PHYS320, Fall 2015

HOMEWORK SET 4

Due September 28, Monday

Please read attached supplementary material and understand 'Kepler's Laws', 'Newtonian Mechanics' and 'Orbital Mechanics'.

Physics 320 | Astrophysics I: Lecture #4 | Prof. Dale E. Gary NJIT

Kepler's Laws

Support for the Heliocentric Model

Although Copernicus set down the basic principles of the heliocentric model, it was regarded as simply an alternative way of thinking about the universe, without any certainty that the Earth really moved. Two later scientists, Galileo and Kepler, gave several strong arguments in favor of the heliocentric model.

Galileo gave observational evidence:

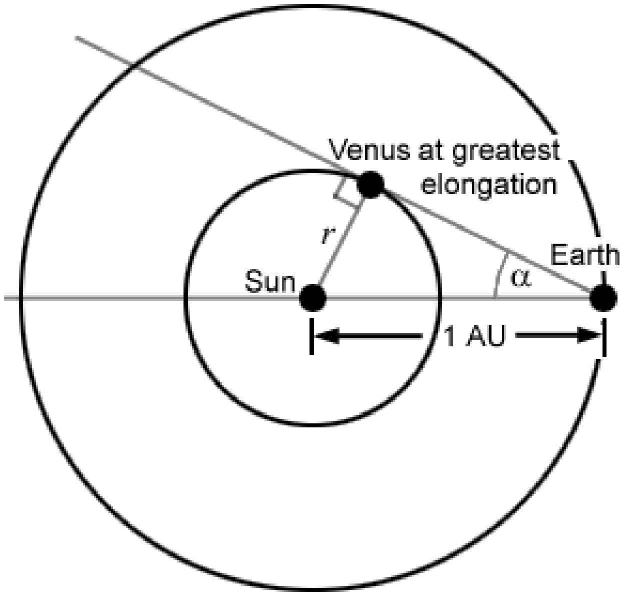
Moons of Jupiter:	gave clear evidence of smaller objects circling larger objects (although no one knew why)
Phases of Venus:	gave clear evidence that Venus circles the Sun
Sunspots on the Sun:	gave clear evidence that heaven is not "perfect"
Craters and Mountains on the Moon:	gave clear evidence that the Moon is another "world"

Kepler gave direct, mathematically rigorous evidence:

Kepler's Three Laws (*qualitative version*)

First Law:	Planets travel in elliptical orbits with the Sun at one focus
Second Law:	Planets move more slowly in their orbits when far from the Sun than when close to the Sun
Third Law:	Planets with larger orbits move more slowly than planets with smaller orbits.

Let's look at how Kepler determined his first Law. First, he showed how to measure the relative distance to the inner planets (Mercury and Venus) by measuring their angle of greatest elongation, α:

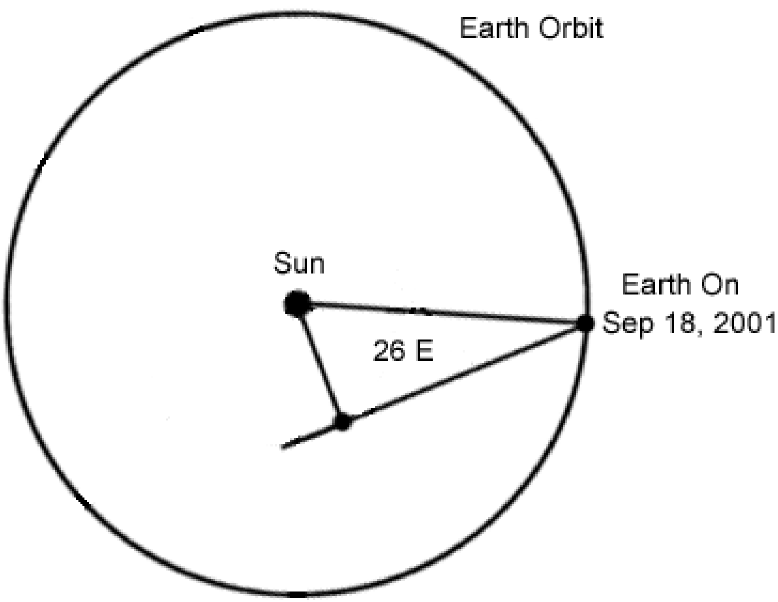


Determining distance of inner planet from Sun:

Greatest Elongation is α, and we want to determine the distance, *r*, of Venus from the Sun, in AU (*Astronomical Units*). The relationship is just $r = \sin \alpha$.

Greatest Elongations for Mercury in 2001-2002 (from JPL Calendar)

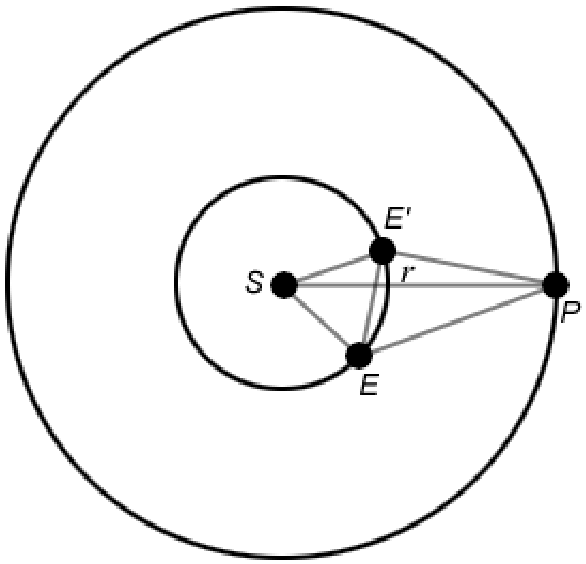
- Sep 18 - Mercury Greatest Eastern Elongation (26 Degrees)
- Oct 29 - Mercury At Its Greatest Western Elongation (18 Degrees)
- Jan 12 - Mercury At Its Greatest Eastern Elongation (19 Degrees)
- Feb 21 - Mercury at Greatest Western Elongation (27 Degrees)
- May 04 - Mercury Greatest Eastern Elongation (20 Degrees)
- Jun 21 - Mercury at Greatest Western Elongation (22 Degrees)



Animation showing the six consecutive greatest elongations of Mercury listed above. The position of Earth is advanced the appropriate number of degrees around its orbit for each date, then the line to Mercury is drawn East or West of the Sun as needed. Mercury is on this ray, at the point that the ray passes closest to the Sun.

By measuring Mercury's greatest elongation from many places along Earth's orbit, any variation in distance of Mercury from the Sun can be determined.

