Math 340/611 - Homework Solutions 6
Differential Equations IVPs and BVPs

1. Not assigned.

2. $\frac{dy}{dt} = (1-t)y$ where $y(0) = 3$

   (a) Euler's method: $y(2) \approx y(0) + 2f(y(0),0) = 3 + 2\cdot 3 = 9.$
   $y(4) \approx y(2) + 2f(y(2),2) = 9 + 2\cdot(-9) = -9.$

   (b) Runge-Kutta: $k_1 = hf(3,0) = 2\cdot 3 = 6$
   $k_2 = hf(3+\frac{1}{2}k_1, 0+\frac{1}{2}2) = 2f(6,1) = 0$
   $k_3 = hf(3+\frac{1}{2}k_2, 0+\frac{1}{2}2) = 2f(3,1) = 0$
   $k_4 = hf(3+k_3, 0+2) = 2f(3,2) = 2\cdot(-3) = -6$
   So, $y(2) \approx 3 + \frac{2}{6}[6+0+0-6] = 3.$

   (c) Adams-Bashforth: $w_2 = w_1 + \frac{h}{2}[3f_1 - f_0] = 9 + \frac{2}{2}[3f(9,2) - f(3,0)]$
   $= 9 + 1[3(-9) - 3] = -21.$
   Adams-Moulton: $w_2 = w_1 + \frac{h}{12}[5f_2 + 8f_1 - f_0]$
   $= 9 + \frac{2}{12}[5f(-21,4) + 8f(9,2) - f(3,0)] = 9 + \frac{1}{6}[5(63) + 8(-9) - 3] = 49.$

3. Not assigned

4. Not assigned

5. Not assigned

6. Not assigned

7. Not assigned

8. (a) For Taylor series method, we note
   $y(0) = 1$
   $y'(0) = 0 + 1 + 0\cdot 1 = 1$
   $y''(x) = 1 + y'(x) + xy'(x) + y$
   $y''(0) = 1 + 1 + 0 + 1 = 3$
   $y'''(x) = y''(x) + xy''(x) + y'(x) + y'(x)$
   $y'''(0) = 3 + 0 + 1 + 1 = 5$
   So, $y(0.1) \approx 1 + 1(0.1) + \frac{1}{2}3(0.1)^2 + \frac{1}{6}5(0.1)^3 = 1.1158\overline{3}$ and
   $y(0.5) \approx 1 + 1(0.5) + \frac{1}{2}3(0.5)^2 + \frac{1}{6}5(0.5)^3 = 1.9791\overline{6}.$

(b) (c) and (d)

t	Euler	Mod_Euler	Runge
0.000000	1.000000	1.000000	1.000000
0.100000	1.100000	1.115500	1.115894
0.200000	1.231000	1.266745	1.267700
0.300000	1.398720	1.461269	1.463033
0.400000	1.610554	1.708937	1.711878
0.500000	1.876031	2.022677	2.027333

9. (a) For the Taylor series method, we note
$y(0) = 1$
$y'(0) = 0 + 1 = 1$
$y''(x) = 1 + y'(x) = 1 + x + y$
$y''(0) = 1 + 0 + 1 = 2$
$y'''(x) = y''(x)$
$y'''(0) = 2$
$y^{(n)}(x) = y^{(n-1)}(x)$ for $n = 3, 4, \ldots$
$y^{(n)}(0) = 2$ for $n = 3, 4, \ldots$
So, $y(x) \simeq 1 + x + \frac{2}{2!}x^2 + \frac{2}{3!}x^3$
$y(0.1) \approx 1 + (0.1) + (0.1)^2 + \frac{1}{3}(0.1)^3 = 1.1101\overline{6}$ and
$y(0.5) \approx 1 + (0.5) + (0.5)^2 + \frac{1}{3}(0.5)^3 = 1.791\overline{3}$.

(b) (c) and (d)

t	Euler	Mod_Euler	Runge	Exact
0.000000	1.000000	1.000000	1.000000	1.000000
0.100000	1.100000	1.110000	1.110342	1.110342
0.200000	1.220000	1.242050	1.242805	1.242806
0.300000	1.362000	1.398465	1.399717	1.399718
0.400000	1.528200	1.581804	1.583648	1.583649
0.500000	1.721020	1.794894	1.797441	1.797443

10. (a) Part a was not assigned.

(b) (c) and (d)

t	Euler	Mod_Euler	Runge	Exact
0.000000	1.000000	1.000000	1.000000	1.000000
0.100000	1.040000	1.040592	1.040601	1.040601

0.200000	1.081184	1.082386	1.082403	1.082403
0.300000	1.123554	1.125382	1.125408	1.125408
0.400000	1.167108	1.169576	1.169612	1.169612
0.500000	1.211842	1.214962	1.215007	1.215007

11. For step size $h = 0.1$:

t	Euler	Mod_Euler	Exact
0.000000	1.000000	1.000000	1.000000
0.100000	1.000000	1.005000	1.004988
0.200000	1.010000	1.019828	1.019804
0.300000	1.029802	1.044064	1.044031
0.400000	1.058934	1.077074	1.077033
0.500000	1.096708	1.118080	1.118034
0.600000	1.142299	1.166240	1.166190
0.700000	1.194824	1.220707	1.220656
0.800000	1.253410	1.280676	1.280625
0.900000	1.317236	1.345413	1.345362
1.000000	1.385561	1.414263	1.414214

For step size $h = 0.2$:

0.000000	1.000000	1.000000	1.000000
0.200000	1.000000	1.020000	1.019804
0.400000	1.040000	1.077372	1.077033
0.600000	1.116923	1.166599	1.166190
0.800000	1.224361	1.281050	1.280625
1.000000	1.355041	1.414625	1.414214

Using Romberg's method on the Euler results gives:
(Use 2 times fine - coarse)

t	Improved_Value	Exact
0.000000	1.000000	1.000000
0.200000	1.020000	1.019804
0.400000	1.077868	1.077033
0.600000	1.167675	1.166190
0.800000	1.282459	1.280625
1.000000	1.416081	1.414214

Using Romberg's method on the Modified-Euler results gives:
(Use (4 times fine - coarse)/3)

t	Improved_Value	Exact
0.000000	1.000000	1.000000
0.200000	1.019771	1.019804
0.400000	1.076975	1.077033
0.600000	1.166120	1.166190
0.800000	1.280551	1.280625
1.000000	1.414142	1.414214

12. Not assigned.

13. Not assigned.

14.

