Mathematical Magic Tricks and Why They Work

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5th Roots of a 10 Digit Number

When a single digit number is raised to the fifth power, the result has the original digit as its final digit.

Digit	Digit to the 5th power
0	0
1	1
2	32
3	243
4	1024
5	3125
6	7776
7	16807
8	32768
9	59049

- Ask the contestant to pick a one or two digit number and raise it to the fifth power (using a calculator).
- Have the contestant tell you the large number they got as the answer.
- You can then tell them the number they started with.

You need only know the above table approximately in order to know the cutoffs for figuring out the first digit.

Number	Number to the 5th power	Cutoff
10	100,000	100,000
20	3,200,000	3,000,000
30	24,300,000	20,000,000
40	102,400,000	100,000,000
50	312,500,000	300,000,000
60	777,600,000	800,000,000
70	1,680,700,000	1,600,000,000
80	3,276,800,000	3,000,000,000
90	5,904,900,000	6,000,000,000

Example: Find the fifth root of 916,132,832.
The last digit is 2 and using the chart above, the first digit is 6.
So the answer is **62**.

(Most calculators do not keep or print out enough digits to handle 3 digit numbers).

1089

The trick

- Pick a three digit number with first digit not equal to last.
- Write the number backwards.
- Subtract the smaller number from the larger. (The result is a 3-digit number – the first digit may just be 0).
- Call the result Apples.
- Write this new number backwards. Call it Oranges.
- Add the final two numbers (Apples and Oranges).
- The result is 1089.
- Ask the contestant to go to a word in a book based on the final answer. You know their word.

Example:
Pick 123
Write 321
Subtract 321-123=198=Apples
Write 891=Oranges
Add 198 + 891=1089

Why it Works

$$
\begin{array}{rrr}
a & b & c \\
- \quad c & b & a \\
\hline
a-1-c & 9 & 10+c-a
\end{array}
$$

Then,

$$
\begin{array}{rrr}
a-1-c & 9 & 10+c-a \\
+ \quad 10+c-a & 9 & a-1-c \\
\hline
9 & 18 & 9 \\
\hline
10 & 8 & 9 \\
\hline
\end{array}
$$

Cube Roots of a 9 Digit Number

When the single digit numbers 0-9 are raised to the third power, the resulting numbers end with 0-9 but in a different order. There is a 1-1 correspondence.

Digit	Digit to the 3rd power
0	0
1	1
2	8
3	27
4	64
5	125
6	216
7	343
8	512
9	729

- Ask the contestant to pick a three digit number and raise it to the third power (using a calculator).
- Have the contestant tell the new, large number.
- You can then tell them the number they started with.

You need only know the above table in order to know the cutoffs for figuring out the first digit.

Number	Number to the 3rd power
100	$1,000,000$
200	$8,000,000$
300	$27,000,000$
400	$64,000,000$
500	$125,000,000$
600	$216,000,000$
700	$343,000,000$
800	$512,000,000$
900	$729,000,000$

Now, the trick is to compute the middle digit. The key here is division by 11. The remainder when a large number is divided by 11 is found by starting at the right and alternately subtracting and adding the digits as one proceeds to the left. This works since 99 is divisible by 11 so that n (a whole number) times 100 has the same remainder when divided by 11 as n does.

Consider the number $10a + b$.
If $a < b$, then the remainder from division by 11 is:

$$10a + b - [10(a-1) + (a-1)] = 11 + b - a = b - a \pmod{11}$$

If a > b, then the remainder from division by 11 is:

$$10a + b - [10a + a] = b - a \pmod{11}$$

If we take a number with a certain remainder when divided by 11 and cube it, there is a one-to-one correspondence between the original remainder when dividing by 11 and the remainder of the cube.

Note: $(11x+r)^3 = 11^3x^3 + 3{*}11^2x^2r + 3{*}11xr^2 + r^3$.

