Math 340 - Homework Solutions 5
Numerical Differentiation and Integration

1. Not assigned.

2. $\frac{1}{2h}[f(x+3h)+f(x-h) \;-\; 2f(x)] =$
$\frac{1}{2h}[f(x)+3hf'(x)+\frac{9h^2}{2!}f''(x)+\frac{27h^3}{3!}f'''(x)+...f(x)-hf'(x)+\frac{h^2}{2!}f''(x)-\frac{h^3}{3!}f'''(x)+...-2f(x)] \;=$
$f'(x) \;+\; \frac{5h}{2}f''(x)+....$ So the difference scheme is a scheme for $f'(x)$ and it has error $\frac{5h}{2}f'''(\xi)$.

3. Not assigned

4. Not assigned

5. We want to solve $f''(x_0) \approx Af(x_0) \;+\; Bf(x_0+h) \;+\; Cf(x_0+3h)$. Expanding the right hand side in Taylor series, we obtain:
$f''(x_0) \approx \; Af(x_0) \;+\; B[f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\frac{h^3}{3!}f'''(x_0)+...] \;+$
$C[f(x_0)+3hf'(x_0)+\frac{9h^2}{2!}f''(x_0)+\frac{27h^3}{3!}f'''(x_0)+...$
Matching up $f(x_0)$, $f'(x_0)$ and $f''(x_0)$, we get
$A+B+C=0$,
$Bh+3Ch=0$ and
$\frac{h^2}{2}(B+9C)=1$.
Solving yields

$$A=\frac{2}{3h^2}, \quad B=-\frac{1}{h^2}, \quad \text{and} \quad C=\frac{1}{3h^2}$$

Thus, $f''(x_0) \;\approx\; \frac{1}{h^2}[\frac{2}{3}f(x_0) - f(x_0-h) + \frac{1}{3}f(x_0+3h)]$. Expanding this in Taylor series shows that $\frac{1}{h^2}[\frac{2}{3}f(x_0)-f(x_0+h)+\frac{1}{3}f(x_0+3h)] \;=\; f''(x_0) \;+\; \frac{4}{3}hf'''(\xi)$.

6. Not assigned

7. Not assigned

8. Not assigned

9. Not assigned

10. (a) Using the Midpoint rule with $h = 2$ gives
$\int_2^6 f(x)dx \approx 2[f(3) + f(5)] = 2[\frac{1}{4} + \frac{1}{6}] = \frac{5}{6} \simeq 0.833.$

(b) Using the Trapezoidal rule with $h = 1$ gives
$\int_2^6 f(x)dx \approx \frac{1}{2}1[f(2) + 2f(3) + 2f(4) + 2f(5) + f(6)] = \frac{1}{2}[\frac{1}{3} + \frac{1}{2} + \frac{2}{5} + \frac{1}{3} + \frac{1}{7}] = \frac{359}{420} \approx 0.855.$

(c) Using Simpson's rule with $h = 1$ gives
$\int_2^6 f(x)dx \approx \frac{1}{3}1[f(2) + 4f(3) + 2f(4) + 4f(5) + f(6)] = \frac{1}{3}[\frac{1}{3} + 1 + \frac{2}{5} + \frac{2}{3} + \frac{1}{7}] = \frac{89}{105} \approx 0.848.$

11. The length of one arch of the sine curve is given by $\int_0^\pi \sqrt{1 + \cos^2(x)}dx$.

h	trapezoid	simpson
1.570796327	7.584475592	7.150712164
0.785398163	3.819943643	3.829178926
0.392699082	3.820197715	3.820282406
0.196349541	3.820197789	3.820197814
0.098174770	3.820197789	3.820197789

12. Not assigned.

13. Not assigned.

14. Using the midpoint rule, we find (with exact = 1.295836866)

h	midpt	error
2.000000000	1.386294361	-0.090457494
1.000000000	1.321755840	-0.025918974
0.500000000	1.302645234	-0.006808368
0.250000000	1.297564013	-0.001727147
0.125000000	1.296270325	-0.000433459
0.062500000	1.295945337	-0.000108471
0.031250000	1.295863991	-0.000027125
0.015625000	1.295843648	-0.000006782
0.007812500	1.295838561	-0.000001695
0.003906250	1.295837290	-0.000000424
0.001953125	1.295836972	-0.000000106

The errors reduce by a factor of 4 approximately at each step so convergence is as expected. The theoretical error is: $\frac{(b-a)f''(\xi)h^2}{24}$ and since $|f''(\xi)| = |-\frac{1}{x^2}| \leq 1$ on [1,3], we find that the error $\leq \frac{(2)(1)h^2}{24} = \frac{h^2}{12}$. So for error to be less than 10^{-3}, we need $h^2 < 0.012 \Rightarrow h < .109$. For error to be less than 10^{-6}, we need $h^2 < 0.000012 \Rightarrow h < .00346$.
Using Romberg's method on this output yields: (factors are 4,16,64,256,1024,4096)

h	midpoint	rom1	rom2	rom3	rom4	rom5	rom6
2.00000	1.386294	1.300243	1.296011	1.295841	1.295837	1.295837	1.295837
1.00000	1.321756	1.296275	1.295843	1.295837	1.295837	1.295837	
0.50000	1.302645	1.295870	1.295837	1.295837	1.295837		
0.25000	1.297564	1.295839	1.295837	1.295837			
0.12500	1.296270	1.295837	1.295837				
0.06250	1.295945	1.295837					
0.03125	1.295864						