h	t	Euler	Mod_Euler	Runge	Err_Eul	ErrModEul	Err_RK4
0.50000	0.000000	0.400000	0.400000	0.400000			
0.50000	0.500000	0.400000	0.420000	0.421056	0.02105	0.0010526	0.0000036820
0.50000	1.000000	0.440000	0.495897	0.500013	0.06000	0.0041027	0.0000129937
0.50000	1.500000	0.536800	0.700993	0.727049	0.19047	0.0262793	0.0002237939
0.50000	2.000000	0.752916	1.457220	1.930134	1.24708	0.5427797	0.0698655608
0.25000	0.000000	0.400000	0.400000	0.400000			
0.25000	0.250000	0.400000	0.405000	0.405063	0.00506	0.0000632	0.0000000634
0.25000	0.500000	0.410000	0.420903	0.421053	0.01105	0.0001497	0.0000002698
0.25000	0.750000	0.431013	0.450378	0.450705	0.01969	0.0003265	0.0000007127
0.25000	1.000000	0.465845	0.499212	0.500002	0.03415	0.0007880	0.0000015308
0.25000	1.250000	0.520098	0.579629	0.581820	0.06172	0.0021891	0.0000021195
0.25000	1.500000	0.604629	0.720006	0.727263	0.12264	0.0072663	0.0000093011
0.25000	1.750000	0.741720	1.000115	1.032012	0.29053	0.0321432	0.0002455818
0.25000	2.000000	0.982411	1.735671	1.989202	1.01758	0.2643285	0.0107982347
0.12500	0.000000	0.400000	0.400000	0.400000			
0.12500	0.125000	0.400000	0.401250	0.401254	0.00125	0.0000039	0.0000000010
0.12500	0.250000	0.402500	0.405055	0.405063	0.00256	0.0000081	0.0000000041
0.12500	0.375000	0.407563	0.411562	0.411576	0.00401	0.0000135	0.0000000097
0.12500	0.500000	0.415349	0.421031	0.421053	0.00570	0.0000212	0.0000000187
0.12500	0.625000	0.426131	0.433865	0.433898	0.00776	0.0000335	0.0000000325

0.12500	0.750000	0.440318	0.450650	0.450704	0.01038	0.0000544	0.0000000533
0.12500	0.875000	0.458494	0.472234	0.472325	0.01383	0.0000907	0.0000000842
0.12500	1.000000	0.481486	0.499844	0.500000	0.01851	0.0001557	0.0000001288
0.12500	1.125000	0.510465	0.535291	0.535565	0.02509	0.0002743	0.0000001892
0.12500	1.250000	0.547108	0.581320	0.581818	0.03470	0.0004979	0.0000002552
0.12500	1.375000	0.593878	0.642278	0.643216	0.04933	0.0009384	0.0000002607
0.12500	1.500000	0.654497	0.725412	0.727273	0.07277	0.0018604	0.0000001080
0.12500	1.625000	0.734816	0.843717	0.847680	0.11286	0.0039647	0.0000023505
0.12500	1.750000	0.844494	1.022849	1.032243	0.18776	0.0094089	0.0000147656
0.12500	1.875000	1.000500	1.320886	1.347270	0.34686	0.0264821	0.0000984757
0.12500	2.000000	1.235109	1.899378	1.998981	0.76489	0.1006217	0.0010188416
0.06250	0.000000	0.400000	0.400000	0.400000			
0.06250	0.125000	0.400625	0.401253	0.401254	0.00062	0.0000004	0.0000000001
0.06250	0.250000	0.403772	0.405062	0.405063	0.00129	0.0000010	0.0000000003
0.06250	0.375000	0.409543	0.411574	0.411576	0.00203	0.0000018	0.0000000006
0.06250	0.500000	0.418149	0.421049	0.421053	0.00290	0.0000031	0.0000000012
0.06250	0.625000	0.429922	0.433893	0.433898	0.00397	0.0000055	0.0000000022
0.06250	0.750000	0.445353	0.450694	0.450704	0.00535	0.0000098	0.0000000036
0.06250	0.875000	0.465147	0.472307	0.472325	0.00717	0.0000179	0.0000000059
0.06250	1.000000	0.490310	0.499967	0.500000	0.00969	0.0000328	0.0000000093
0.06250	1.125000	0.522293	0.535504	0.535565	0.01327	0.0000611	0.0000000142
0.06250	1.250000	0.563234	0.581702	0.581818	0.01858	0.0001159	0.0000000209
0.06250	1.375000	0.616381	0.642990	0.643216	0.02683	0.0002265	0.0000000267
0.06250	1.500000	0.686877	0.726809	0.727273	0.04039	0.0004637	0.0000000165
0.06250	1.625000	0.783307	0.846662	0.847682	0.06437	0.0010203	0.0000000918
0.06250	1.750000	0.921042	1.029745	1.032257	0.11121	0.0025135	0.0000007917
0.06250	1.875000	1.130262	1.339906	1.347362	0.21710	0.0074621	0.0000060332
0.06250	2.000000	1.478358	1.968724	1.999928	0.52164	0.0312762	0.0000719798
0.03125	0.000000	0.400000	0.400000	0.400000			
0.03125	0.125000	0.400939	0.401254	0.401254	0.00031	0.0000000	0.0000000000
0.03125	0.250000	0.404415	0.405063	0.405063	0.00064	0.0000001	0.0000000000
0.03125	0.375000	0.410553	0.411575	0.411576	0.00102	0.0000002	0.0000000000
0.03125	0.500000	0.419587	0.421052	0.421053	0.00146	0.0000005	0.0000000001
0.03125	0.625000	0.431885	0.433897	0.433898	0.00201	0.0000010	0.0000000001
0.03125	0.750000	0.447986	0.450702	0.450704	0.00271	0.0000019	0.0000000002
0.03125	0.875000	0.468664	0.472321	0.472325	0.00366	0.0000038	0.0000000004

0.03125	1.000000	0.495035	0.499993	0.500000	0.00496	0.0000074	0.0000000006
0.03125	1.125000	0.528726	0.535551	0.535565	0.00683	0.0000142	0.0000000010
0.03125	1.250000	0.572173	0.581790	0.581818	0.00964	0.0000277	0.0000000015
0.03125	1.375000	0.629160	0.643161	0.643216	0.01405	0.0000553	0.0000000020
0.03125	1.500000	0.705850	0.727158	0.727273	0.02142	0.0001151	0.0000000019
0.03125	1.625000	0.812948	0.847425	0.847682	0.03473	0.0002573	0.0000000034
0.03125	1.750000	0.970703	1.031613	1.032258	0.06155	0.0006452	0.0000000426
0.03125	1.875000	1.222252	1.345405	1.347368	0.12511	0.0019634	0.0000003549
0.03125	2.000000	1.677219	1.991372	1.999995	0.32278	0.0086277	0.0000045365