The situation for the outer planets is harder, but can be done. Kepler did it by observing the outer planet at pairs of times separated by one sidereal rotation of the planet. Here is an outline of how this is done, for one pair of observations of Mars:

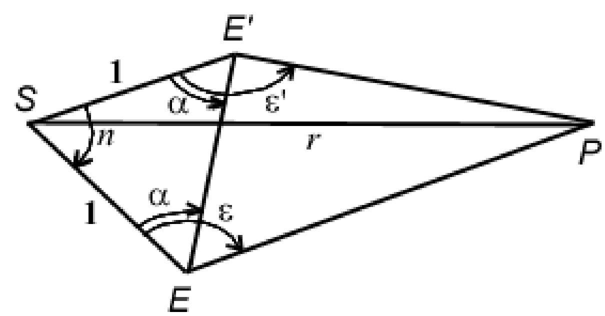


Determining distance of outer planet from Sun:

Take two measurements of the elongation (angle from the Sun) of Mars, one sidereal period (687 days) apart. Earth is at location *E'* at the time of the first observation, goes once around its orbit and arrives back at location *E* (almost completing two orbits) after Mars has gone around once.

The figure below shows the situation at a larger scale, with the angles labeled. The two elongation angles are ε and ε', and we also know the angle *n*, which is just the number of degrees less than two full orbits that the Earth makes in 687 days. You should be able to show that n = 42.89 degrees.

Since triangle ΔSEE' is isoceles, we can determine α, (you should be able to show that it is 68.56 degrees) and hence the length *EE'* (use the Law of Sines to show that it is 0.73 AU). Subtract α from ε and ε', which allows us to solve for triangle Δ*EPE'*. Finally, using the Law of Cosines for triangle Δ*SPE'*, we can determine the distance *r*.



Kepler repeated this procedure for many pairs of measurements of the planet Mars, taken by the excellent observer, Tycho Brahe, and was able to show that the distance *r* varies with time in the way expected if the path were an ellipse! You can see that this is not easy, and requires extremely good observations of elongation angles.

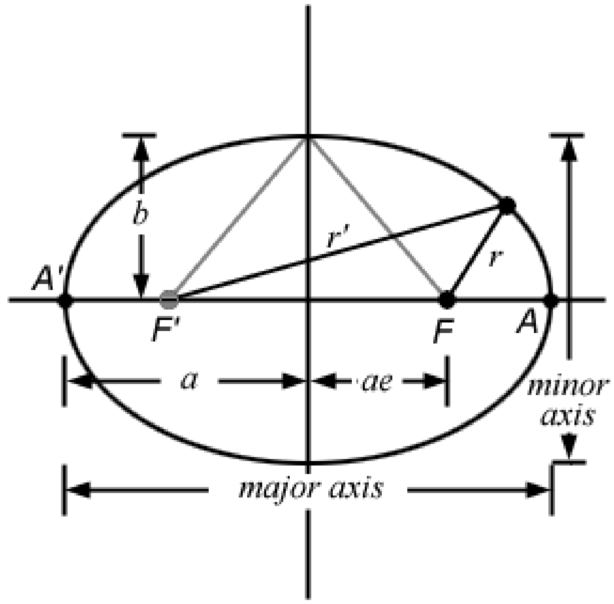
C. Kepler's Laws

1. Ellipses

In order to talk quantitatively about what Kepler discovered, we first need to remind ourselves of the properties of an ellipse.

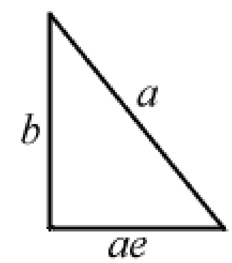
Properties of the Ellipse:

- Semi-major axis a = half of the long axis of ellipse

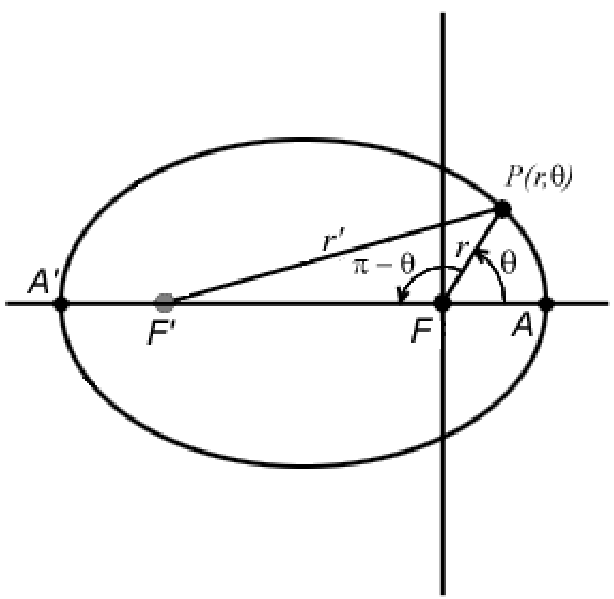


- Semi-minor axis b = half of the short axis of ellipse
- Eccentricity e = distance of focus *F* from center, in units of a. The eccentricity ranges from 0 (a circle) to 1 (a parabola).
- Sum of distances of a point on the ellipse from the two foci (r and r') is a constant: $r + r' = 2a$.

The triangle at right is 1/2 of the triangle formed by the light gray lines in the figure above. Convince yourself that the lengths of the sides are as shown. By the Pythagorean Theorem, show that the following relation holds: $b^2 = a^2(1 - e^2)$.



We would like to have an equation for the ellipse, and it is most convenient to express it in polar coordinates, relative to an origin at the focus *F*. A point on the ellipse *P*(r, θ) will then have polar coordinates *r* and θ, as shown in the diagram below. In the case of planetary orbits θ is called the *true-anomaly*.



Equation for an Ellipse, in Polar Coordinates:

The coordinates of a point on the ellipse are (r,θ). The triangle Δ*FPF'* has sides of length r, r', and $2ae$. Thus, by the Law of Cosines:

$$r'^2 = r^2 + (2ae)^2 - 2r(2ae)\cos(\pi - \theta)$$

but remember $r + r' = 2a$, or $r'^2 = (2a - r)^2$., so combining these and solving for *r*, we have the polar equation for an ellipse:

$$r = a(1 - e^2) / (1 + e \cos \theta)$$

2. Orbits as Ellipses

The above properties belong to all ellipses, but when the ellipse represents a planetary orbit, some of these variables have special significance. Here are some of them:

- As Kepler found, planets have an orbit that is an ellipse with the Sun at one focus. In the above drawings, the Sun would be at focus *F*. There would be *nothing* at all at focus *F* '.
- When the planet is at position *A* on the ellipse (closest to the Sun), it is at *perihelion*.
- When the planet is at position *A*' in its orbit (farthest from the Sun), it is at *aphelion*

3. Conic Sections

The equation for an ellipse is again:

$$r = a(1 - e^2) / (1 + e \cos \theta) \qquad \text{(equation for an ellipse)}$$

and if we set the eccentricity, e, to zero, the equation reduces to

$$r = a = constant \qquad \text{(equation for a circle -- } e = 0\text{)}$$

while if we set e to 1, the equation reduces to

$$\begin{aligned} r &= a(1 - e)(1 + e) / (1 + e \cos \theta) \\ &= 2p / (1 + \cos \theta) \qquad \text{(equation for a parabola -- } e = 1\text{)} \end{aligned}$$

Thus, the eccentricity parameter determines how "stretched-out" the ellipse is. Note that the eccentricity can even be greater than one, in which case we have the equation

$$r = a(e^2 - 1) / (1 + e \cos \theta) \qquad \text{(equation for a hyperbola -- } e > 1\text{)}$$

Note that the above equation cannot be derived from the equation of the ellipse, as we could the limiting cases for $e = 0$ and $e = 1$, but rather must be derived from the cartesian equation for a hyperbola (see Wikipedia, for example). The circle, ellipse, parabola, and hyperbola are all *conic sections*, and they all represent possible orbit shapes. We will see later how the total energy of the orbiting object will determine which of these shapes it follows.