Original Number	Remainder of 3rd power
$11x + 0$	0
$11x + 1$	1
$11x + 2$	8
$11x + 3$	$27 \Rightarrow 5$
$11x + 4$	$64 \Rightarrow 9$
$11x + 5$	$125 \Rightarrow 4$
$11x + 6$	$216 \Rightarrow 7$
$11x + 7$	$343 \Rightarrow 2$
$11x + 8$	$512 \Rightarrow 6$
$11x + 9$	$729 \Rightarrow 3$
$11x + 10$	$1000 \Rightarrow 10$

Example: Find the cube root of 395,446,904
The last digit must be 4, since $4^3 = 64$.
The first digit must be 7, since $7^3 = 343$ but $8^3 = 512$.
The remainder of 395,446,904 when divided by 11 is

$$4 - 0 + 9 - 6 + 4 - 4 + 5 - 9 + 3 = 6$$

so the original number has remainder 8 when divided by 11.
The remainder when dividing the original number by 11 is last-middle+first, or

(first + last) - middle, so we have

$$7 + 4 - middle = 8 \Rightarrow middle = 3$$

And the original number is 734.

What's the Card on Top of the Last Stack?

- Turn over a card and count cards from that number up to 10. For example, if a four is turned over, count 4, 5, 6, ..., 10. (Face cards count as 10). Then turn over the stack. Next, turn over another card and repeat. Do this until there are no more cards left in the deck. Keep the cards in the last stack if not finished in your hand.
- Have the contestant choose 3 stacks to leave on the table, adding the other cards to those in your hand.
- Let the contestant choose 2 stacks from which to turn over the top card.
- Add these together and add 19. Remove this many cards from your hand. The number of cards remaining in your hand is the top card of the remaining stack.

Why it Works

The number of cards in each full stack is 11 minus the value of the first card. E.g., if the card is 10, there is one card. If the top card is a 5, there are six cards in the stack. Thus, in the 3 stacks chosen (if the first cards are a, b and c) the number of cards is:

$$(11 - a) + (11 - b) + (11 - c) = 33 - (a + b + c)$$

So, in your hand, there are $52 - [33 - (a + b + c)] = 19 + (a + b + c)$ cards.

Removing $19 + (a + b)$ cards leaves c cards in your hand.

A variation of this trick is to count up to 13, letting Jacks equal 11, Queens equal 12 and Kings equal 13. Now how many cards do you remove from your hand?

The 3 stacks have $14 - a$, $14 - b$ and $14 - c$ cards respectively, so in your hand there are $52 - [42 - (a + b + c)] = 10 + (a + b + c)$ cards. Thus, count off the sum of the first two cards and add 10.

Pick a 4 Digit Number and I'll Find Your Card

- Have the volunteer pick a 4 digit number at least 1010, call it A.
- Add the digits, call this B.
- Subtract. (Take $A - B$).
- You are left with a 4 digit number.
- Take cards of different suits to represent the four digits you have. Take an ace for any ones, nines for any zeroes and the obvious ones for the others.
- Have the volunteer give you 3 of the cards.
- Announce the card they still have in their hand and have them show it to the audience

Why it Works

To find the remainder when dividing a whole number by 9, just add the digits and this sum has the same remainder. Thus, subtracting the sum of the digits from the number itself will always leave a number which is divisible by 9. The four cards chosen by the volunteer must sum to a multiple of 9 so when given 3 of the cards, you know the fourth. (Of course, the suit is just the 4th suit).

Another way to see this is to consider the 4 digit number as $abcd$ or
$1000\,a + 100\,b + 10\,c + d \ = \ 999a + 99b + 9c + (a + b + c + d)$.
Now subtract the sum of the digits $a + b + c + d$ to get
$999a + 99b + 9c$, which is divisible by 9. So the 4 digit number you end up with is divisible by 9. (This also shows why a number and the sum of its digits have the same remainder when divided by 9).

Kruskal's Trick

- Deal a deck of cards face up in rows of 6.
- Have the volunteer select a card in the first row.
- Have the volunteer count the face value of the card number of cards to get to another card (ace counts as one) and continue this way through the deck (letting Jacks count as 2, Queens as 4 and Kings as 6).
- Have to person keep in mind the last card they stop at before the deck ends.
- Tell them that card.