0.01562	0.000000	0.400000	0.400000	0.400000			
0.01562	0.125000	0.401096	0.401254	0.401254	0.00015	0.0000000	0.0000000000
0.01562	0.250000	0.404738	0.405063	0.405063	0.00032	0.0000000	0.0000000000
0.01562	0.375000	0.411062	0.411576	0.411576	0.00051	0.0000000	0.0000000000
0.01562	0.500000	0.420316	0.421053	0.421053	0.00073	0.0000000	0.0000000000
0.01562	0.625000	0.432885	0.433898	0.433898	0.00101	0.0000002	0.0000000000
0.01562	0.750000	0.449334	0.450704	0.450704	0.00137	0.0000004	0.0000000000
0.01562	0.875000	0.470476	0.472324	0.472325	0.00184	0.0000008	0.0000000000
0.01562	1.000000	0.497486	0.499998	0.500000	0.00251	0.0000017	0.0000000000
0.01562	1.125000	0.532091	0.535561	0.535565	0.00347	0.0000034	0.0000000001
0.01562	1.250000	0.576900	0.581811	0.581818	0.00491	0.0000067	0.0000000001
0.01562	1.375000	0.636013	0.643202	0.643216	0.00720	0.0000136	0.0000000001
0.01562	1.500000	0.716219	0.727244	0.727273	0.01105	0.0000286	0.0000000002
0.01562	1.625000	0.829581	0.847618	0.847682	0.01810	0.0000645	0.0000000001
0.01562	1.750000	0.999688	1.032095	1.032258	0.03256	0.0001630	0.0000000024
0.01562	1.875000	1.279446	1.346867	1.347368	0.06792	0.0005018	0.0000000211
0.01562	2.000000	1.815791	1.997749	2.000000	0.18420	0.0022512	0.0000002784

0.00781	0.000000	0.400000	0.400000	0.400000			
0.00781	0.125000	0.401175	0.401254	0.401254	0.00007	0.0000000	0.0000000000
0.00781	0.250000	0.404901	0.405063	0.405063	0.00016	0.0000000	0.0000000000
0.00781	0.375000	0.411318	0.411576	0.411576	0.00025	0.0000000	0.0000000000
0.00781	0.500000	0.420684	0.421053	0.421053	0.00036	0.0000000	0.0000000000
0.00781	0.625000	0.433390	0.433898	0.433898	0.00050	0.0000000	0.0000000000
0.00781	0.750000	0.450016	0.450704	0.450704	0.00068	0.0000000	0.0000000000
0.00781	0.875000	0.471395	0.472325	0.472325	0.00092	0.0000002	0.0000000000
0.00781	1.000000	0.498735	0.500000	0.500000	0.00126	0.0000004	0.0000000000
0.00781	1.125000	0.533814	0.535564	0.535565	0.00175	0.0000008	0.0000000000

0.00781	1.250000	0.579335	0.581817	0.581818	0.00248	0.0000016	0.0000000000
0.00781	1.375000	0.639569	0.643213	0.643216	0.00364	0.0000033	0.0000000000
0.00781	1.500000	0.721655	0.727266	0.727273	0.00561	0.0000071	0.0000000000
0.00781	1.625000	0.838434	0.847666	0.847682	0.00924	0.0000161	0.0000000000
0.00781	1.750000	1.015476	1.032217	1.032258	0.01678	0.0000409	0.0000000001
0.00781	1.875000	1.311852	1.347242	1.347368	0.03551	0.0001267	0.0000000013
0.00781	2.000000	1.900671	1.999426	2.000000	0.09932	0.0005736	0.0000000171
0.00390	0.000000	0.400000	0.400000	0.400000			
0.00390	0.125000	0.401214	0.401254	0.401254	0.00003	0.0000000	0.0000000000
0.00390	0.250000	0.404982	0.405063	0.405063	0.00008	0.0000000	0.0000000000
0.00390	0.375000	0.411447	0.411576	0.411576	0.00012	0.0000000	0.0000000000
0.00390	0.500000	0.420868	0.421053	0.421053	0.00018	0.0000000	0.0000000000
0.00390	0.625000	0.433644	0.433898	0.433898	0.00025	0.0000000	0.0000000000
0.00390	0.750000	0.450359	0.450704	0.450704	0.00034	0.0000000	0.0000000000
0.00390	0.875000	0.471859	0.472325	0.472325	0.00046	0.0000000	0.0000000000
0.00390	1.000000	0.499365	0.500000	0.500000	0.00063	0.0000001	0.0000000000
0.00390	1.125000	0.534686	0.535565	0.535565	0.00087	0.0000002	0.0000000000
0.00390	1.250000	0.580570	0.581818	0.581818	0.00124	0.0000004	0.0000000000
0.00390	1.375000	0.641381	0.643215	0.643216	0.00183	0.0000008	0.0000000000
0.00390	1.500000	0.724440	0.727271	0.727273	0.00283	0.0000017	0.0000000000
0.00390	1.625000	0.843006	0.847678	0.847682	0.00467	0.0000040	0.0000000000
0.00390	1.750000	1.023736	1.032248	1.032258	0.00852	0.0000102	0.0000000000
0.00390	1.875000	1.329189	1.347337	1.347368	0.01817	0.0000318	0.0000000001
0.00390	2.000000	1.948271	1.999855	2.000000	0.05172	0.0001446	0.0000000011
0.00195	0.000000	0.400000	0.400000	0.400000			
0.00195	0.125000	0.401234	0.401254	0.401254	0.00001	0.0000000	0.0000000000
0.00195	0.250000	0.405023	0.405063	0.405063	0.00004	0.0000000	0.0000000000
0.00195	0.375000	0.411511	0.411576	0.411576	0.00006	0.0000000	0.0000000000
0.00195	0.500000	0.420960	0.421053	0.421053	0.00009	0.0000000	0.0000000000
0.00195	0.625000	0.433771	0.433898	0.433898	0.00012	0.0000000	0.0000000000
0.00195	0.750000	0.450532	0.450704	0.450704	0.00017	0.0000000	0.0000000000
0.00195	0.875000	0.472091	0.472325	0.472325	0.00023	0.0000000	0.0000000000
0.00195	1.000000	0.499682	0.500000	0.500000	0.00031	0.0000000	0.0000000000
0.00195	1.125000	0.535124	0.535565	0.535565	0.00044	0.0000000	0.0000000000
0.00195	1.250000	0.581193	0.581818	0.581818	0.00062	0.0000001	0.0000000000
0.00195	1.375000	0.642295	0.643216	0.643216	0.00092	0.0000002	0.0000000000

0.00195	1.500000	0.725851	0.727272	0.727273	0.00142	0.0000004	0.0000000000
0.00195	1.625000	0.845331	0.847681	0.847682	0.00235	0.0000010	0.0000000000
0.00195	1.750000	1.027963	1.032255	1.032258	0.00429	0.0000025	0.0000000000
0.00195	1.875000	1.338169	1.347360	1.347368	0.00919	0.0000079	0.0000000000
0.00195	2.000000	1.973581	1.999964	2.000000	0.02641	0.0000363	0.0000000001