4. Kepler's Laws

We repeat once more Kepler's Laws, but being a bit more quantitative:

Kepler's Three Laws (*quantitative version*)

First Law:	Planets travel in elliptical orbits with the Sun at one focus, and obey the equation $r = c / (1 + e \cos \theta)$, where $c = a(1 - e^2)$ for $0 < e < 1$. (Comets and other bodies can have hyperbolic orbits, where $c = a(e^2 - 1)$, for $e > 1$.)
Second Law:	The radius vector of a planet sweeps out equal areas in equal times (planet travels fastest when near perihelion).
Third Law:	The square of the orbital period of a planet is proportional to the cube of its semi-major axis: $P^2 = ka^3$, where k is a constant. Note: Kepler showed that this relationship also held for the newly discovered moons of Jupiter by Galileo, but with a different value for the constant k! One consequence of this law is that inner planets travel faster than outer planets, also.

D. What we have Learned

Careful measurements (by Tycho Brahe) were used by Kepler to measure the orbits of the planets and show that *The Earth Moves!* Kepler's Laws prove *quantitatively* that the true situation is given by a heliocentric model in which the planets revolve around the Sun. Kepler showed that the planets move in ellipses. We learned the important terms for ellipse characteristics: focus, semi-major axis, semi-minor axis, eccentricity, and for elliptical orbits, the terms perihelion and aphelion. We examined the characteristics and mathematical formulas for conic sections (ellipse, parabola, and hyperbola).

Kepler thought that there was a clockwork universe of crystal spheres, arranged harmonically (in certain ratios, related to regular geometric solids), but we will see next time that Newton was able to put it all on a firm physical foundation with his Law of Universal Gravitation.

Physics 320 | Astrophysics I: Lecture #5 | Prof. Dale E. Gary NJIT

Newtonian Mechanics

A: Dynamics

We are now going to study orbits in some detail, but first we need to review some basic mechanics such as you learned in your Freshman Physics class. In particular, we need to review:

- motion (coordinates, vectors, velocity, acceleration)
- linear momentum
- forces

We will especially be working in polar coordinates, which are the natural coordinate system for orbital motion. In the next lecture we will review additional topics in basic mechanics--angular momentum and energy.

1. Coordinates and Vectors

Recall that in three dimensions, a vector equation really represents three equations, one for each spatial dimension. A vector equation like

$$\boldsymbol{r} = \boldsymbol{r}_o + \boldsymbol{v}_o t$$

really represents the three equations

$$x = x_o + v_{ox}\, t$$
$$y = y_o + v_{oy}\, t$$
$$z = z_o + v_{oz}\, t$$

where the coordinates of the vectors are

$$\boldsymbol{r} = (x, y, z)$$
$$\boldsymbol{r}_o = (x_o, y_o, z_o)$$
$$\boldsymbol{v}_o = (v_{ox}, v_{oy}, v_{oz}).$$

Legal vector operations are addition and subtraction, e.g.,

$$\boldsymbol{v} + \boldsymbol{u} = \boldsymbol{u} + \boldsymbol{v}$$
$$(\boldsymbol{v} - \boldsymbol{u}) + \boldsymbol{w} = \boldsymbol{v} - (\boldsymbol{u} - \boldsymbol{w})$$

of multiplication by a scalar, e.g.,

$$a\boldsymbol{v} = \boldsymbol{v}a$$
$$a(\boldsymbol{v} + \boldsymbol{u}) = a\boldsymbol{v} + a\boldsymbol{u} \text{ , etc.}$$

Ordinary multiplication of two vectors generally has no meaning, but there are two special ways to "multiply" vectors that are defined: the *dot product* and the *cross product*.

a. Dot Product (or scalar product)

$$\boldsymbol{v} \cdot \boldsymbol{u} = v_x u_x + v_y u_y + v_z u_z = |\boldsymbol{v}||\boldsymbol{u}| \cos\theta \qquad \text{(a scalar)}$$

The meaning is " the component of $\boldsymbol{v}$ in the direction of $\boldsymbol{u}$ times the magnitude of $\boldsymbol{u}$," or equivalently, " the component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$ times the magnitude of $\boldsymbol{v}$."

Example: A box sliding down an incline

b. Cross Product (or vector product)

$$\boldsymbol{v} \times \boldsymbol{u} = \begin{vmatrix} \underline{\boldsymbol{x}} & \underline{\boldsymbol{y}} & \underline{\boldsymbol{z}} \\ v_x & v_y & v_z \\ u_x & u_y & u_z \end{vmatrix} = (v_y u_z - u_y v_z)\,\underline{\boldsymbol{x}} - (v_x u_z - u_x v_z)\,\underline{\boldsymbol{y}} + (v_x u_y - u_x v_y)\,\underline{\boldsymbol{z}} \text{ (a vector)}$$

where $\underline{\boldsymbol{x}}$, $\underline{\boldsymbol{y}}$, and $\underline{\boldsymbol{z}}$ are unit vectors.

The magnitude is

$$|\,\boldsymbol{v} \times \boldsymbol{u}\,| = vu \sin\theta \qquad \text{(area of parallelogram)}$$

and the direction is perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{u}$.

c. Unit Vectors

Note that polar coordinates, (r, θ), are related to rectangular (2-D) coordinates (x, y) by

$$r = (x^2 + y^2)^{1/2} \;;\;\; \theta = \tan^{-1}(y / x)$$

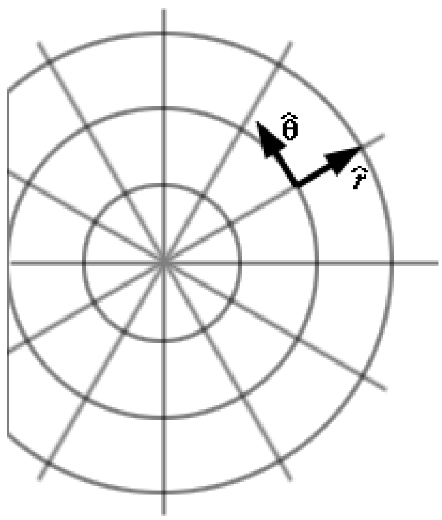
or conversely

$x = r \cos\theta$; $y = r \sin\theta$.

