Abstract

We study the coercivity properties and the norm dependence on the wavenumber $k$ of certain regularized combined field boundary integral operators that we recently introduced for the solution of two and three-dimensional acoustic scattering problems with Neumann boundary conditions. We show that in the case of circular and spherical boundaries, our regularized combined field boundary integral operators are $L^2$ coercive for large enough values of the coupling parameter, and that the norms of these operators are bounded by constant multiples of the coupling parameter. We establish that the norms of the regularized combined field boundary integral operators grow modestly with the wavenumber $k$ for smooth boundaries and we provide numerical evidence that these operators are $L^2$ coercive for two-dimensional starlike boundaries. We present and analyze a fully discrete collocation (Nyström) method for the solution of two-dimensional acoustic scattering problems with Neumann boundary conditions based on regularized combined field integral equations. In particular, for analytic boundaries and boundary data, we establish pointwise superalgebraic convergence rates of the discrete solutions.

Keywords: Helmholtz equations, Regularized Combined Field Integral Equations, Coercivity, Numerical range, Trigonometric interpolation, Collocation methods.

1 Introduction

Numerical methods based on integral equation formulations have certain advantages over those that use volumetric formulations for the solution of acoustic homogeneous scattering problems, largely owing to 1) the reduction in dimensionality that is achieved from posing the scattering problems on the lower-dimensional boundary of the scatterers and 2) their explicit enforcement of the radiation conditions through choices of outgoing Green’s functions in the boundary layer potentials that enter the integral formulations. In addition, well-conditioned integral equations are available for acoustic scattering problems with various boundary conditions. For instance, in the case of Dirichlet boundary conditions the scattered field can be represented as a suitable linear combination of single and double layer potentials leading to classical Combined Field Integral Equations (CFIE) [7]. However, in the case of Neumann boundary conditions, the same approach leads to CFIE that contain hypersingular operators, e.g. the normal derivatives of double layer potentials [12]. These hypersingular operators are pseudodifferential operators of order one (they behave like a derivative) and thus the spectra of the CFIE operators accumulate at infinity. These unfavorable spectral distributions lead upon discretizations to linear systems that require increasingly larger numbers of Krylov-subspace iterations for refined discretizations. The derivative-like effect of the hypersingular operators can be mitigated by the use of the regularizing operators that are pseudo-inverses of the
hypersingular operators \cite{3–5, 7, 9, 10, 13, 35, 40}; the resulting combined field regularized integral equations (CFIER) are uniquely solvable across the frequency spectrum and behave in practice as integral equations of the second kind.

The aforementioned CFIER formulations constitute important alternatives to the classical CFIE, especially in the very low and high-frequency regimes where solvers based on certain CFIER consistently outperform those based on CFIE in terms of computational cost with no undue compromise in accuracy \cite{9}. However, besides evidence that solvers based on CFIER formulations work very well in practice for high-frequency problems, no significant attempts have been made at understanding the dependence on the wavenumber of the norms and condition numbers of the boundary integral operators that enter those formulations. This situation is in contrast to the case of Dirichlet boundary conditions wherein several important recent papers have investigated the explicit wavenumber dependence in the high-frequency regime of the norms of the classical CFIE operators introduced in \cite{7} together with that of their inverses \cite{16, 17, 27, 38}. An extremely interesting and important fact that was emphasized in these works is the $L^2$ coercivity of the CFIE operators which was first rigorously established for the case of circular and spherical geometries in \cite{27}, then conjectured to be true based on supporting numerical evidence for other smooth and non-smooth two-dimensional geometries \cite{11}. Recently, the availability of frequency-uniform coercive boundary integral equations was rigorously established \cite{45} in the case of Dirichlet boundary conditions. The new integral equations introduced in \cite{45}, although different from the classical CFIE in \cite{7}, are still integral equations of the second kind. The availability of boundary integral equations whose underlying operators are coercive is important as the coercivity property facilitates error analysis of Galerkin discretizations of boundary integral equations in non-smooth domains.

The combined field regularized integral equations (CFIER) formulations that we use in this paper consist of suitable combinations of single layer potentials and double layer potentials that act on certain regularizing operators. The regularizing operators that we use are pseudoinverses of Neumann-to-Dirichlet maps that in addition are coercive. More specifically, these regularizing operators are either single layer operators with purely imaginary wavenumbers or the principal symbols of such operators in the sense of pseudodifferential operators. We note that a similar approach was advocated in \cite{4, 5, 35}, yet the choice of regularizing operators in those references is different from ours: the choice of the regularizing operators advocated in the former reference relies on use of the principal symbol of the Neumann-to-Dirichlet map for circular geometries combined with certain quadratic partitions of unities while the choice presented in the latter reference consists of (local) Padé approximations of complexified versions of the principal symbol of the Neumann-to-Dirichlet map. Although various regularizing operators leading to various types of CFIER have been proposed in the literature, they all rely on the Calderón’s identities. Regularizing operators that correspond to single layer potentials with zero wavenumbers were introduced in \cite{3, 40}, whereas regularizing operators that are boundary layer potentials corresponding to complex wavenumbers were introduced in \cite{2, 22}. The latter operators lead to CFIER with superior spectral properties that outperform their counterparts that rely on boundary layer potential with zero wavenumbers, especially in the challenging high frequency regime \cite{9}. In this paper we aim to justify the excellent spectral properties of complexified CFIER operators in the high-frequency regime.