0.00097	0.000000	0.400000	0.400000	0.400000			
0.00097	0.125000	0.401244	0.401254	0.401254	0.00000	0.0000000	0.0000000000
0.00097	0.250000	0.405043	0.405063	0.405063	0.00002	0.0000000	0.0000000000
0.00097	0.375000	0.411543	0.411576	0.411576	0.00003	0.0000000	0.0000000000
0.00097	0.500000	0.421006	0.421053	0.421053	0.00004	0.0000000	0.0000000000
0.00097	0.625000	0.433835	0.433898	0.433898	0.00006	0.0000000	0.0000000000
0.00097	0.750000	0.450618	0.450704	0.450704	0.00008	0.0000000	0.0000000000
0.00097	0.875000	0.472208	0.472325	0.472325	0.00011	0.0000000	0.0000000000
0.00097	1.000000	0.499841	0.500000	0.500000	0.00015	0.0000000	0.0000000000
0.00097	1.125000	0.535344	0.535565	0.535565	0.00022	0.0000000	0.0000000000
0.00097	1.250000	0.581505	0.581818	0.581818	0.00031	0.0000000	0.0000000000
0.00097	1.375000	0.642755	0.643216	0.643216	0.00046	0.0000000	0.0000000000
0.00097	1.500000	0.726560	0.727273	0.727273	0.00071	0.0000001	0.0000000000
0.00097	1.625000	0.846503	0.847682	0.847682	0.00117	0.0000002	0.0000000000
0.00097	1.750000	1.030102	1.032257	1.032258	0.00215	0.0000006	0.0000000000
0.00097	1.875000	1.342741	1.347366	1.347368	0.00462	0.0000019	0.0000000000
0.00097	2.000000	1.986647	1.999991	2.000000	0.01335	0.0000091	0.0000000000

(a) We print (above) the Euler, Improved Euler and Runge Kutta 4th order solutions for the various step sizes. (We don't print all values as that is very long)

(b) We present figures for Euler, Modified Euler and Fourth Order Runge Kutta.

We print a log-log plot of error at time of 2 units. Notice the slope for Euler's method is about 1 for $O(h)$ convergence, the slope for Modified Euler is about 2 for $O(h^2)$ convergence and the slope is about 4 for Runge Kutta for $O(h^4)$ convergence.

(c)

h	t	RK4
0.0625	0.000000	0.400000
0.062500	0.062500	0.400313
0.062500	0.125000	0.401254