Unit vectors are vectors of length 1, e.g. the $\underline{\boldsymbol{x}}$, $\underline{\boldsymbol{y}}$, and $\underline{\boldsymbol{z}}$ above. Unit vectors in polar coordinates are:

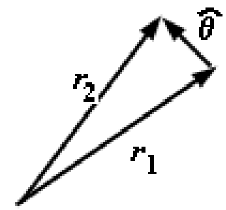
$$\underline{\boldsymbol{r}} = \quad \cos\theta\, \underline{\boldsymbol{x}} + \sin\theta\, \underline{\boldsymbol{y}} ;$$
$$\underline{\theta} = -\sin\theta\, \underline{\boldsymbol{x}} + \cos\theta\, \underline{\boldsymbol{y}}$$

and have directions in the r and θ directions, respectively. Graphically:



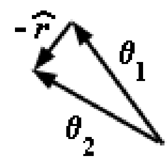
Note that

$d\underline{\boldsymbol{r}}/d\theta = \underline{\theta}$



and

$d\underline{\theta}/d\theta = -\underline{\boldsymbol{r}}$



d. Time Derivatives, Velocity and Acceleration

In rectangular coordinates, the 2-dimensional position, velocity, and acceleration are as follows:

$$\boldsymbol{r} = x\underline{\boldsymbol{x}} + y\underline{\boldsymbol{y}} \qquad \text{(position)}$$

$$\boldsymbol{v} = d\boldsymbol{r} / dt = \boldsymbol{r}' \quad \text{(prime notation)}$$
$$= dx/dt\, \underline{\boldsymbol{x}} + dy/dt\, \underline{\boldsymbol{y}} = v_x \underline{\boldsymbol{x}} + v_y \underline{\boldsymbol{y}} \qquad \text{(velocity)}$$
$$\boldsymbol{a} = d\boldsymbol{v} / dt = d^2\boldsymbol{r} / dt^2 = \boldsymbol{r}'' \quad \text{(prime notation)}$$
$$= d^2x/dt^2 \underline{\boldsymbol{x}} + d^2y/dt^2 \underline{\boldsymbol{y}} = a_x \underline{\boldsymbol{x}} + a_y \underline{\boldsymbol{y}} \quad \text{(acceleration)}$$

In terms of polar coordinates, things are a little more complicated:

$$\boldsymbol{r} = r\underline{\boldsymbol{r}}$$

$$\boldsymbol{v} = (d/dt)\, r\underline{\boldsymbol{r}} = r'\underline{\boldsymbol{r}} + r\, d\underline{\boldsymbol{r}}/dt = r\underline{\boldsymbol{r}} + r\, d\underline{\boldsymbol{r}}/d\theta\; d\theta/dt = r'\underline{\boldsymbol{r}} + r\theta'\underline{\theta}$$

$$\Longrightarrow v_r = r' ; \quad v_\theta = r\theta'$$

$$\boldsymbol{a} = d\boldsymbol{v} / dt = (d/dt)(r'\boldsymbol{r} + r\theta'\underline{\theta}) = r''\underline{\boldsymbol{r}} + r'\, d\underline{\boldsymbol{r}}/d\theta\, d\theta/dt + r'\theta'\underline{\theta} + r\theta''\underline{\theta} + r\theta'\, d\underline{\theta}/d\theta\, d\theta/dt$$

$$= r''\underline{\boldsymbol{r}} + r'\theta'\underline{\theta} + r'\theta'\underline{\theta} + r\theta''\underline{\theta} - r\theta'^2\underline{\boldsymbol{r}} = (r'' - r\theta'^2)\underline{\boldsymbol{r}} + (r\theta'' + 2r'\theta')\underline{\theta}$$

$$\Longrightarrow a_r = r'' - r\theta'^2 ; \quad a_\theta = r\theta'' + 2r'\theta'$$

B. Newton's Laws

I. Law of Inertia

Newton's first law is basically a statement of conservation of linear momentum, $\boldsymbol{p} = m\boldsymbol{v}$. The law states:

"The velocity of a body remains constant unless the body is acted on by an outside force."

or

"A body at rest tends to remain at rest, a body in motion tends to remain in motion, unless acted on by an outside force.."

or

$$d\boldsymbol{p}/dt = 0 \Longrightarrow m\, d\boldsymbol{v}/dt = 0 \Longrightarrow m\boldsymbol{a} = 0$$

II. Force Law

Newton's second law defines the force on a body in terms of its effect in accelerating the body. The law states:

"The acceleration imparted to a body is proportional to and in the direction of the force applied, and inversely proportional to the mass of the body."

or

$$\boldsymbol{F} = m\boldsymbol{a}$$

Note that if m = constant, this can be written

$$\boldsymbol{F} = m\, d\boldsymbol{v}/dt = d/dt\, m\boldsymbol{v} = d\boldsymbol{p}/dt$$

This can be thought of as the definition of force. If you pull or push a 1 kg body, and it is observed to accelerate by 1 m/s, you have applied a force of 1 N (newton).

III. Action and Reaction

Newton's third law is basically a statement of conservation of total linear momentum for a system of particles. The law states:

"For every force acting on a body, there is an equal and opposite force exerted by the body."

or

$$\boldsymbol{F}_1 = -\boldsymbol{F}_2$$

or

$$\boldsymbol{P} = \Sigma\, \boldsymbol{p}_\mathrm{i} = \text{constant}$$

C. Law of Universal Gravitation

Newton was interested in explaining the motion of the Moon. He knew its distance fairly accurately, and that it orbited the Earth with a nearly uniform circular motion.

In uniform circular motion, the magnitude of $\boldsymbol{v}$ is constant, but the direction changes. This turning of the velocity is due to a central force, called the centripetal force, which gives the body a centripetal acceleration (towards the center of its circular path). The centripetal acceleration can be found by considering the velocity at two points on the circle, and taking the limit,

$$\lim_{\Delta t \Rightarrow 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = a$$

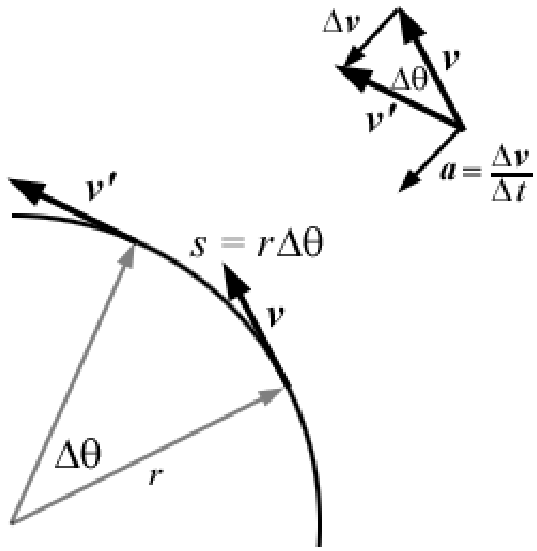


Diagram showing centripetal acceleration for an object in uniform circular motion. The centripetal acceleration needed to "turn" the velocity from v to v' is $\Delta v/\Delta t$, where the relationships between s, Δv and $\Delta\theta$ are as shown.