How you Do it and Why it Works

- Start at any card and do as the volunteer.
- It is very likely that at some card your paths will intersect.
- From there until the end, your paths are the same.

Why Your Paths Should (Usually) Cross

- You have a 1 in 6 chance of picking the same first card.
- You and the player will pick approximately 48 / 5.4 (average value of the cards 1,2,2,3,4,4,5,6,6,7,8,9,10) or about 9 cards.
- There is approximately a 1 in 5.4 chance of landing on the same spot – or a 4.4/5.4 of not landing on the same next spot.
- Thus, the probability of not picking the same first card and not landing on any of the other cards of the volunteer is roughly:
$\frac{5}{6}\left(\frac{4.4}{5.4}\right)^9 \simeq 0.132$

Fibonacci Sums

- Have 2 volunteers pick a single digit number.
- List the numbers lower, then higher. (a and b).
- Next make a Fibonacci sequence of 10 numbers from them:
  $a,\ b,\ a+b, b\ +\ (a+b), ...$ Have them use a calculator to add the 10 numbers.
- You tell them the sum faster than they can add it.

Why it Works

The numbers obtained are: a
b
$a + b$
$a + 2b$
$2a + 3b$
$3a + 5b$
$5a + 8b$
$8a + 13b$
$13a + 21b$
$21a + 34b$
which sums to $55a + 88b$ or 11 time the 7th number. So just multiply the 7th number by 11 in your head.

To multiply a multi-digit number by 11, start at the right and let the last digit be the last digit of your answer. Then move to the left adding pairs of numbers to get the digits to the left one at a time. (Pretend there is a zero to the left of the first digit.) At most, you need to "carry" one.

E.g., 16837642 * 11 gives: 2, then 6, then 0, then 4 , then 1, then 2, then 5, then 8, then 1. So the answer is 185214062.

Base 2

1	3	5	7	9	11	13	15	17	19	21	23
25	27	29	31	33	35	37	39	41	43	45	47
49	51	53	55	57	59	61	63				
2	3	6	7	10	11	14	15	18	19	22	23
26	27	30	31	34	35	38	39	42	43	46	47
50	51	54	55	58	59	62	63				
4	5	6	7	12	13	14	15	20	21	22	23
28	29	30	31	36	37	38	39	44	45	46	47
52	53	54	55	60	61	62	63				
8	9	10	11	12	13	14	15	24	25	26	27
28	29	30	31	40	41	42	43	44	45	46	47
56	57	58	59	60	61	62	63				
16	17	18	19	20	21	22	23	24	25	26	27
28	29	30	31	48	49	50	51	52	53	54	55
56	57	58	59	60	61	62	63				
32	33	34	35	36	37	38	39	40	41	42	43
44	45	46	47	48	49	50	51	52	53	54	55
56	57	58	59	60	61	62	63				

Base 3

1	4	7	10	13	16	19	22	25	28	31	34	37	40
43	46	49	52	55	58	61	64	67	70	73	76	79	
2	5	8	11	14	17	20	23	26	29	32	35	38	41
44	47	50	53	56	59	62	65	68	71	74	77	80	
3	4	5	12	13	14	21	22	23	30	31	32	39	40
41	48	49	50	57	58	59	66	67	68	75	76	77	
6	7	8	15	16	17	24	25	26	33	34	35	42	43
44	51	52	53	60	61	62	69	70	71	78	79	80	
9	10	11	12	13	14	15	16	17	36	37	38	39	40
41	42	43	44	63	64	65	66	67	68	69	70	71	
18	19	20	21	22	23	24	25	26	45	46	47	48	49
50	51	52	53	72	73	74	75	76	77	78	79	80	
27	28	29	30	31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50	51	52	53	
54	55	56	57	58	59	60	61	62	63	64	65	66	67
68	69	70	71	72	73	74	75	76	77	78	79	80	