Inspiring Students to Want to Learn Math

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Why Students Lose Interest	Why Students Stay Interested

Computer Storage and the Internet

Computer chips have many circuits on them. Each circuit can be open or closed (on/off; 1/0). How much information can be expressed by a given number of circuits, N?

Of course, 2^N.

The trick works because each set of numbers represents 1 of 6 circuits being ON. There are 6 lists, so $2^6 = 64$ pieces of information can be encoded. Naturally, some ways of making up these lists are more useful than others.

First list - numbers with the "1" circuit on.
Second - numbers with the "2" circuit on.
Third - numbers with the "4" circuit on.
Fourth - numbers with the "8" circuit on.
Fifth - numbers with the "16" circuit on.
Sixth - numbers with the "32" circuit on.

It's the same as writing a number in base 2.

For example, consider the number
$37 = 1(32)+0(16)+0(8)+1(4)+1(2)+0(1)$.
Thus, 37 is the sum of the first numbers on the 32, 4 and 1 lists.

Understanding this topic is useful for understanding the rudiments of computer chip design.

Extensions:
How would this work in base 3 (-1, 0 or 1 are the values of the circuits) or other bases? There are more lists but fewer numbers per list (to go up to a specific maximum value) in the trick as you go to higher bases.

Triangulation

Finding the epicenter of an earthquake or your position using a GPS system involves, first, finding the distance from known locations (seismograph stations or satellites). This gives circles in 2-D, spheres in 3-D. Then one needs to find the intersections of the circles or spheres.

2 circles (typically) intersect at 2 points, so 3 are needed.
2 spheres intersect in a circle, 4 spheres are needed.

Earthquakes cause propagation of compression (Pressure, P) waves which travel about 7 km/sec and stress waves (S) which travel at about 3 km/sec.

Suppose an earthquake occurs and at Station A, the P wave arrives 15 seconds before the S wave. At Station B, the P wave arrives 20 seconds before the S wave. Station B is 100 km East of Station A. Find the possible locations of the epicenter of the earthquake. (A third station's data is needed to decide which of the two locations is correct).

$a^2 + h^2 = r_0^2$

$b^2 + h^2 = r_1^2$

$d = a + b$

Subtracting$\Rightarrow a^2 - b^2 = r_0^2 - r_1^2$

So $(a+b)(a-b) = r_0^2 - r_1^2$

$a - b = \frac{1}{d} \left(r_0^2 - r_1^2\right)$ and $a + b = d$.

Now add to get

$$2a = \frac{r_0^2 - r_1^2}{d} + d \Rightarrow a = \frac{r_0^2 - r_1^2 + d^2}{2d}$$

$$b = d - a = \frac{r_1^2 - r_0^2 + d^2}{2d}$$

and

$$h = \sqrt{r_0^2 - a^2} = \sqrt{r_1^2 - b^2}$$

For our example,

$$r_0 \; = \; 4 \text{ km/s } 15 \text{ s} \quad = \; 60\text{km},$$

$$r_1 \; = \; 4 \text{ km/s } 20 \text{ s} \quad = \; 80\text{km and}$$

$$d \; = \; 100\text{km}.$$

Then,

$$a = \frac{3600 - 6400 + 10000}{200} = 36 \text{ km}.$$

$$b = \frac{6400 - 3600 + 10000}{200} = 64 \text{ km}.$$

$$h = \sqrt{3600 - 1296} = \sqrt{2304} = 48 \text{ km}.$$

Mortgages

Suppose you want to borrow
D dollars at
interest rate r per year ($r/12$ per month) for
y years and
your monthly payment is M dollars per month.
Find a relation among the variables D, r, M and y.

After one month, you owe

$$D\left(1 + \frac{r}{12}\right)$$

dollars and you pay off M dollars so you owe

$$D\left(1 + \frac{r}{12}\right) - M$$

Let $G = \left(1 + \frac{r}{12}\right)$ represent the growth term. After another month, you owe

$$(DG - M)G - M$$

Thus, you owe:

$$\begin{aligned} O_1 &= DG - M \quad \text{after 1 month} \\ O_2 &= (DG - M)G - M = DG^2 - MG - M \\ O_3 &= [(DG - M)G - M]G - M \\ &= DG^3 - MG^2 - MG - M \\ O_4 &= DG^4 - MG^3 - MG^2 - MG - M \end{aligned}$$

Each month, we multiply the previous result by G and subtract M

$$O_n = GO_{n-1} - M$$

We can notice (or prove) a pattern giving after $12y$ months:

$$DG^{12y} - M(G^{12y-1} + G^{12y-2} + ... + G^2 + G + 1).$$

Since $1 + G + G^2 + ... + G^{12y-1} = \frac{1-G^{12y}}{1-G}$

After y years, the mortgage should be paid off so the amount remaining is zero. This requires

$$D\left(1 + \frac{r}{12}\right)^{12y} - \frac{M\left(1-(1 + \frac{r}{12})^{12y}\right)}{1-(1 + \frac{r}{12})} = 0$$