From the drawings above, we have

$$s = r\,\Delta\theta = v\,\Delta t \quad \text{and } \Delta v = v\,\Delta\theta \implies \Delta v/\Delta t = v^2/r$$

so

$$a = v^2/r \qquad \text{(centripetal acceleration)}$$

Note that the velocity here is only in the theta direction, $v_\theta = r\, d\theta/dt$, so we can recognize the second term in our earlier, general expression for radial acceleration as the centripetal acceleration, complete with minus sign to indicate that it points inward toward the center:

$$a_r = r'' - r\theta'^2 = r'' - v_\theta^2/r$$

The force is $\boldsymbol{F} = m\boldsymbol{a}$, so the Moon must experience a force $F = mv^2/r$, where $m = M_{Moon}$ = mass of the Moon, v = orbital velocity of the Moon, and $r = D_{Moon}$ = distance to the Moon from the center of the Earth (actually from the center of the Earth-Moon system, as we will see later).

It may seem hard to measure the velocity of the Moon--how would you do it?

Just use its period and the length of its orbit ($2\pi D_{Moon}$) to get $v = 2\pi D_{Moon}/P$, which gives

$$F = \frac{mv^2}{r} = \frac{M_{\text{Moon}}(2\pi D_{\text{Moon}}/P)^2}{D_{\text{Moon}}} = \frac{M_{\text{Moon}} 4\pi^2 D_{\text{Moon}}}{P^2}$$

But recall that Kepler gave the relationship between the period and radius (semi-major axis) of an orbit, which in this case is: $P^2 = kD_{\text{Moon}}^3$, which gives

$$F = \frac{M_{\text{Moon}} 4\pi^2}{kD_{\text{Moon}}^2} \quad \text{==> inverse square law } F \alpha\ 1/r^2.$$

From Newton's third law, he knew that the force of the Earth on the Moon must be balanced by an equal and opposite force of the Moon on the Earth, so the force had to be proportional to <u>both</u> masses:

$$F\ \alpha \frac{M_{\text{Moon}} M_{\text{Earth}}}{r^2} = \frac{GMm}{r^2}$$

Of course, the force is attractive, directed along the radius vector, so the vector force is

$$\boldsymbol{F} = -\frac{GMm}{r^2}\underline{\boldsymbol{r}}$$

Newton now took at leap of insight, and considered this law to be valid everywhere, i.e. it is a Universal Law. In particular, he could measure the force on an object at the surface of the Earth:

$$\boldsymbol{F} = -\frac{GM_{\text{Earth}}m}{R_{\text{Earth}}^2}\underline{\boldsymbol{r}} = -mg\underline{\boldsymbol{r}}$$

which allowed him to relate his constant, *G*, to the acceleration of gravity at Earth's surface:

$$g = \frac{GM_{\text{Earth}}}{R_{\text{Earth}}^2}$$

If his law was Universal, then the magnitude of the force on the Moon would be

$$F = \frac{GM_{\text{Earth}}M_{\text{Moon}}}{D_{\text{Moon}}^2} = gM_{\text{Moon}}\frac{R_{\text{Earth}}^2}{D_{\text{Moon}}^2} = g'\,M_{\text{Moon}}$$

Could it be so simple? Is the Moon merely "falling" with this value of acceleration? To find out, Newton needed to show that the acceleration needed to keep the Moon in its orbit was related to the acceleration of a stone falling at the Earth's surface, according to this relationship (working with accelerations rather than forces avoids the need to know the Moon's mass). To show this, Newton considered the distance the Moon must "fall" towards Earth in one second.

The value of g', above, is

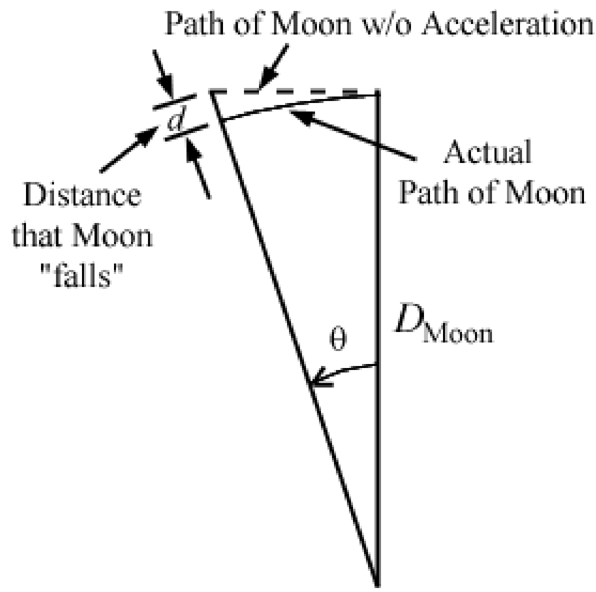
$$g' = 9.8\text{ m/s}^2\,(R_{\text{Earth}}/D_{\text{Moon}})^2 = 2.71 \times 10^{-3}\text{ m/s}^2.$$

Thus, in 1 s, the Moon will fall a distance

$$d = 1/2\, g't^2 = g'/2 = 1.36 \times 10^{-3}\text{ m} \quad (1.36\text{ mm}).$$

Let's see if this matches the orbit of the Moon. The sidereal period of the Moon is $P =$ $27.32\text{ days} = 2.36 \times 10^6\text{ s}$, so the angular velocity is $2\pi/\mathrm{P} = 2.66 \times 10^{-6}\text{ s}^{-1}$. Thus, in 1 s, the Moon moves through an angle of 2.66×10^{-6} radians. The distance that the Moon falls is shown in the figure below, and is related to the distance to the Moon by

$$d = D_{\text{Moon}}/\cos\theta - D_{\text{Moon}} = -D_{\text{Moon}}\,(1 - 1/\cos\theta)\,.$$



We cannot use most calculators with such a small angle (most will just give $\cos 2.66 \times 10^{-6} = 1$), but we can expand as a power series, which for such a small angle will allow us to keep only the leading terms. We will need two expansions:

$$\cos\theta = 1 - \theta^2/2 + \theta^4/24 - \ldots$$
$$(1 - x)^n = 1 + nx + 1/2n(n-1)x^2 + \ldots$$

where the second expansion is used to expand the result of the first expansion, in the form $(1 - \theta^2/2)^{-1}$. Performing these expansions and keeping only the leading term, we find that

$$d = D_{\mathrm{Moon}}\theta^2/2 = (384{,}000 \text{ km})\,(2.66 \times 10^{-6})^2 / 2 = 1.36 \times 10^{-3} \text{ m}.$$

The same answer! So Newton finally understood the truth! The Moon is falling towards Earth just like any stone or *apple* would do. The Law of Gravitation truly is Universal!