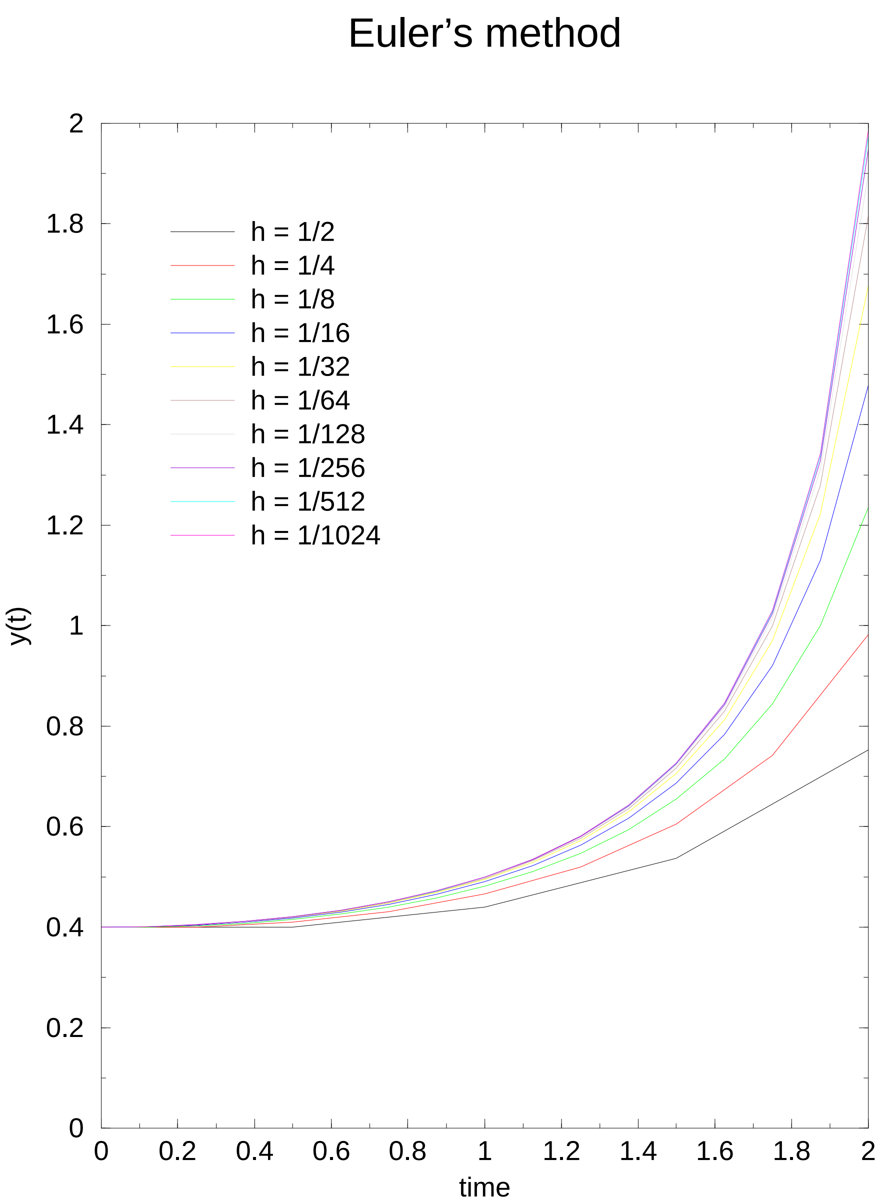

Figure 1: Problem 14 - Euler

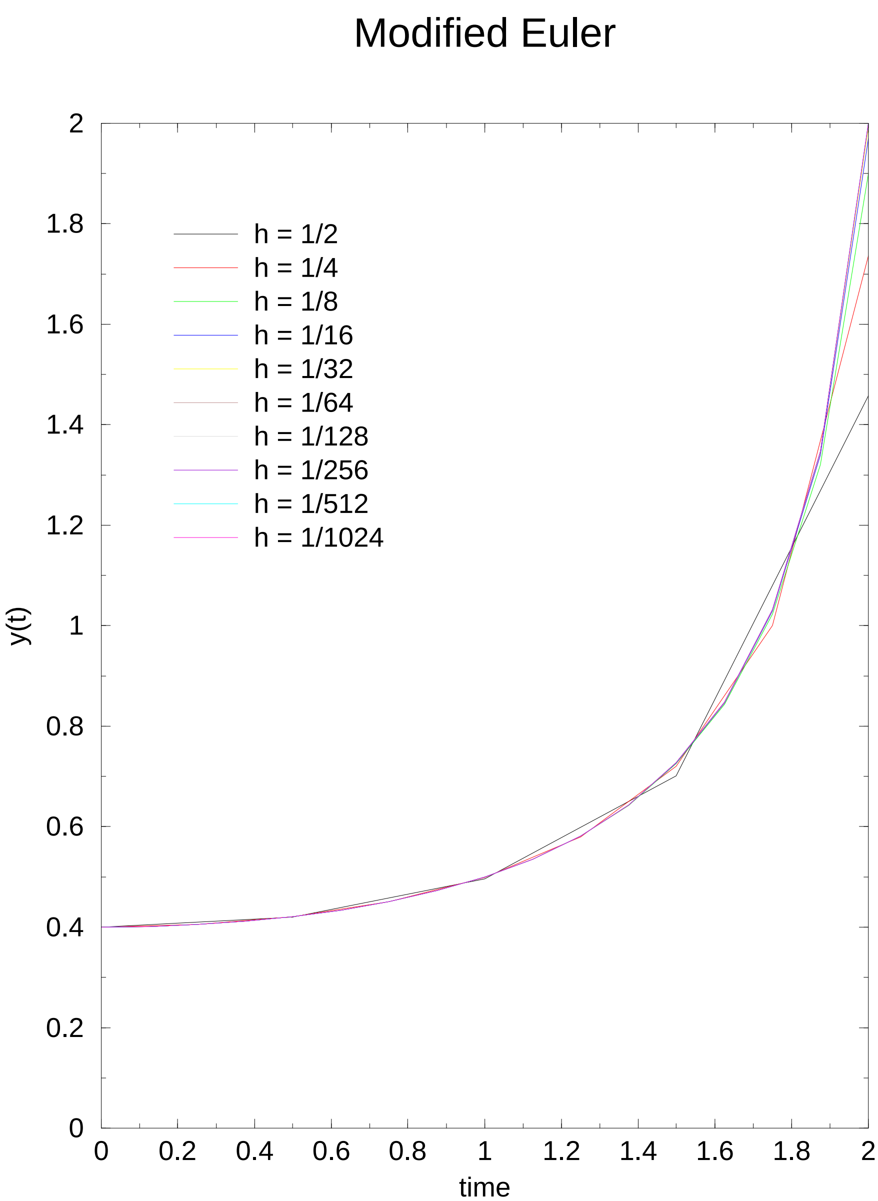

Figure 2: Problem 14 - Modified Euler

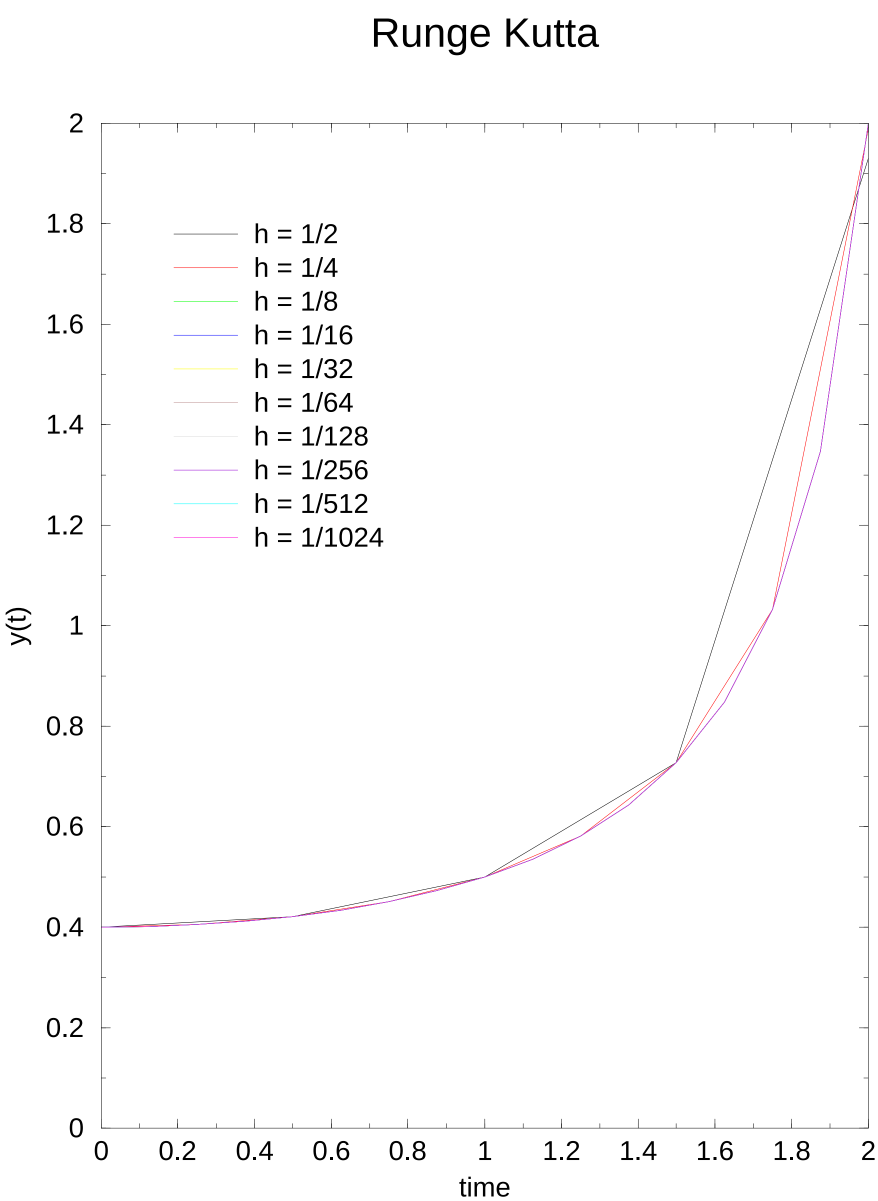

Figure 3: Problem 14 - Runge Kutta Fourth Order

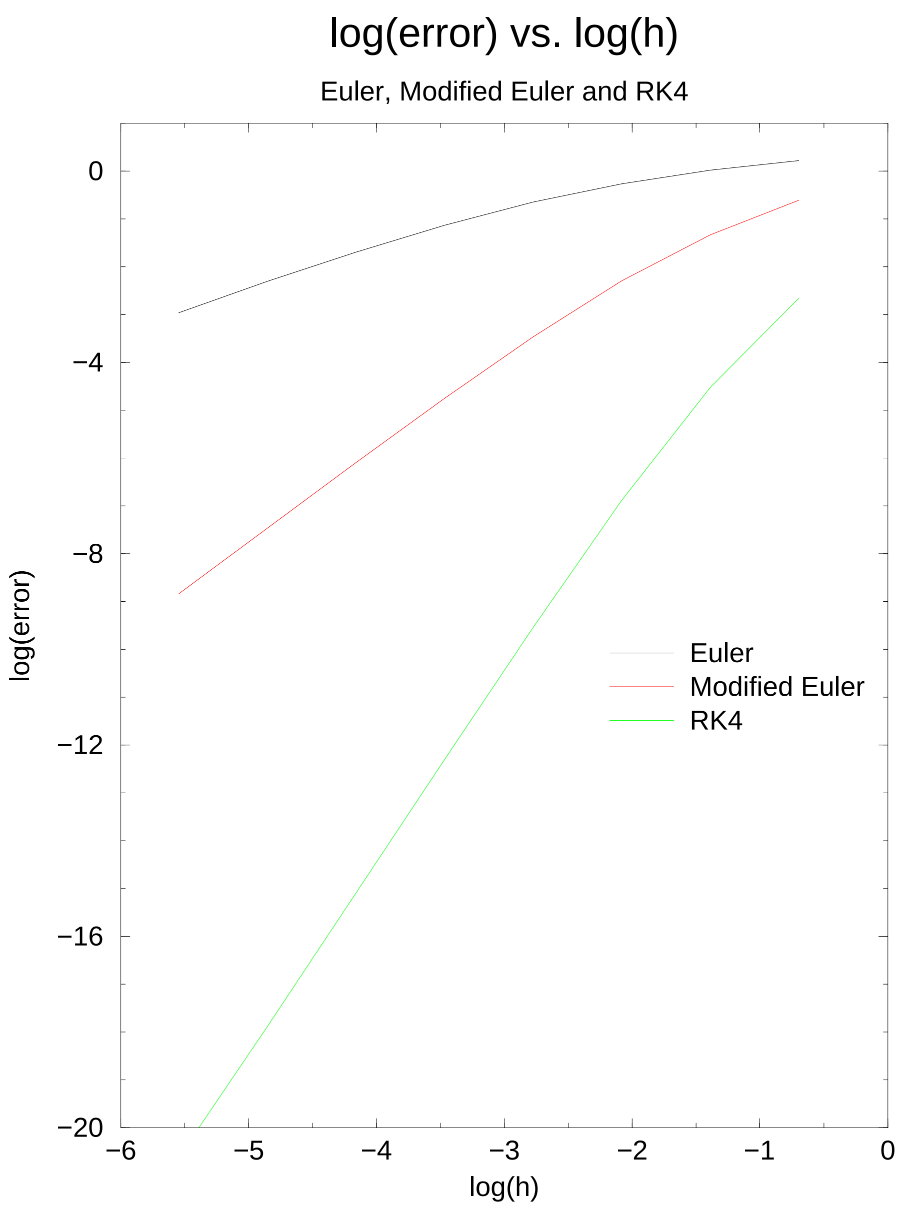

Figure 4: Problem 14 - Error Plot

```
0.062500 0.187500 0.402832
0.062500 0.250000 0.405063
0.062500 0.312500 0.407968
0.062500 0.375000 0.411576
0.062500 0.437500 0.415922
0.062500 0.500000 0.421053
0.062500 0.562500 0.427023
0.062500 0.625000 0.433898
0.062500 0.687500 0.441760
0.062500 0.750000 0.450704
0.062500 0.812500 0.460846
0.062500 0.875000 0.472325
0.062500 0.937500 0.485308
0.062500 1.000000 0.500000
0.062500 1.062500 0.516650
0.062500 1.125000 0.535565
0.062500 1.187500 0.557127
0.062500 1.250000 0.581818
0.062500 1.312500 0.610250
0.062500 1.375000 0.643216
0.062500 1.437500 0.681758
0.062500 1.500000 0.727273
0.062500 1.562500 0.781679
0.062500 1.625000 0.847682
0.062500 1.687500 0.929219
0.062500 1.750000 1.032257
0.062500 1.812500 1.166285
0.062500 1.875000 1.347362
0.062500 1.937500 1.604997
0.062500 2.000000 1.999928
0.062500 2.062500 2.680256
0.062500 2.125000 4.125590
0.062500 2.187500 9.181191
0.062500 2.250000 254.616682
0.062500 2.312500 3643911537432353300000.000000
0.062500 2.375000 inf
```

Since the exact solution blows up at $t = \sqrt{5}$ we see the Runge

Kutta 4th order solution blowing up too.

15. Not assigned.

16. $y'' - y' + y = t^2$ becomes, letting $u_1 = y'$