15. The difference between consecutive values should decrease by about a factor of 4, since error is $O(h^2)$. This is true since:
$I(h) \simeq Ch^2$
$I(h/2) \simeq C(h/2)^2 \simeq Ch^2/4$
$I(h/4) \simeq C(h/4)^2 \simeq Ch^2/16$
$I(h/8) \simeq C(h/8)^2 \simeq Ch^2/64$ etc.
The differences decrease by a factor of 4: $\frac{3}{4}Ch^2$, $\frac{3}{16}Ch^2$, $\frac{3}{64}Ch^2$ etc.

    These differences in our example are:

Points	Difference	Factor
4	-0.04137	1.95
8	-0.02118	1.98
16	-0.01071	1.99
32	-0.00538	

    Since halving the grid spacing reduces the error by only a factor of 2, it is only converging with order h, not h^2, so Midpoint rule is not converging as it should.

16. Not assigned.

17.

h	trapezoidal	error
1.000000000	0.500000000	0.166666666
0.500000000	0.603553391	0.063113275
0.250000000	0.643283046	0.023383620
0.125000000	0.658130222	0.008536445
0.062500000	0.663581197	0.003085470
0.031250000	0.665558936	0.001107730
0.015625000	0.666270811	0.000395855
0.007812500	0.666525657	0.000141009
0.003906250	0.666616549	0.000050118
0.001953125	0.666648882	0.000017785
0.000976562	0.666660362	0.000006304
0.000488281	0.666664434	0.000002233

The errors decrease by about a factor of 3 (not 4, as expected). This may be due to the fact that the second derivative blows up at the endpoint $x = 0$ so the error analysis does not apply.

18. (a) Using the Trapezoidal rule with $h = 2$ gives $\int_0^6 f(x)dx \approx \frac{1}{2}2[f(0)+ 2f(2) + 2f(4) + f(6)] = \frac{1}{2}2[0 + 2(48) + 2(-64) + 0] = -32$.

(b) Using the Trapezoidal rule with $h = 6$ gives $\int_0^6 f(x)dx \approx \frac{1}{2}6[f(0)+ f(6)] = \frac{1}{2}6[0 + 0] = 0$.

(c) Trapezoidal method has error order 2 $O(h^2)$, so using Romberg's method gives 3^2 times the fine minus one times the coarse, over $3^2 - 1$. The improved result is $[(9)(-32) - (1)(0)]/8 = -36$.

(d) Simpson's rule gives: $\frac{h}{3}[f(0) + 4f(3) + f(6)]$ with $h = 3$, yielding an integral of 0. The error term with (composite) Simpson's rule is: $O(h^4)$ or $\frac{(b-a)h^4}{180} f''''(\xi)$.

19. Not assigned.

20. (a) With $h = 2$, we have $f''(2) \simeq \frac{f(4)-2f(2)+f(0)}{2^2} = \frac{3-6+8}{4} = \frac{5}{4}$.
With $h = 1$, we have $f''(2) \simeq \frac{f(3)-2f(2)+f(1)}{1^2} = \frac{2-6+5}{1} = 1$.

(b) Richardson extrapolation yields: $f''(2) \simeq \frac{4\ fine\ -\ coarse}{3} = \frac{(4)(1)-\frac{5}{4}}{3} = \frac{11}{12}$

(c) Trapezoidal Rule gives $\int_0^4 f(x)dx \simeq \frac{1}{2}[8 + 2(5) + 2(3) + 2(2) + 3] = \frac{31}{2}$

Simpson's Rules gives $\int_0^4 f(x)dx \simeq \frac{1}{3}[8 + 4(5) + 2(3) + 4(2) + 3] = \frac{45}{3} = 15.$

21. (a) Trapezoidal Rule:
$\int_0^1 f(x)dx \approx \frac{1}{2}0.25[f(0) + 2f(\frac{1}{4}) + 2f(\frac{1}{2}) + 2f(\frac{3}{4}) + f(1)]$
$= \frac{1}{2}0.25[1 + 2(0.939413) + 2(0.778801) + 2(0.569783) + 0.367879]$
$= 0.74298$

(b) Simpson's Rule:
$\int_0^1 f(x)dx \approx \frac{1}{3}0.25[f(0)+4f(\frac{1}{4})+2f(\frac{1}{2})+4f(\frac{3}{4})+f(1)] = 0.74686.$
So the error in part (a) is about $|0.74686 - 0.74298| = 0.00388$.

(c) The error bound for the Trapezoidal Rule is:
$\frac{(b-a)h^2}{12} f''(\xi) \leq \frac{1}{12}(\frac{1}{4})^2 2 = \frac{1}{96} = 0.0104.$
(Since $f''(x) = -2e^{-x^2} + 4x^2e^{-x^2}$, which has magnitude no bigger than 2 on [0,1]).