Physics 320 | Astrophysics I: Lecture #6 | Prof. Dale E. Gary NJIT

Orbital Mechanics I

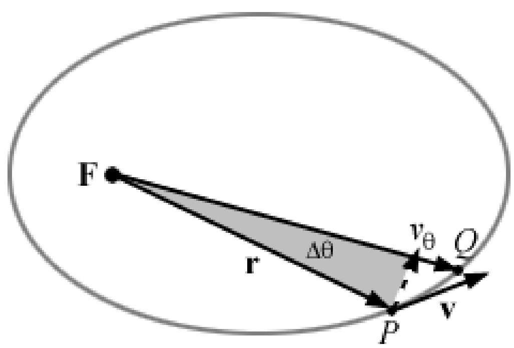
A: Physical Interpretation of Kepler's Laws

Kepler's first law states that the planets move in elliptical orbits around the Sun, with the Sun at one focus. Elliptical orbits are indeed a property of inverse square law central forces, as we will show shortly.

Let us examine Kepler's second and third laws in view of Newton's Law of Universal Gravitation.

1. Law of Areas and Angular Momentum

Kepler's second law states that the radius vector between the Sun and an orbiting planet sweeps out equal areas in equal times. Consider a planet moving along its elliptical orbit at a distance r, with velocity $\mathbf{v}$, as in the figure below.



After a time Δt, it moves an angular distance from point P to point Q of

$$\Delta\theta = v_\theta \, \Delta t \, / \, r \, .$$

During this time, the radius vector sweeps out the triangle FPQ, whose area is

$$\lim_{t \to 0} \frac{\Delta A}{\Delta t} = \frac{dA}{dt} = \frac{r\, v_\theta}{2} = 1/2\, r^2\theta'$$

According to Kepler's law, $dA/dt = \text{constant}$, and in particular after one complete period P, the area swept out is the total area of the ellipse,

$$dA/dt = A/P = \pi ab/P = \text{constant} = rv_\theta/2.$$

There are two places in its orbit where the radial velocity, v_r, of a planet goes to zero, and it has only $v = v_\theta$ --these are at aphelion and perihelion. At these locations, the speeds obey the relation

$$v = 2A/Pr = 2\pi ab/Pr$$

but at perihelion, $r = a(1 - e)$ and at aphelion $r = a(1 + e)$, so

$$v_{\text{peri}} = 2\pi ab/Pa(1 - e)$$
$$v_{\text{ap}} = 2\pi ab/Pa(1 + e)$$

but remember our relation $b = a\,(1 - e)^{1/2}$, so these become

$$v_{\text{peri}} = (2\pi a/P)[(1 + e)/(1 - e)]^{1/2}$$

$$v_{\text{ap}} = (2\pi a/P)[(1 - e)/(1 + e)]^{1/2}$$

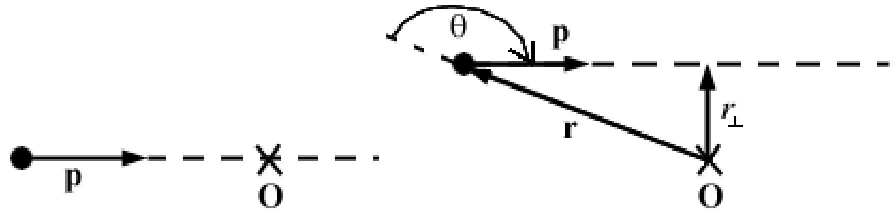
Example: What are v_{peri} and v_{ap} for Earth orbit?

$$P = 365.26 \text{ days} = 3.156 \times 10^7 \text{ s}$$
$$e = 0.0167,\ a = 1 \text{ AU} = 1.496 \times 10^8 \text{ km}$$
$$v_{\text{peri}} = (2\pi a/P)[(1 + e)/(1 - e)]^{1/2} = 30.28 \text{ km/s}$$
$$v_{\text{ap}} = (2\pi a/P)[(1 - e)/(1 + e)]^{1/2} = 29.28 \text{ km/s}$$

Again, this result shows that planets move faster near the Sun, but the Earth's orbit is so nearly circular that the speed does not change much. App for planetary orbits.

Angular Momentum:

What does all of this have to do with angular momentum? Recall that angular momentum is a measure of rotational motion about a center of rotation--usually the center of mass (but if an object is "pinned," the center of rotation is about that pinning point).



This system has zero angular momentum   This system has non-zero angular momentum

The angular momentum is given by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

where $\mathbf{p} = m\mathbf{v}$ is the linear momentum. In magnitude, this is $|\mathbf{L}| = L = rp \sin \theta = rp_{\mathrm{perp}}$. But in polar coordinates, $p_{\mathrm{perp}} = mv_\theta$, so

$$L = mrv_\theta = mr^2\dot{\theta}$$

so this is the appropriate expression for the angular momentum of a planet about the Sun. The key is to examine how the angular momentum changes around the orbit, i.e.,

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r}\times\mathbf{p}) = \dot{\mathbf{r}}\times\mathbf{p} + \mathbf{r}\times\dot{\mathbf{p}} = \mathbf{v}\times\mathbf{p} + \mathbf{r}\times\mathbf{F}$$

but $\mathbf{v} \times \mathbf{p} = \mathrm{m}\mathbf{v} \times \mathbf{v} = 0$

$d\mathbf{L}/dt = \mathbf{r} \times \mathbf{F}$ (this is the torque on the planet)

For any central force, in particular for Newton's Law of Universal Gravitation, where $\mathbf{F} = -(GMm/r^2)\,\underline{\boldsymbol{r}}$, we are going to have $\mathbf{r} \times \mathbf{F} = 0$ also! Thus,

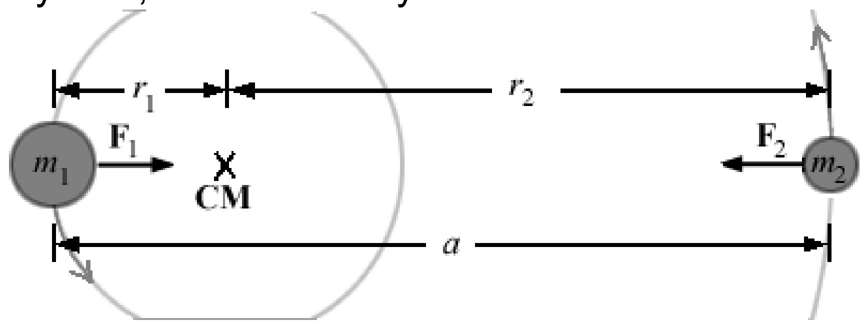
$d\mathbf{L}/dt = 0$; so $\mathbf{L}$ = constant.

In fact, from the above expression, $L = mrv_\theta$. Finally, we see that the statement of Kepler's second law is that same as the statement of conservation of angular momentum:

$dA/dt = L/2m$ = constant

2. Kepler's Third Law

For the general case of two masses interacting according to Newton's Law of Universal Gravitation, the two masses actually orbit about the center of mass of the system, not necessarily the center of the more massive body.