After introducing the complexified CFIER operators and having established their invertibility in appropriate functional spaces, we study the wavenumber dependence of the $L^2$ norms of the CFIER operators in the case of two and three-dimensional smooth geometries. In particular, we establish that these norms can grow at most as $Ck^{1/2}$ in terms of the wavenumber $k$, where $C$ are
constants that depend only on the geometry and the coupling constants. A spectral analysis allows us to obtain sharper versions of these results in the case of circular and spherical geometries. We show in this case that the norms of the CFIER operators are bounded from above by constants $C$ that depend only on the coupling parameters. Furthermore, we rigorously establish that for high enough frequencies and sufficiently large coupling constants—that is for coupling constants larger than $Ck^{1/3}$, the CFIER operators are coercive in $L^2(\Gamma)$ in the case of circular and spherical geometries $\Gamma$, and that the coercivity constants do not depend on the wavenumbers $k$. We recall that if $H$ is a Hilbert space, an operator $A : H \rightarrow H'$ (where $H'$ is the dual of $H$) is coercive if there exists a constant $\gamma > 0$ such that $\gamma \|u\|_H^2 \leq \Re \langle Au, u \rangle$ for all $u \in H$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H$ and $H'$. The question of the $L^2$ coercivity of the CFIER operators for general boundaries remains an open issue. We provide numerical evidence that the coercivity property holds for two-dimensional starlike boundaries and in the case of strictly-convex boundaries the coercivity property is uniform in the wavenumber. We have recently extended the CFIER formulations to the case of non-smooth boundaries [10]; we will study the coercivity of the CFIER operators in non-smooth domains in a future contribution. We note that integral equation formulations whose operators are compact perturbations of coercive operators are available for the solution of acoustic and electromagnetic scattering problems [28, 29], yet those integral equations are not of the second kind.

We present a collocation (Nyström) method for the discretization of our CFIER operators that follows the discretization method introduced in [30]. This collocation algorithm is based on global trigonometric approximations, splitting of the kernels of integral operators into singular and smooth components, and explicit quadratures of products of singular parts (logarithms) and trigonometric polynomials [34, 36]. We establish the convergence of the fully discrete method in appropriate Sobolev spaces which implies pointwise convergence of the discrete solutions. In the case of analytic boundaries, we establish superalgebraic convergence of the collocation method. The main ingredients in the error analysis proof are the mapping properties of the boundary integral operators that enter CFIER and Sobolev spaces bounds of the error in trigonometric interpolation. We note that unlike the classical CFIE operators that map Sobolev spaces $H^{p+1}$ into $H^p$, the CFIER operators map $H^p$ into $H^p$, which simplifies the error analysis; however, one has to deal with compositions of discrete operators. Using these high-order discretizations of the CFIER operators, we provide numerical evidence of the $L^2$ coercivity of the CFIER operators for smooth, starlike boundaries. Specifically, just as in [11], we compute numerical ranges of the discretizations of the CFIER operators corresponding to very fine meshes, and we highlight the fact that the discrete numerical ranges do not contain the origin, which means that the continuous CFIER operators are coercive.

The paper is organized as follows: in Section 2 we introduce our Regularized Combined Field Integral Equations; in Section 3 we establish upper bounds on the norms of the CFIER operators and their coercivity in the case of circular and spherical boundaries; in Sections 4 we describe the Nyström method and we derive error estimates for the discrete solutions; in Section 5 we present numerical results that confirm the high-order nature of our solvers and we provide numerical evidence of the coercivity of the CFIER operators for smooth, starlike boundaries.

2 Regularized Combined Field Integral Equations

Let us consider the time-harmonic acoustic scattering problem for a two-dimensional sound-hard bounded obstacle $\Omega$. In this case, the scattered field $u^s$ satisfies the Helmholtz equation together
with Neumann conditions on the boundary $\Gamma$ of $\Omega$ and Sommerfeld radiation conditions at infinity,

$$
\Delta u^s + k^2 u^s = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \Omega \quad (2.1)
$$

$$
\frac{\partial u^s}{\partial n} = -\frac{\partial u^{inc}}{\partial n} \quad \text{on} \quad \Gamma \quad (2.2)
$$

$$
\lim_{|r| \to \infty} r^{1/2}(\partial u^s / \partial r - iku^s) = 0, \quad (2.3)
$$

where $u^{inc}$ denotes the incident acoustic field. We assume that the boundary $\Gamma$ is a smooth and closed curve in two-dimensions. The classical Indirect Combined Field Integral Equation formulation (ICFIE), which was originally introduced by Brakhage and Werner [7] and Panich [40], assumes a representation of the scattered field as a combination of single and double layer potentials of the form

$$
u^s(z) = (u_{SL}\phi)(z) + i\eta(u_{DL}\phi)(z)$$

$$= \int_{\Gamma} G_k(z - y)\phi(y)ds(y) + i\eta \int_{\Gamma} \frac{\partial G_k(z - y)}{\partial n(y)} \phi(y)ds(y) \quad (2.4)$$

where $z \in \mathbb{R}^2 \setminus \Omega$, $\phi$ denotes a (smooth) function defined on $\Gamma$ which we will refer to as the density, $\eta$ is a coupling parameter $\eta \neq 0$, and $G_k$ denotes the outgoing free-space Green’s function