$\begin{pmatrix} y \\ u_1 \end{pmatrix}' = \begin{pmatrix} u_1 \\ t^2 + u_1 - y \end{pmatrix}$ with $y(2) = 2$ and $u_1(2) = 3$. Euler's method gives: $y(3) \simeq y(2) + 1 \cdot 3 = 5$
$u_1(3) \simeq u_1(2) + 1 \cdot (2^2 + 3 - 2) = 8$
$y(4) \simeq y(3) + 1 \cdot 8 = 13$
$u_1(4) \simeq u_1(3) + 1 \cdot (3^2 + 8 - 5) = 20$

17. Let $\begin{aligned} u_1 &= y \\ u_2 &= y' = u_1' \\ u_3 &= y'' = u_2'' \end{aligned} \quad \Rightarrow \quad \begin{aligned} u_1' &= u_2 \\ u_2' &= u_3 \\ u_3' &= 3t - 3u_3u_1 + 6u_2^2 - 2u_1 \end{aligned}$

18. For consistency, we must consider whether w_{i+1} differs from the true value y_{i+1} (assuming all previous values are exact) with error $O(h^2)$ or higher order. I.e. local truncation error $\frac{y_{i+1} - w_{i+1}}{h} \to 0$ as $h \to 0$.
$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!}y_i'' + \frac{h^3}{3!}y_i''' + ...$
$w_{i+1} = \frac{3}{2}y_i - \frac{1}{2}y_{i-1} + \frac{1}{2}hy_i'$
$= \frac{3}{2}y_i - \frac{1}{2}(y_i - hy_i' + \frac{h^2}{2!} - \frac{h^3}{3!}y_i''' + ...) + \frac{1}{2}hy_i'$.
Comparing y_{i+1} and w_{i+1}, we have

$$\frac{y_{i+1} - w_{i+1}}{h} = \frac{\frac{3}{4}h^2 + ...}{h} = O(h)$$

So the method is consistent.

For stability, we assume $f(t, y) = 0$ and obtain the difference equation: $w_{n+1} = \frac{3}{2}w_n - \frac{1}{2}w_{n-1}$.
We let $w_n = \rho^n$, yielding $\rho^{n+1} - \frac{3}{2}\rho^n - \frac{1}{2}\rho^{n-1} = 0$.
Thus, $\rho^2 - \frac{3}{2}\rho - \frac{1}{2} = 0$. So $\rho = 1$ or $\rho = \frac{1}{2}$ and the method is strongly stable.