Recall the equation for center of mass

$$\mathbf{r}_{cm} = \Sigma m_i \mathbf{r}_i / \Sigma m_i$$

For a two mass system, we will refer to the separation of the two masses as $a = r_1 + r_2$, where r_1 is the distance of mass m_1 from the center, and r_2 is the distance of mass m_2 from the center. Consider the case when the two masses are in circular orbits. During their motion, the two planets must be acted on by a centripetal force given by

$$F_1 = m_1 v_1^2/r = 4\pi^2 m_1 r_1 / P^2$$

and

$$F_2 = m_2 v_2^2/r = 4\pi^2 m_2 r_2 / P^2$$

where we have used $v = 2\pi r / P$. Now, by Newton's third law, these two forces must be equal in magnitude (and opposite in direction), which means $m_1 r_1 = m_2 r_2$. This actually proves that the center of the circular motion is the center of mass. From this and $a = r_1 + r_2$, we have

$$r_1 = [m_2/(m_1 + m_2)]a.$$

Also, by Newton's Law of Universal Gravitation we have the expression for the force:

$$F_1 = F_2 = F = Gm_1 m_2/a^2,$$

so using the expression for F_1, we have

$$Gm_1 m_2/a^2 = 4\pi^2 m_1 r_1 / P^2 = (4\pi^2 m_1 / P^2)\ [m_2/(m_1 + m_2)]\ a,$$

or

$$P^2 = [4\pi^2/G(m_1 + m_2)]a^3$$

which, as promised, is the expression corresponding to Kepler's third law.

Note that the center of mass is also called the *barycenter*. The two masses orbit the barycenter with the same period--you use the separation between the masses, a, not the distances of the masses r_1 and r_2 from the center of mass, to determine the period.

3. Orbital Velocity

We will now use these results to derive a particularly simple equation for the orbital velocity for any point on an *elliptical* orbit. Since most orbits are elliptical, this will be a very useful equation.

We decompose the velocity into its two components:

$$v_r = dr/dt = r' \quad \text{and} \quad v_\theta = r\,(d\theta/dt) = r\theta'$$

Going back to our equation for an ellipse:

$$r = a(1 - e^2) / (1 + e \cos \theta)$$

we can explicitly take the derivative and get the radial component of the velocity as

$$\begin{aligned} v_r &= dr/dt = a(1 - e^2)\, d/dt\, [(1 + e \cos \theta)]^{-1} \\ &= ae(1 - e^2) \sin \theta\, d\theta/dt \,/\, (1 + e \cos \theta)^2 \end{aligned}$$

But note that earlier we had $rv_\theta = r^2 d\theta/dt = 2\pi ab/P = 2\pi a^2(1 - e^2)^{1/2}/P$, so

$$d\theta/dt = 2\pi a^2(1 - e^2)^{1/2}/Pr^2$$

Substitution of this into the equation for v_r, gives

$$\begin{aligned} v_r &= ae(1 - e^2) \sin \theta\, [2\pi a^2(1 - e^2)^{1/2}/Pr^2]/\, (1 + e \cos \theta)^2 \\ &= [2\pi a \,/\, P(1 - e^2)^{1/2}]\, (e \sin \theta). \end{aligned}$$

The corresponding perpendicular component of the velocity is

$$\begin{aligned} v_\theta &= r\, d\theta/dt = 2\pi a^2(1 - e^2)^{1/2}/Pr \\ &= [2\pi a \,/\, P(1 - e^2)^{1/2}]\, (1 + e \cos \theta). \end{aligned}$$

We simply sum the squares of these components to get the total magnitude of the velocity

$$v^2 = v_r^2 + v_\theta^2 = (2\pi a / P)^2 (1 + 2e \cos \theta + e^2) / (1 - e^2).$$

It is useful to substitute from the equation of an ellipse for the quantity $e \cos \theta$:

$$e \cos \theta = a(1 - e^2)/r - 1$$

which gives:

$$v^2 = (2\pi a / P)^2 [(2a/r)(1 - e^2) + e^2 - 1] / (1 - e^2) = (2\pi a / P)^2 (2a/r - 1).$$

Finally, from Kepler's third law, $P^2 = [4\pi^2/G(m_1+ m_2)]a^3$, we have

$$v^2 = [(4\pi^2 a^2)\, G(m_1+ m_2) / 4\pi^2 a^3](2a/r- 1) = G(m_1+ m_2)\, (2/r - 1/a)$$

This final equation for the velocity of an elliptical orbit

$$v^2 = G(m_1+ m_2)\, (2/r - 1/a)$$

is called the *vis viva equation*.

What have we learned?

We found that Kepler's second law (Law of Equal Areas), is equivalent to conservation of angular momentum $L = mrv_\theta$, so that $dL/dt = 0$ for any orbit. This is a consequence of the central force nature of the gravitational force--only a perpendicular force could change a bodies' angular momentum, and since there is none, the angular momentum cannot change. We obtained simple expressions for the speed of a planet or other orbiting body at perihelion and aphelion:

$$v_{\text{peri}} = (2\pi a/P)[(1 + e)/(1 - e)]^{1/2}$$
$$v_{\text{ap}} = (2\pi a/P)[(1 - e)/(1 + e)]^{1/2}.$$

We also noted, using Newton's third law (Law of equal action and reaction), that two bodies orbit their combined center of mass (the *barycenter*) rather than the center of either body. From this and Newton's Law of Universal Gravitation ($F = Gm_1m_2/a^2$), we proved Kepler's third law in its quantitative form:

$$P^2 = [4\pi^2/G(m_1 + m_2)]a^3.$$

Applying Kepler's third law, we were able to obtain a more general equation for orbital speed, valid at *any* point in the orbit, the *vis viva* equation:

$$v^2 = G(m_1 + m_2)\,(2/r - 1/a).$$

Orbital Mechanics II

B: Orbits and Energy

1. Conservation of Energy

Recall the concepts of potential energy *U*, and kinetic energy *K*, the sum of which gives the total energy

$$E = K + U$$

where $K = 1/2\ mv^2$. But what is the potential energy appropriate to our planetary system?

Remember that only differences in potential energy are important. If we raise a mass at the surface of the Earth by a distance *h*, we do work against gravity

$$W = \int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r} = -mg\Delta r = -mgh$$

and the negative of this work is the change of potential energy

$$\Delta U = -W = -\int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r} = mgh$$

Raising a mass from the floor to the desk raises the potential energy by mgh, and raising it by the same height from the desk to a shelf also raises U by mgh -- only the difference ΔU matters. Thus, we are free to choose our zero of potential energy anywhere we wish.

We will choose $\Delta U = 0$ at r = infinity, which means the potential energy *anywhere* in the system is *negative*. Now what is the potential energy at position r? We first place a test mass of mass m at infinity, and then move it radially inward to distance r from the center (from mass M) with the force of gravity acting all the way along this path. We have

$$U = -\int_{\inf}^{r} -\frac{GMm}{r^2}\,\underline{\boldsymbol{r}}\cdot d\mathbf{r}$$

which evaluates to

$$U = GMm\int_{\inf}^{r}\frac{dr}{r^2} = -\frac{GMm}{r}\Big|_{\inf}^{r}$$

or finally,

$$U = -GMm/r \qquad \text{(Gravitational potential energy)}$$

As advertised, the potential energy is negative for any r, and approaches zero as r approached infinity. The total energy for an elliptical orbit, then, is

$$\begin{aligned} E &= K + U = 1/2\ mv^2 - GMm/r \\ &= 1/2\ mG(M+m)[2/r - 1/a] - GMm/r \end{aligned}$$

$$E = -\ GMm/2a$$

where we have used the *vis viva* equation (which is why this is valid only for elliptical orbits), and we have made the approximation that $M >> m$.