$$G_k(z - y) = i\frac{1}{4}H_0^{(1)}(k|z - y|). \quad (2.5)$$

Taking the normal derivative of both sides of (2.4), using the trace formulas for the normal derivative of layer potentials for smooth domains together with the sound-hard boundary condition the classical ICFIE is given by

$$
-\frac{\phi(x)}{2} + (K'_k\phi)(x) + i\eta(N_k\phi)(x) = -\frac{\partial u^{inc}(x)}{\partial n(x)}, \quad x \text{ on } \Gamma \quad (2.6)
$$

where $K'_k$ denotes the semi-sum of the interior and exterior Neumann traces of the single layer potential $u_{SL}$ on $\Gamma$. Specifically, the operator $K'_k$ is defined as

$$(K'_k\phi)(x) = \int_{\Gamma} \frac{\partial G(x - y)}{\partial n(x)} \phi(y)ds(y), \quad x \text{ on } \Gamma. \quad (2.7)$$

The operator $N_k$ in equations (2.6) denotes the Neumann trace of the double layer potential $u_{DL}$ on $\Gamma$ given in terms of finite part integrals [20]

$$
(N_k\phi)(x) = \int_{\Gamma} \frac{\partial^2 G_k(x - y)}{\partial n(x)\partial n(y)} \phi(y)ds(y)$$

$$= k^2 \int_{\Gamma} G_k(x - y)(n(x) \cdot n(y))\phi(y)ds(y) + \int_{\Gamma} \partial_s G_k(x - y)\partial_s \phi(y)ds(y) \quad (2.8)$$

where $\partial_s$ denotes the tangential derivative on $\Gamma$.

In what follows we recall the definition of Sobolev spaces $H^p(\Gamma)$ according to [31, 42], as we make frequent use of these spaces. Our presentation follows very closely that in [31]. We start with the space $L^2[0, 2\pi]$ of $2\pi$ periodic functions that are square integrable. A function $\varphi \in L^2[0, 2\pi]$ can
be expressed in terms of its Fourier series \( \varphi(t) = \sum_{m=-\infty}^{\infty} a_m e^{imt} \) where \( a_m = \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) e^{-imt} dt \). Then, for a given \( 0 \leq p < \infty \), we denote by \( H^p[0,2\pi] \) the space of functions \( \varphi \in L^2[0,2\pi] \) with the property
\[
\sum_{m=-\infty}^{\infty} (1 + m^2)^p |a_m|^2 < \infty
\]
in terms of the Fourier coefficients \( a_m \) of \( \varphi \). The space \( H^p[0,2\pi] \) is called a Sobolev space and it is equipped with the scalar product defined by
\[
(\varphi, \psi)_p = \sum_{m=-\infty}^{\infty} (1 + m^2)^p a_m b_m
\]
in terms of the Fourier coefficients \( a_m \) and \( b_m \) of \( \varphi \) and \( \psi \) respectively. For \( 0 \leq p < \infty \) we denote by \( H^{-p}[0,2\pi] \) the dual space of \( H^p[0,2\pi] \), that is the space of bounded linear functionals on \( H^p[0,2\pi] \).

If the curve \( \Gamma \) is represented by a smooth (infinitely differentiable) \( 2\pi \) periodic parametrization \( \Gamma = \{ x(t) : t \in [0, 2\pi] \} \), then for \( 0 \leq p < \infty \), the space \( H^p(\Gamma) \) is defined as the space of functions \( \varphi \in L^2(\Gamma) \) such that \( \varphi \circ x \in H^p[0,2\pi] \). It was shown in [31] that this definition of \( H^p(\Gamma) \) is invariant with respect to the parametrization.

Having reviewed the definition of Sobolev spaces \( H^p(\Gamma) \), we return to the issue of the unique solvability of (2.6) in the appropriate Sobolev spaces \( H^{1/2}(\Gamma) \). This is settled in a classical manner via Fredholm theory. Indeed, the operator \( N_k \) is strongly elliptic as it satisfies a Gårding type inequality in \( H^{1/2}(\Gamma) \) — its principal part corresponding to \( N_0 \) is coercive in the space of functions in \( H^{1/2}(\Gamma) \) with zero mean [23, 39] — and the operator \( K_k \) is compact in the same space for smooth surfaces, and therefore the operators in (2.6) are Fredholm of index 0. The injectivity of these operators is established in connection with the fact that solutions of the Helmholtz equation in \( \Omega \) with zero impedance boundary conditions on \( \Gamma \) are trivial [20] from which their invertibility follows from the Fredholm theory.