Since the scheme is both consistent and stable, it is convergent.

19. For consistency, we must find a and b such that w_{i+1} differs from the true value y_{i+1} (assuming all previous values are exact) with error $O(h^2)$ or higher order. I.e. local truncation error $\frac{y_{i+1} - w_{i+1}}{h} \to 0$ as $h \to 0$.
$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!}y_i'' + \frac{h^3}{3!}y_i''' + ...$

$w_{i+1} \;=\; a[y_i \;-\; hy_i' \;+\; \frac{h^2}{2!}y_i'' \;-\; \frac{h^3}{3!}y_i''' \;+\; \ldots]$
$+\; bh[y_i' \;-\; hy_i'' \;+\; \frac{h^2}{2!}y_i''' \;+\; \ldots \;+\; 4y_i' \;+\; y_i' \;+\; hy_i'' \;+\; \frac{h^2}{2!}y_i''' \;+\; \ldots]$
where we have kept terms through y_i'''.
Matching up the y_i terms gives $a \;=\; 1$.
Matching up the y_i' terms gives $1 \;=\; -a + 6b$ where we've factored out h from both sides. This yields $b \;=\; \frac{1}{3}$. The scheme will be consistent.

For stability, we have $w_{n+1} \;=\; w_{n-1}$ where we let $f \equiv 0$. Thus, $\lambda^2 - 1 \;=\; 0$ and $\lambda \;=\; \pm 1$. So the scheme is weakly stable.

Using the values of $a = 1$ and $b = \frac{1}{3}$ we find that evaluating $\frac{y_{i+1} - w_{i+1}}{h}$ the y_i'', y_i''' and y_i'''' terms cancel out. We must take the $y_i^{(5)}$ terms into account to get $\frac{y_i^{(5)}(\frac{1}{5!}h^5 + \frac{1}{5!}h^5)}{h} \simeq O(h^4)$.

20. Not assigned.

21. (a) $w_{i+1} \;=\; \frac{18}{11}w_i \;-\; \frac{9}{11}w_{i-1} \;+\; \frac{2}{11}w_{i-2} \;+\; ahf(t_{i+1}, w_{i+1})$
To study the stability, we assume $f(t, y) \;=\; 0$ and obtain:
$\rho^{n+1} \;=\; \frac{18}{1}\rho^n - \frac{9}{11}\rho^{n-1} \;+\; \frac{2}{11}\rho^{n-2}$ or
$\rho^{n+1} \;-\; \frac{18}{1}\rho^n + \frac{9}{11}\rho^{n-1} \;-\; \frac{2}{11}\rho^{n-2} \;=\; 0$ or
$\rho^3 \;-\; \frac{18}{1}\rho^2 + \frac{9}{11}\rho \;-\; \frac{2}{11} \;=\; 0$

(b) For consistency, expand both sides to get:
$w_{i+1} \;=\; w_i \;+\; hw_i' \;+\; \frac{h^2}{2!}w_i'' \;+\; \frac{h^3}{3!}w_i''' \;+\; \ldots$
$=\; \frac{18}{11}w_i \;-\; \frac{9}{11}(w_i - hw_i' \;+\; \frac{h^2}{2!}w_i'' \;-\; \frac{h^3}{3!}w_i''' \;+\; \ldots \;+\; \frac{2}{11}(w_i - 2hw_i' \;+\; \frac{(2h)^2}{2!}w_i'' \;-\; \frac{(2h)^3}{3!}w_i''' \;+\; \ldots) \;+\; ahw_i'$.
Matching up the w_i terms gives: $1 \;=\; 1$.
Matching up the hw_i' terms gives: $1 \;=\; \frac{9}{11} - \frac{4}{11} + a \;\;\Rightarrow\;\; a \;=\; \frac{6}{11}$.

(c) The cubic equation in part (a) becomes $(\rho - 1)(\rho^2 - \frac{7}{11} + \frac{2}{11}) \;=\; 0$, which has roots 1 and $\frac{7}{22} \pm \frac{i\sqrt{39}}{22}$ which have magnitude < 1. Thus, the method is strongly stable.

22. Not assigned.

23. Solve the equaiton for $y(t_{n+1})$ which we'll abbreviate by y_{n+1} yielding $y_{n+1} \;=\; 4y_n - 3y_{n-1} - 2hy_{n-1}'$ or $w_{n+1} \;=\; 4w_n - 3w_{n-1} - 2hf(t_{n-1}, w_{n-1})$.
To study the residual, we consider $y_{n+1} - w_{n+1}$ where y_{n+1} is the exact value. Write out Taylor series:

$y_{n+1} - w_{n+1} = y_n + hy'_n + \frac{h^2}{2!}y''_n + \frac{h^3}{3!}y'''_n + ... - [4y_i - 3(y_n - hy'_n + \frac{h^2}{2!}y''_n - \frac{h^3}{3!}y'''_n + ...) - 2h(y'_n - hy''_n + \frac{h^2}{2!}y'''_n - \ldots)]$ $= 0y_n + 0hy'_n + 0h^2y''_n + \frac{h^3}{3!}y'''_n(1+1+6) + \ldots$. So the residual is $O(h^3)$ and the local truncation error is $\frac{y_{n+1}-w_{n+1}}{h} = O(h^2)$ with leading term $\frac{4}{3}h^2y'''_n$. The method is consistent since the local truncation error has order at least 1.

24. $w_{n+1} = 4w_n - 3w_{n-1} - 2hf(t_{n-1}, w_{n-1})$. To study stability, we let $f \equiv 0$. Thus, we get $\lambda^2 - 4\lambda + 3 = 0 \Rightarrow \lambda = 1, 3$. Since a root has magnitude greater than 1, the method is unstable.