Note that the total energy is negative, and is just a constant. Thus, energy is conserved along the orbit, as of course it must be.

2. Total Energy for any Orbit

Now we will examine the total energy for any orbit, not limited to elliptical orbits. To do this, we need to use conservation of angular momentum. It is possible to show, although we will not do so, that the angular momentum and eccentricity are related by

$$L^2 / GMm^2r = 1 + e\cos\theta,$$

so solving for r, we get

$$r = (L^2 / GMm^2) / (1 + e\cos\theta).$$

Note that this has the same form as the general expression for the polar equation for a conic section. Let us now repeat the calculation of total energy in the same way as before:

$$v_r = dr/dt = (L^2 / GMm^2)\, d/dt\, [(1 + e \cos\theta)]^{-1}$$
$$= (L^2 / GMm^2)\, e \sin\theta\, d\theta/dt \,/\, (1 + e \cos\theta)^2$$

Recall that $|\mathbf{L}| = |\mathbf{r} \times \mathbf{p}| = mrv_\theta$, so that $rv_\theta = L/m$. But since $v_\theta = r\, d\theta/dt$, we have $r^2 d\theta/dt = L / m,$ so. Putting this into the above equation,

$$v_r = (GMm / L)\, (e \sin\theta).$$

The corresponding perpendicular component of the velocity is even simpler to derive

$$v_\theta = r\, d\theta/dt = r^2 d\theta/dt \,/\, r = L / rm = (GMm / L)\, (1 + e \cos\theta).$$

So the total velocity is

$$v^2 = v_r^2 + v_\theta^2 = (GMm / L)^2\, (1 + 2e \cos\theta + e^2)$$

and finally the kinetic energy is

$$K = 1/2\, mv^2 = 1/2\, m(GMm / L)^2\, (1 + 2e \cos\theta + e^2)$$

while the potential energy is

$$U = -GMm/r = -(GMm)^2 m/L^2 (1 + e \cos\theta)$$
$$= -1/2\, m(GMm / L)^2\, (2 + 2e \cos\theta)$$

We finally come to the rather simple expression for total energy, for an orbit of the form of any conic section:

$$E = K + U = 1/2\, m(GMm / L)^2\, (e^2 - 1) \qquad \text{(Total energy for any orbit)}$$

It is instructive to solve this equation for the eccentricity, to get

$$e = [1 + 2L^2 E \,/\, (GMm)^2 m]^{1/2}$$

In particular, note that

- for $E > 0$, we have $e > 1$ ==> hyperbola
- for $E = 0$, we have $e = 1$ ==> parabola
- for $E < 0$, we have $e < 1$ ==> ellipse

3. Planetary Motion and Effective Potential

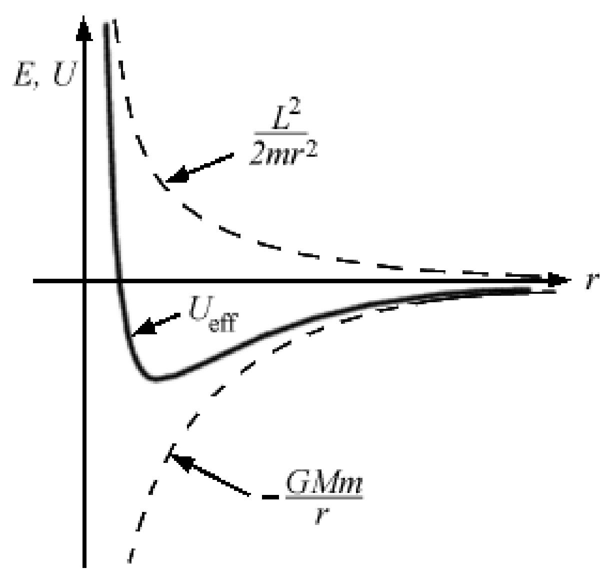
We can consider the kinetic energy as having a radial part and an angular part:

$$K = 1/2\, mv^2 = 1/2\, mv_r^2 + 1/2\, mv_\theta^2$$
$$K = K_r + K_\theta$$

where $v_\theta^2 = L^2 / m^2 r^2$ ==> $K_\theta = 1/2\, mv_\theta^2 = 1/2\, L^2 / mr^2$. Consider a body moving inward through the solar system. As r decreases, the angular kinetic energy increases for smaller r due to conservation of angular momentum. Since the total energy is conserved, this increasing angular kinetic energy comes in part from the radial kinetic energy. The system acts as though there were a radial force opposing the inward motion. Let us write the total energy as

$$E = K_r + K_\theta + U(r) = 1/2\, mv_r^2 + 1/2\, L^2 / mr^2 - GMm/r = K_r + U_{\text{eff}}$$

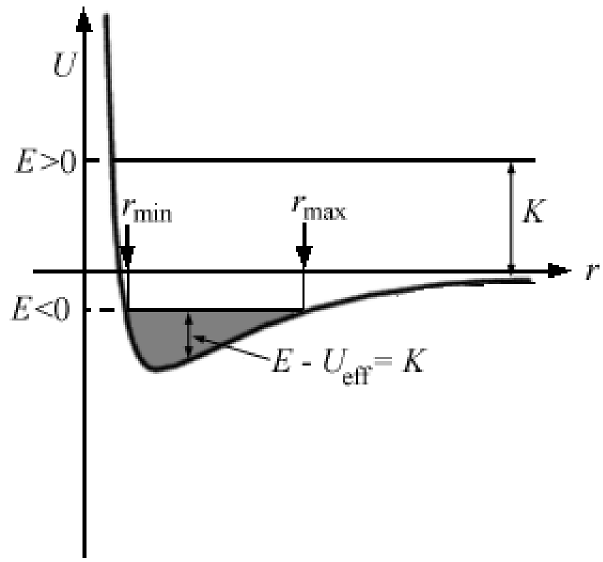
so we call this combination an effective potential. Graphically, we have:



There is a nice way to interpret orbits using this diagram. Since E = constant on any orbit, if $E < 0$ we have a bound orbit, with two turning points where $K_r = 0$,

$$r_{\text{min}} = r_{\text{peri}}$$
$$r_{\text{max}} = r_{\text{ap}}$$

Note that *any* orbit with $E < 0$ has two turning points. For $E = 0$, there is only one turning point, and the object reaches infinity with zero energy. Finally, for $E > 0$, there is again one turning point, but the object reaches infinity with energy left over.



Effective potential in general relativity (strongly curved space-time).