Although the integral equations (2.6) are uniquely solvable for all values of the wavenumber \( k \), they are not integral equations of the second kind. Indeed, the operator \( N_k \) is a pseudodifferential operator of order 1 [39] and thus the eigenvalues of the integral operators that enter the formulation (2.6) accumulate at infinity. In order to mitigate the “derivative” effect of \( N_k \) we use recently introduced Regularized Combined Field Integral equations [9]. The derivation of these integral equation formulations starts with the Green’s representation formula
\[
\begin{align*}
 u^s(z) &= \int_{\Gamma} \left( -G_k(z - y) \frac{\partial u^s(y)}{\partial n(y)} + \frac{\partial G_k(z - y)}{\partial n(y)} u^s(y) \right) ds(y), \quad z \in \mathbb{R}^2 \setminus \Omega \quad (2.9)
\end{align*}
\]
and it is based on the use of the Neumann-to-Dirichlet operator (NtD) \( Y_k \) that maps the Neumann trace of \( u^s \) on \( \Gamma \) to the Dirichlet trace of \( u^s \) on \( \Gamma \), so that \( Y_k(\partial u^s/\partial n|_{\Gamma}) = u^s|_{\Gamma} \) and \( Y_k : H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma) \). With these notations equations (2.9) becomes
\[
\begin{align*}
 u^s(z) &= \int_{\Gamma} \left( -G_k(z - y)(\partial u^s/\partial n)(y) + \frac{\partial G_k(z - y)}{\partial n(y)} Y_k(\partial u^s/\partial n)(y) \right) ds(y), \quad z \in \mathbb{R}^2 \setminus \Omega \quad (2.10)
\end{align*}
\]
Applying the exterior Neumann trace to equations (2.10) we obtain
\[
\begin{align*}
 \frac{1}{2} \partial u^s/\partial n &= -K_k(\partial u^s/\partial n) + (N_k Y_k)(\partial u^s/\partial n) \quad (2.11)
\end{align*}
\]
from which it follows that $Y_k$ is a right pseudo-inverse (parametrix) of $N_k$ in the sense that $N_kY_k = I/2 + K$ where $K$ is a compact operator in $H^{-1/2}(\Gamma)$. For a general domain $\Omega$ the NtD operator $Y_k$ is not known, yet operators $R$ that approximate the NtD operator in the sense that $R$ differ from $Y_k$ by smoother operators can be obtained with relative ease. For example, the single layer operator $S_k$ is such an operator in the light of Calderón’s identities [15]. For a given density $\psi \in H^{-1/2}(\Gamma)$ the operator $S_k : H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$ is defined as the Dirichlet trace of the single layer potential $u_{SL}\psi$

$$ (S_k\psi)(x) = \int_{\Gamma} G_k(x - y)\psi(y)ds(y), \ x \text{ on } \Gamma. \quad (2.12) $$

Using such operators $R$ that approximate the NtD operator $Y_k$ in the sense defined above, and taking a cue from the form of equations (2.10), we look for scattered fields $u^s$ in the form

$$ u^s(z) = -\int_{\Gamma} \left( G_k(z - y)\mu(y) + i\eta - \frac{\partial G_k(z - y)}{\partial n(y)}(R\mu)(y) \right) ds(y) \quad (2.13) $$

for a density function $\mu$ defined on $\Gamma$. Representations (2.13) lead to the (Indirect) Regularized Combined Field Integral Equation formulations:

$$ (A_{k,R}\mu)(x) = \frac{\mu(x)}{2} - (K_{k,R}\mu)(x) - \eta(N_k \circ R)\mu(x) = \frac{\partial u^{inc}(x)}{\partial n(x)}, \ x \text{ on } \Gamma. \quad (2.14) $$

Since the operators $R$ are chosen to be pseudo-inverses of the operator $N_k$, i.e. $N_kR = \alpha I + Compact in the natural trace space $H^{m-\frac{1}{2}}(\Gamma)$, $m \geq 0$ of boundary valued problems for the Helmholtz equation, then (a) the operators $R$ have the following mapping property $R : H^{m-\frac{1}{2}}(\Gamma) \rightarrow H^{m+\frac{1}{2}}(\Gamma)$ and (b) the operator in the left hand side of equations (2.14) is itself a compact perturbation of a multiple of identity, and thus the integral equations (2.14) are of the second kind. On account of the Fredholm theory, the operators $A_{k,R}$ are invertible if and only if they are injective. If in addition to being pseudo-inverses of $N_k$, the operators $R$ are coercive with respect to the duality pairing $\langle H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma) \rangle$, then the unique solvability property of the operators $A_{k,R}$ in equations (2.14) is ensured:

**Theorem 2.1** Assume that the operators $R$ the left-hand side of (2.14) (1) are right pseudo-inverses of the operators $N_k$ in $H^{m-\frac{1}{2}}(\Gamma)$, $m \geq 0$ and (2) satisfy the following (coercivity) property

$$ \langle R\psi, \psi \rangle \geq \alpha \|\psi\|^2_{H^{-1/2}(\Gamma)}, \ \alpha > 0, \ \forall \psi \in H^{-1/2}(\Gamma). $$

Then, if $\eta \neq 0$, the operators $A_{k,R}$ are invertible in $H^{m-\frac{1}{2}}(\Gamma)$, $m \geq 0$.