25. For this problem we iterated until consecutive results differred by less than 10^{-8}.

    (a) Using functional iteration:

```
Beginning t = 0 y = 1 end time = 0.200000 step size = 0.010000
t Solution   error   iterations
0.000000 1.000000
0.010000 1.027559
0.020000 1.055899
0.030000 1.085066 0.000000 4
0.040000 1.115109 0.000000 4
0.050000 1.146083 0.000000 4
0.060000 1.178047 0.000000 4
0.070000 1.211067 0.000000 4
0.080000 1.245214 0.000000 4
0.090000 1.280568 0.000000 4
0.100000 1.317218 0.000000 4
0.110000 1.355263 0.000000 4
0.120000 1.394813 0.000000 4
0.130000 1.435992 0.000000 4
0.140000 1.478939 0.000000 4
0.150000 1.523814 0.000001 4
0.160000 1.570798 0.000001 4
0.170000 1.620098 0.000001 4
0.180000 1.671956 0.000001 4
0.190000 1.726651 0.000002 5
0.200000 1.784511 0.000002 5
```

(b) To use Newton's method, we write the equation as:
$x - \frac{3h}{8}e^x - w_i - \frac{h}{24}(19e^{w_i} - 5e^{w_{i-1}} + e^{w_{i-2}}) = 0$
where $x = e^{w_{i+1}}$. The first guess for w_{i+1} is just w_i. This yields:

```
Beginning t = 0 y = 1 end time = 0.200000 step size = 0.010000
t       Solution    Error   iteration
0.000000 1.000000
0.010000 1.027559
0.020000 1.055899
0.030000 1.085066 0.000000 2
0.040000 1.115109 0.000000 2
0.050000 1.146083 0.000000 2
0.060000 1.178047 0.000000 2
0.070000 1.211067 0.000000 2
0.080000 1.245214 0.000000 2
0.090000 1.280568 0.000000 2
0.100000 1.317218 0.000000 2
0.110000 1.355263 0.000000 2
0.120000 1.394813 0.000000 2
0.130000 1.435992 0.000000 2
0.140000 1.478939 0.000000 2
0.150000 1.523814 0.000001 2
0.160000 1.570798 0.000001 2
0.170000 1.620098 0.000001 2
0.180000 1.671956 0.000001 2
0.190000 1.726651 0.000002 2
0.200000 1.784511 0.000002 2
```

26. (a) For the ODE $y' = -y$, the method becomes:
$w_{i+1} = w_i + \frac{h}{4}\left(-w_i + 3(-w_i - \frac{2h}{3}(-w_i))\right)$ This gives, $w_{i+1} = (1 - h + \frac{h^2}{2})w_i$. Thus, the inequality of interest is: $|(1 - h + \frac{h^2}{2})| \leq 1$. h must satisfy this inequality for the method to be stable.

(b) For consistency, we write out the local truncation error: $\frac{y_{n+1} - w_{n+1}}{h}$ and see if the result has order h^p with $p \geq 1$. Here we get: $y_{n+1} - w_{n+1} = y_n + hy'_n + \frac{h^2}{2!}y''_n + \frac{h^3}{3!}y'''_n + ...$
$- [\frac{1}{2}y_n + \frac{1}{2}(y_n - hy'_n + \frac{h^2}{2!}y''_n - \frac{h^3}{3!}y'''_n + ...)]$
$\frac{h}{4}[y'_n + 5(y'_n - hy''_n + \frac{h^2}{2!}y'''_n - \ldots)]$.
The y_n and y'_n terms cancel out and the leading term is $\frac{3}{2}h^2y''_n$

to give local truncation error of $\frac{3}{2}hy_n'' + \ldots$. So the method is consistent.

For stability, we let $f \equiv 0$ to obtain $\lambda^{i+1} = \frac{1}{2}\lambda^i + \frac{1}{2}\lambda^{i-1}$ or $\lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = 0$, which has roots 1 and $\frac{-1}{2}$. Since these rotts have magnitude smaller than or equal to 1, the scheme is stable.

27. Not assigned.

28. For $y' = \lambda y$, wehave Runge-Kutta becoming:
$k_1 = hf(y)$
$k_2 = hf(y + \frac{1}{2}k_1)$
$k_3 = hf(y + \frac{1}{2}k_2)$
$k_4 = hf(y + k_3)$
Thus, for this particular ODE, we have $k_1 = h\lambda y$
$k_2 = h\lambda(y + \frac{1}{2}k_1)$
$k_3 = h\lambda(y + \frac{1}{2}k_2)$
$k_4 = h\lambda(y + k_3)$
Plugging in gives, $k_1 = h\lambda y$
$k_2 = h\lambda y + \frac{1}{2}h^2\lambda^2 y$
$k_3 = h\lambda y + \frac{1}{2}h^2\lambda^2 y + \frac{1}{4}h^3\lambda^3 y$
$k_4 = h\lambda y + h^2\lambda^2 y + \frac{1}{2}h^3\lambda^3 y + \frac{1}{4}h^4\lambda^4 y$
Using Runge-Kutta $w_{i+1} = w_i + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$ becomes
$w_{i+1} = w_i[1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3 + \frac{1}{24}h^4\lambda^4]$

29. Not assigned.

30. Not assigned.

31. Not assigned.

32. Not assigned.

33. The second order Taylor series method is:
$w_{i+1} = w_i + hw_i' + \frac{h^2}{2}w_i''$.
For the ODE, $y' = -2y$, we have, $w_i' = -2w_i$ and $w_i'' = -2w_i' = 4w_i$.
So, the scheme becomes: $w_{i+1} = w_i[1 - 2h + 2h^2]$.
For stability we need $|1 - 2h + 2h^2| < 1$, so $-1 < 1 - 2h + 2h^2 < 1$ and solving this yields that the left inequality is always true, while the right inequality is true if $0 < h < 1$. Thus, the method is stable if $h < 1$.

34. Not assigned.

35. Not assigned.

36. Not assigned.

37. Not assigned.

38. Not assigned.

39. Not assigned.

40. (a) $y'(x_i) \simeq \frac{y(x_{i+1}) - y(x_{i-1})}{2h}$
$y''(x_i) \simeq \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2}$

(b) The ODE can be discretized as: $\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} + x_i \left(\frac{y(x_{i+1}) - y(x_{i-1})}{2h} \right) - x_i^2 y_i = 2$
leading to the system of linear equations:
At $x = 0.25$, $\frac{y(0.5) - 2y(0.25) + y(0)}{0.25^2} + 0.25 \left(\frac{y(0.5) - y(0)}{2(0.25)} \right) - 0.25^2 y(0.25) = 2(0.25)$
At $x = 0.5$, $\frac{y(0.75) - 2y(0.5) + y(0.25)}{0.25^2} + 0.5 \left(\frac{y(0.75) - y(0.25)}{2(0.25)} \right) - 0.5^2 y(0.5) = 2(0.5)$
At $x = 0.75$, $\frac{y(1) - 2y(0.75) + y(0.5)}{0.25^2} + 0.75 \left(\frac{y(1) - y(0.5)}{2(0.25)} \right) - 0.75^2 y(0.75) = 2(0.75)$
Plugging in the boundary conditions and combining like terms gives:

$$\begin{pmatrix} -32\frac{1}{16} & 16\frac{1}{2} & 0 \\ 15 & -32\frac{1}{4} & 17 \\ 0 & 14\frac{1}{2} & -32\frac{9}{16} \end{pmatrix} \begin{pmatrix} y(0.25) \\ y(0.5) \\ y(0.75) \end{pmatrix} = \begin{pmatrix} -15 \\ 1 \\ 19 \end{pmatrix}$$

41. Not assigned.