22. Not assigned.

23. Not assigned.

24.

	$O(h)$		$O(h^3)$		$O(h^5)$
$I(h)$	2.3965	$\frac{3\ fine - coarse}{2}$	3.1912	$\frac{27\ fine - coarse}{26}$	2.998809
$I(h/3)$	2.9263	$\frac{3\ fine - coarse}{2}$	3.005935		
$I(h/9)$	2.9795				

25. Not assigned.

26. Not assigned.

27.

	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$
$I(h)$	1.570796	2.004560	1.999984	1.999999
$I(h/2)$	1.896119	2.000270	1.999999	
$I(h/4)$	1.974232	2.000016		
$I(h/8)$	1.993570			

First we take (4 fine - coarse)/3;
then (16 fine - coarse)/15;
then (64 fine - coarse)/ 63.

28. Not assigned.

29. Trapezoidal rule gives $\frac{1}{2}2[f(0)+f(2)] = 4$ and Simpson's rule gives $\frac{1}{3}[f(0)+4f(1)+f(2)] = 2$. Solving for $f(1)$ gives $f(1) = \frac{1}{2}$.

30. Let $\int_0^{3h} f(x)dx \simeq Af(0) + Bf(h) + Cf(2h) + Df(3h)$. Then,
$\int_0^{3h} 1dx = 3h = A + B + C + D \Rightarrow A + B + C + D = 3h$
$\int_0^{3h} xdx = \frac{9h^2}{2} = Bh + 2Ch + 3Dh \Rightarrow B + 2C + 3D = \frac{9h}{2}$
$\int_0^{3h} x^2dx = 9h^3 = Bh^2 + 4Ch^2 + 9Dh^2 \Rightarrow B + 4C + 9D = 9h$
$\int_0^{3h} x^3dx = \frac{81}{4}h^4 = Bh^3 + 8Ch^2 + 27Dh^2 \Rightarrow B + 8C + 27D = \frac{81}{4}h$
This gives, using Gaussian elimination: $A = D = \frac{3}{8}h$, $B = C = \frac{9}{8}h$.
Thus, we have $\int_0^{3h} f(x)dx \simeq \frac{3}{8}h[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$.

31. Not assigned.

32. (a) Using the Trapezoidal Rule, $T = \frac{0.2}{2}[6.050+2(7.389)+2(9.025)+2(11.023)+2(13.464)+16.445] = 10.4297$

 (b) The correct integral is $e^{2.8} - e^{1.8} = 10.3950$. So the error is -0.0347.

 (c) The error bound for the Trapezoidal rule for this case is: $\frac{h^2 max(f''(\xi))(b-a)e^{2.8}}{12} = \frac{(0.2)^2e^{2.8}1}{12} = 0.0548$

 (d) If we did not know the function, we would need to approximate max $f''(\xi)$. Using the values of f (which yield the largest second derivative) at 2.6, 2.8, and 3.0, we get max $f''(\xi) \simeq \frac{f(3.0)-2f(2.8)+f(2.6)}{0.2^2} \simeq 16.5$ for an error bound of 0.055.

 (e) If we want the error to be less than 5x10^{-6}, we need $\frac{h^2e^{2.8}1}{12} < 5\text{x}10^{-6}$, which gives $h < 0.00191$.

33. Not assigned.

34. Not assigned.

35. (a) Not assigned.

 (b)

h	trapezoidal	simpson
0.500000000	0.000365555	0.000293192
0.250000000	0.000518910	0.000574242
0.125000000	0.049093359	0.065284841
0.062500000	0.025224992	0.017268869
0.031250000	0.014109339	0.010404122

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0.015625000 0.010592389 0.009420073
0.007812500 0.013196441 0.014064459
0.003906250 0.013473018 0.013565210
0.001953125 0.013492485 0.013498974
0.000976562 0.013492484 0.013492484
0.000488281 0.013492485 0.013492486
0.000244141 0.013492486 0.013492486
0.000122070 0.013492486 0.013492486
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Notice that quite a small step size is required and that Simpson's Rule does not do much better than Trapezoidal.

36. Not assigned.

37. Not assigned.

38. $\int_0^1 f(x)dx = c_0 f(0) + c_1 f(x_1)$ has 3 unknowns (c_0, c_1 and x_1). We match up both sides for $f(x) = 1$, $f(x) = x$ and $f(x) = x^2$ giving:
$c_0 + c_1 = 1$
$c_0(0) + c_1 x_1 = \frac{1}{2}$ and
$c_0(0)^2 + c_1 x_1^2 = \frac{1}{3}$.
Plugging the second equation into the third gives $x_1 = \frac{2}{3}$. Then $c_1 = \frac{3}{4}$ and $c_0 = \frac{1}{4}$. The final result is: $\int_0^1 f(x)dx = \frac{1}{4}f(0) + \frac{3}{4}f(\frac{2}{3})$.

39. Not assigned.

40. Not assigned.