**Proof.** Give the assumption (1) and the fact that the operators $K_{k,R}'$ are compact, it follows that the operators $A_{k,R}$ are multiples of identity plus compact operators. Using the Fredholm property of the operators $A_{k,R}$, the invertibility of $A_{k,R}$ is equivalent to their injectivity. We prove next that the coercivity condition (2) in the Theorem implies the injectivity of the operators $A_{k,R}$. To this end, let $\mu$ be a solution of the homogeneous equation (2.14), then use the representation (2.13) to define a field $u$ everywhere in $\mathbb{R}^2 \setminus \Gamma$; we denote by $u^+$ the values of the field $u$ in the exterior of $\Omega$ and by $u^-$ the values of the field $u$ in the interior of $\Omega$. It follows then that $u^+$ is a radiative solution to the Helmholtz problem in the exterior of $\Omega$ with $\partial u^+/\partial n = 0$ on $\Gamma$ whence $u^+ = 0$ in the exterior of $\Omega$. The jump relations for the traces of layer potentials classically give

$$ -u_- = i\eta \ R\mu, \ \frac{\partial u_-}{\partial n} = \mu \text{ on } \Gamma. $$
from which we get using Green’s identities

\[ i \eta \int_{\Gamma} (\mathcal{R}\mu) \tilde{\mu} ds = - \int_{\Gamma} u_\perp \frac{\partial u_\perp}{\partial n} ds = \int_{D} (|\nabla u_\perp|^2 - k^2 |u_\perp|^2) dx. \] (2.15)

Using the assumption (2) and the fact that the quantity in the right-hand side of equations (2.15) is real, we obtain that \( \mu = 0 \), and thus the proof is complete. ■

One class of operators \( \mathcal{R} \) that satisfy conditions (1) and (2) in Theorem 2.1 consists of operators of the form \( \mathcal{R} = S_{ik_1}, \ k_1 > 0 \), where \( S_{ik_1} \) is the single layer potential defined in equations (2.12) corresponding to the purely imaginary wavenumber \( ik_1 \). This choice was introduced in our previous effort [9], where we proved that \( \mathcal{R} = S_{ik_1}, \ k_1 > 0 \) satisfies conditions (1) and (2) in Theorem 2.1. Given this choice of regularizing operators \( \mathcal{R} \), we denote the corresponding regularized integral operators by \( A_{k,S_{ik_1}} \) whose definition is given by:

\[
(A_{k,S_{ik_1}}\mu)(x) = \frac{\mu(x)}{2} - (K_k\mu)(x) - i\eta(N_k \circ S_{ik_1})\mu(x). \] (2.16)

Besides the operators \( \mathcal{R} = S_{ik_1} \), another natural choice of operators \( \mathcal{R} \) that satisfy the properties (1) and (2) in Theorem 2.1 is given by \( \mathcal{R} = PS_{ik_1} \) where \( PS_{ik_1} \) represents the principal symbol of the operator \( S_{ik_1} \) in the sense of pseudodifferential operators [47]. Before giving the explicit definition of these operators, we briefly review the concept of principal symbols of pseudodifferential operators on a smooth and periodic curve \( \Gamma \). Given that we can assume that the smooth and closed curve \( \Gamma \) can be represented via a smooth and \( 2\pi \) periodic parametrization, we restrict our exposition to periodic pseudodifferential operators, following the presentation in [42]. In this context, linear and bounded operators \( A \) between Sobolev spaces of \( 2\pi \) periodic functions, that is \( A : H^s[0,2\pi] \to H^s[0,2\pi] \), have the following spectral representation [42]

\[
(Au)(t) = \sum_{n \in \mathbb{Z}} \sigma(t,n) \hat{u}(n)e^{int}, \quad \sigma(t,n) = e_{-n}(t)(Ae_n)(t),
\]

where \( e_n(t) = e^{int} \) and \( \hat{u}(n) \) are the Fourier coefficients of the function \( u \), that is

\[
\hat{u}(n) = \frac{1}{2\pi} \int_{0}^{2\pi} u(x)e^{-inx} dx.
\]

A linear operator \( A \) of the kind described above is a periodic pseudodifferential operator of order \( \alpha \), denoted by \( A \in OPS(\alpha) \), if its symbol \( \sigma(t,n) \) is the restriction to \( \mathbb{R} \times \mathbb{Z} \) of a function \( \sigma(t,\xi) \) which is smooth in \( \mathbb{R} \times \mathbb{R} \), \( 2\pi \) periodic in \( t \) and satisfies the inequalities

\[
\left| \left( \frac{\partial}{\partial t} \right)^k \left( \frac{\partial}{\partial \xi} \right)^l \sigma(t,\xi) \right| \leq c_{k,l}(1 + |\xi|)^{\alpha - l}, \quad k, l \in \mathbb{N},
\]

in which case we say that \( \sigma(t,n) \in \Sigma^\alpha \). It follows easily form their definition that operators \( A \in OPS(\alpha) \) are bounded from \( H^s([0,2\pi]) \) to \( H^{s-\alpha}([0,2\pi]) \) [42, 47]. If the symbol \( \sigma \) of an operator \( A \in OPS(\alpha) \) is such that there exist \( \alpha = \alpha_0 > \alpha_1 > \ldots > \alpha_j \to -\infty \) with the property that \( \sigma - \sum_{j=0}^{N-1} \sigma_j \in \Sigma^{\alpha_N} \) for all \( N \in \mathbb{N} \), then \( \sigma_0 \) is called the principal symbol of \( \sigma \). Using a global smooth and periodic parametrization of the closed curve \( \Gamma \) in terms of the arclength, i.e. a diffeomorphism \( x : [0,2\pi] \to \Gamma \), the operator \( A \) can be lifted to an operator acting on periodic functions \( \tilde{u} \) defined on \( \Gamma \) by \( A^* \tilde{u} = x^{-1} \circ A(u \circ x) \). Using the change of variables formula for the
principal symbol of $A$ [47], the principal symbol of the pseudodifferential operator $A^\sigma$ can be also defined.