Then,

$$D\left(1 + \frac{r}{12}\right)^{12y} + \frac{12M}{r}\left(1-(1 + \frac{r}{12})^{12y}\right) = 0$$

$$D + \frac{12M}{r}\left((1 + \frac{r}{12})^{12y} - 1\right) = 0 \Rightarrow$$

$$D = \frac{12M}{r}\left[1-\left(1 + \frac{r}{12}\right)^{12y}\right]$$

Similarly, we can solve for M and to obtain:

$$M = \frac{rD}{12\left(1-\left(1 + \frac{r}{12}\right)^{12y}\right)}$$

This tells how the monthly payment depends on interest rate, years and amount borrowed. Solving for y gives:

$$y = \frac{\ln(1 - \frac{rD}{12M})}{12\ln(1 + r/12)}$$

To compute r in terms of the other variables requires numerical methods for nonlinear equations.

Example: What is the most expensive home you can afford if the interest rate is 6%, the term is 30 years and the monthly payment you can afford is \$1,000 per month?

$$D = \frac{12M}{r}\left[1 - \left(1 + \frac{r}{12}\right)^{12y}\right]$$

$$= \frac{12 \cdot 1000}{0.06}\left[1 - \left(1 + \frac{0.06}{12}\right)^{12 \cdot 30}\right]$$

$$= 200,000\left[1 - (1.005)^{-360}\right]$$

$$\approx .833958(200,000) = \$166,791$$

Notice that if the loan is perpetual, you can afford a house worth \$200,000.

Computing the Probability of Winning a Series

<u>Equal probabilities</u>

In 1494, Luca Pacioli wrote a book titled "Summe de arithmetic" in which he discussed a game of balla up to 6 points. How should the stakes be divided if the game stops at 5-3?

The problem was only solved in the mid-1600s by Pascal and Fermat.

If we assume balla was a fair game of chance. Then Player 1 would win under the following scenarios:
where 1 means Player 1 wins the point
and 2 means Player 2 wins the point.

scenario	*probability*
1	1/2
21	1/4
221	1/8
Total	0.875

If balla is to be considered a game of skill, then the best we can surmise is that Player 1 has a 5/8 chance of winning a given point. The table becomes:

scenario	*probability*
1	5/8
21	15/64
221	45/512
Total	485/512 $\approx$ .9473

If the game ends earlier, there are more scenarios. Say the score was 3-1, then Player 1 wins if he/she wins:
the next 3 points
2 of the next 3 points and the 4th point
2 of the next 4 points and the 5th point
2 of the next 5 points and the 6th point
2 of the next 6 points and the 7th point

The number of ways to do this is:

$$P^3 + \binom{3}{2} P^2QP + \binom{4}{2} P^2Q^2P$$

$$+ \binom{5}{2} P^3Q^2P + \binom{6}{2} P^4Q^2P$$

where $Q = 1 - P$.

Using Matrices (more operations but easier to program)

Better Foul Shooting

Problem: Find the "best" trajectory for shooting a foul shot in basketball.

- No air resistance
- Height vs. t depends only on gravity and initial vertical component of velocity.
- Horizontal velocity constant
- Basket is 10 ft. high. Foul line is 15 ft from basket. $g = 32$ ft/sec^2. h_0 is the height from which ball is released.

Equations

$$-\frac{1}{2}gt^2 + v_0 \sin(\theta)t + h_0 = 10$$

$$v_0 \cos(\theta)t = 15$$

$$\text{Minimize} \quad \frac{1}{2}mv^2$$

Solve the second equation for t:

$$t = \frac{15}{v_0 \cos(\theta)}$$

Plug this into the first equation to get:

$$\frac{-16(15)^2}{v_0^2 \cos^2(\theta)} + 15\frac{v_0 \sin(\theta)}{v_0 \cos(\theta)} = 10 - h_0$$

or

$$\frac{-16(15)^2}{v_0^2 \cos^2(\theta)} + 15 \tan(\theta) = 10 - h_0$$

Solving for v_0 yields:

$$\frac{-3600}{v_0^2 cos^2(\theta)} = 10 - h_0 - 15 \tan(\theta)$$

or

$$v_0^2 = \frac{-3600}{cos^2(\theta)[10 - h_0 - 15 \tan(\theta)]}$$

We take the positive square root, when it exists. When v_0 is a minimum, so is $\frac{1}{2}mv_0^2$.