
3	3.0	9	9
4	5.1	10	11

This leads to the equations for $Y = A + bx$ of: $\begin{pmatrix} 3 & 4 \\ 4 & 10 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 5.1 \\ 11.0 \end{pmatrix}$.
Then we finish by noting that $y = e^A e^{bx}$. ($A = 0.5$ and $b = 0.9$).

33. $A^TAx = A^Tb \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

Thus, $\begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$, giving $x_1 = 9/7$ and $x_2 = 4/7$.

34. Not assigned.

35. Not assigned.