The connection between boundary integral operators of potential theory and pseudodifferential operators is well known [14, 46, 47]. For instance, it can be shown that the single layer operator $S_{ik_1}$ is a periodic pseudodifferential operator of order $-1$ on $\Gamma$ whose principal symbol equals $\sigma^0_{S_{ik_1}} = \frac{1}{2}(k_1^2 + |\xi|^2)^{-\frac{1}{2}}$ (see Appendix for a proof of this fact), that is:

$$
(\mathcal{R}\phi)(x(t)) = (PS_{ik_1}\phi)(x(t)) = \sum_{n \in \mathbb{Z}} \sigma^0_{k_1}(n)\hat{\psi}(n)e^{i\eta x}, \quad \sigma^0_{k_1}(n) = \frac{1}{2}(k_1^2 + n^2)^{-\frac{1}{2}}, \quad (2.17)
$$

where $\hat{\psi}(n)$ are the Fourier coefficients of the function $\phi(x(t))|x'(t)|$. The difference between $S_{ik_1}$ and $PS_{ik_1}$ is a pseudodifferential operator of order $-3$ (since the next term in the asymptotic expansion of the symbol $\sigma_{S_{ik_1}}$ is $\Sigma^{-3}$ [35]) and hence $S_{ik_1} - PS_{ik_1} : H^{m-\frac{3}{2}}(\Gamma) \rightarrow H^{m+\frac{3}{2}}(\Gamma)$ (we prove this fact in an Appendix). Thus, we have $N_k \circ PS_{ik_1} = N_k \circ S_{ik_1} - N_k \circ (S_{ik_1} - PS_{ik_1})$. The last operator in this identity maps $H^{m-\frac{3}{2}}(\Gamma)$ to $H^{m+\frac{3}{2}}(\Gamma)$ and thus is compact as an operator from $H^{m-\frac{3}{2}}(\Gamma)$ to itself. Consequently, since condition (1) in Theorem 2.1 is satisfied by $\mathcal{R} = S_{ik_1}$, it is also satisfied given the choice $\mathcal{R} = PS_{ik_1}$. Furthermore, the operator $PS_{ik_1}$ is strongly elliptic since its symbol satisfies $\frac{1}{2}(k_1^2 + |\xi|^2)^{-\frac{1}{2}} \geq \frac{1}{|\mathcal{R}|}$ for large enough $|\xi|$, and thus it is coercive on $H^{-\frac{3}{2}}(\Gamma)$ based on Gårding inequality [42, 47] (another proof of properties (1) and (2) can be obtained based on representations of the operators in parametric form that we present in Section 4). Thus, this second choice of regularizing operator $\mathcal{R} = PS_{ik_1}$ also satisfies conditions (1) and (2) in Theorem 2.1. Given the choice $\mathcal{R} = PS_{ik_1}$, we denote the corresponding regularized integral operators by $A_{k, PS_{ik_1}}$; their definition is given by:

$$
(A_{k, PS_{ik_1}}\mu)(x) = \frac{\mu(x)}{2} - (K_{k_1}^\sigma)(x) - i\eta(N_k \circ PS_{ik_1})\mu(x). \quad (2.18)
$$

We denote the equations (2.14) when the choice of the regularizing operators is $\mathcal{R} = S_{ik_1}$ by the label ICFIE-R —which corresponds to the operators $A_{k, S_{ik_1}}$ defined in equation (2.16), and by the label ICFIE-RPS when the choice of regularizing operators is $\mathcal{R} = PS_{ik_1}$ —which corresponds to the operators $A_{k, PS_{ik_1}}$ defined in equation (2.18). We mention that the idea of using regularizing operators based on principal symbols of pseudodifferential operators was first introduced in [35]. The regularizing operators in [35] consist of suitable modifications via quadratic partitions of unity on $\Gamma$ of the principal symbol of the Neumann-to-Dirichlet operator for circular geometries, which is different from our choice. In the next Section we investigate certain norm properties of the ICFIE-R and ICFIE-RPS operators and their inverses in connection with the wavenumbers $k$ and $k_1$ and the coupling parameter $\eta$.

## 3 Wavenumber dependence of the norms of the regularized combined field integral operators

The selection of the coupling parameter $\eta$ and the wavenumber $k_1$ in the definition of the regularizing operators $\mathcal{R} = S_{ik_1}$ and $\mathcal{R} = PS_{ik_1}$ in the ICFIE-R and ICFIE-RPS operators (2.16) and (2.18) is guided by considerations the spectral properties of these operators. In our previous effort [9], we presented numerical evidence that the choice $\eta = 2$ and $k_1 = k$ leads to small numbers of
Krylov subspace iterations. In general, the choice of these parameters is driven by the spectral properties of the ensuing combined field integral operators in the case of circular (or spherical) geometries [3, 32]. In this case, all of the acoustic boundary integral operators that enter the formulations (2.16) and (2.18) are diagonalizable in the (complete) basis given by the Fourier harmonics \( \{ e^{im\theta}, m \in \mathbb{Z} \} \) and the eigenvalues of these operators can be computed explicitly. In this section we investigate the spectral properties of the ICFIE-R and ICFIE-RPS operators, with an emphasis on explicit wavenumber estimates on the condition numbers of these operators in the case of circular and spherical geometries. In this case, we show that in the high-frequency regime, and for certain choices of the coupling parameter \( \eta \) and the wavenumber \( k_1 \) in the definition of the regularizing operators, the ICFIE-R and ICFIE-RPS operators defined in equations (2.16) and (2.18) enjoy similar condition number properties to their CFIE sound-soft counterparts. The latter properties are as follows: the norms of the sound-soft CFIE operators can be uniformly bounded in \( k \) with constants that depend on the coupling constant [6], and for certain choices of the coupling constants, the sound-soft CFIE operators are \( L^2 \) coercive with coercivity constants that do not depend on \( k \) [27].

First, our numerical evidence suggests that the choice \( k_1 = k \) in the definition of the regularizing operators \( R = S_{ik_1} \) and \( R = PS_{ik_1} \) leads to operators ICFIE-R and ICFIE-RPS with superior spectral properties, and this is the choice that we use in the analysis that follows. We note that this choice is not essential, as the results that we will establish hold for the more general case when \( k_1 = ck \), where \( c > 0 \) is a constant that does not depend on \( k \). Our analysis is based on the spectral properties of the boundary integral operators \( S_k, K'_k, N_k [32] \), whose eigenvalues/eigenvectors can be computed explicitly in the case when \( \Gamma = S^1 \), where \( S^1 \) denotes the unit circle in two-dimensions:

\[
S_k e^{im\theta} = \frac{i\pi}{2} j_{|m|}(k) H^{(1)}_{|m|}(k)e^{im\theta},
\]

\[
K'_k e^{im\theta} = -\frac{1}{2} + \frac{i\pi k}{2} j_{|m|}'(k) H^{(1)}_{|m|}(k) e^{im\theta},
\]

\[
N_k e^{im\theta} = \frac{i\pi}{2} k^2 j_{|m|}'(k)(H^{(1)}_{|m|}(k))' e^{im\theta} \tag{3.1}
\]

in terms of the Bessel and Hankel functions of order \(|m|\) and argument \( k \) and their derivatives with respect to \( k \). The integral operators ICFIE-R and ICFIE-RPS defined in equations (2.16) and (2.18) are diagonalizable in the complete basis of Fourier harmonics and we get that the eigenvalues of the ICFIE-R operators are given by

\[
A_{k,S_{ik}} e^{im\theta} = a_m(k) e^{im\theta}, a_m(k) = 1 - \frac{i\pi k}{2} j_{|m|}'(k) H^{(1)}_{|m|}(k) + \frac{i\pi^2 k^2}{4} j_{|m|}'(k)(H^{(1)}_{|m|}(k))' j_{|m|}(ik) H^{(1)}_{|m|}(ik) \tag{3.2}
\]

while the eigenvalues of the ICFIE-RPS operators are given by

\[
A_{k,PS_{ik}} e^{im\theta} = p_m(k) e^{im\theta}, p_m(k) = 1 - \frac{i\pi k}{2} j_{|m|}'(k) H^{(1)}_{|m|}(k) + \frac{\eta \pi k^2}{4} j_{|m|}'(k)(H^{(1)}_{|m|}(k))'(m^2 + k^2)^{-\frac{1}{2}}. \tag{3.3}
\]
3.1 Uniform bounds on the norms of the operators ICFIE-R and ICFIE-RPS in the case of circular and spherical geometries

In this subsection we investigate the uniform boundedness in wavenumber $k$ of the operators ICFIE-R and ICFIE-RPS in the case when $\Gamma = S^1$. We start with a useful result

**Lemma 3.1** There exist two constants $C_1 > 0$ and $C_2 > 0$ and a number $k_0 > 0$ such that

$$C_1(m^2 + k^2)^{-1/2} \leq i J_m(ik) H_m^{(1)}(ik) \leq C_2(m^2 + k^2)^{-1/2}$$

for all $k > k_0$ and all $m \geq 0$.

**Proof.** We begin by using the representation of the functions $J_m(ik)$ and $H_m^{(1)}(ik)$ in terms of the Bessel functions of the third kind $J_m(ik)H_m^{(1)}(ik) = \frac{2}{\pi i} I_m(k) K_m(k)$ where both $I_m(k) > 0$ and $K_m(k) > 0$. Using the uniform asymptotic expansions as $\nu \to \infty$ (Formulas 9.7.7 and 9.7.8 in [1])

$$I_\nu(\nu z) \sim \frac{1}{\sqrt{2\pi \nu}} \frac{e^{\nu z}}{(1 + z^2)^{1/4}}(1 + O(\nu^{-1}))$$

$$K_\nu(\nu z) \sim \frac{1}{\sqrt{\pi \nu z}} e^{-\nu z} (1 + O(\nu^{-1}))$$

(3.4)

where $\mu = \sqrt{1 + z^2 + \ln \frac{z}{1 + \sqrt{1 + z^2}}}$, we get that there exists a constant $M_0 > 0$ such that

$$\frac{1}{4}(m^2 + k^2)^{-1/2} \leq I_m(k)K_m(k) \leq (m^2 + k^2)^{-1/2}, \text{ for all } m > M_0, k > 0.$$  

Using the asymptotic expansion (Formula 9.7.5 in [1]) which is valid for $\nu$ fixed and $z \to \infty$

$$I_\nu(z)K_\nu(z) \sim \frac{1}{2z} \left(1 - \frac{1}{2} \frac{\mu - 1}{(2z)^2} + O \left(\frac{\mu}{z^4}\right)\right), \quad \mu = 4\nu^2$$

(3.6)

we get that for a fixed $m$ there exists $k_m$ such that

$$\frac{1}{4k} \leq I_m(k)K_m(k) \leq \frac{1}{k} \text{ for all } k \geq k_m.$$  

(3.7)

It follows from the estimate (3.7) that there exist $k_0 = \max_{m \leq M_0} k_m$ and $c = (1 + M_0^2/k_0^2)^{1/2}$ such that

$$\frac{1}{4}(m^2 + k^2)^{-1/2} \leq \frac{1}{4k} \leq I_m(k)K_m(k) \leq \frac{1}{k} \leq c(m^2 + k^2)^{-1/2}, \text{ for all } 0 \leq m \leq M_0, k \geq k_0.$$  

(3.8)

We obtain by combining estimates (3.5) and (3.8) that there exist constants $C_1 = \frac{1}{2\pi}$, $C_2 = \frac{2}{\pi} \max(1, (1 + M_0^2/k_0^2)^{1/2})$, and $k_0 > 0$ such that

$$C_1(m^2 + k^2)^{-1/2} \leq i J_m(ik) H_m^{(1)}(ik) = \frac{2}{\pi} I_m(k)K_m(k) \leq C_2(m^2 + k^2)^{-1/2}$$

(3.9)

for all $m \geq 0$ and all $k \geq k_0$.  

We prove the following result which can be viewed as the sound-hard counterpart of a result established in [6] in the sound-soft case (all the norms in the subsequent results are the $2$ norm unless specified otherwise):
Theorem 3.2 When $\Gamma$ is the unit circle in $\mathbb{R}^2$, then there exist $k_1 > 0$ and $C > 0$ independent of $k$ such that

$$\|A_{k, S_{ik}}\|_{L^2(\mathbb{S}^1) \rightarrow L^2(\mathbb{S}^1)} \leq C(1 + |\eta|), \quad \|A_{k, P_{S_{ik}}}\|_{L^2(\mathbb{S}^1) \rightarrow L^2(\mathbb{S}^1)} \leq C(1 + |\eta|)$$

for all $k_1 \leq k$.

Proof. Given that in the case $\Gamma = \mathbb{S}^1$ the set $\{e^{im\theta} : m \in \mathbb{Z}\}$ constitutes an orthonormal basis of $L^2(\Gamma)$ and that both operators $A_{k, S_{ik}}$ and $A_{k, P_{S_{ik}}}$ are diagonalizable in this basis, the result of the theorem follows immediately once we established that there exists $k_1 > 0$ such that

$$|a_m(k)| \leq C(1 + |\eta|), \quad |p_m(k)| \leq C(1 + |\eta|), \text{ for all } m \in \mathbb{Z}, \ k \geq k_1. \quad (3.10)$$

We note that it is enough to prove the estimates (3.10) in the case $m \geq 0$. We establish first the bound on $|p_m(k)|$. In order to show that there exists $k_1 > 0$ such that $|p_m(k)|$ is uniformly bounded in $k$ for all $k > k_1$, we use several estimates established in Lemma 3.9, Proposition 3.10 in [6], Lemma 4.9, and Lemma 4.10 in [27]. First, we use the result in Proposition 3.10 in [6] which established that there exists a constant $C$ independent of $k$ such that for all $m \geq 0$ and for any $k \geq 1$

$$|kJ'_m(k)H^1_m(k)| \leq C. \quad (3.11)$$

Using asymptotic expansions for fixed $m$ as $k \rightarrow \infty$ of Bessel and Hankel functions [1] (Formulas 9.2.11, 9.2.11, 9.2.12)

$$J_m(k) = \sqrt{\frac{2}{\pi k}} \cos(k - \frac{m\pi}{2} - \frac{\pi}{4}) + O(k^{-1})$$

$$J'_m(k) = -\sqrt{\frac{2}{\pi k}} \sin(k - \frac{m\pi}{2} - \frac{\pi}{4}) + O(k^{-1})$$

$$Y'_m(k) = \sqrt{\frac{2}{\pi k}} \cos(k - \frac{m\pi}{2} - \frac{\pi}{4}) + O(k^{-1}) \quad (3.12)$$

we obtain that for a fixed $m$, the quantities $k^2J'_m(k)(H^1_m(k))'(m^2 + k^2)^{-\frac{1}{2}}$ are bounded for all $k$ sufficiently large. In what follows, we will denote by $\nu$ the orders of the Bessel functions. Since for all orders $\nu \geq 0$ and all $k > 0$, the functions $J'_\nu(k)(H^1_\nu(k))'$ are continuous in $k$ and $\nu$, it follows that it suffices to investigate the uniform boundedness of the products $k^2J'_\nu(k)(H^1_\nu(k))'(\nu^2 + k^2)^{-\frac{1}{2}}$ for large enough orders $\nu$. We proceed, just as in the proof of Proposition 3.10. in [6], by distinguishing two cases.

Case 1: $\nu > k \geq 1/2$. We use the result from Lemma 3.9. in [6] which established that $kJ'_\nu(k)$ is a positive increasing function of $k$ for $0 < k < \nu$. It follows that

$$\frac{k^2(J'_\nu(k))^2}{(\nu^2 + k^2)^{\frac{1}{2}}} \leq \nu^{-1}k^2(J'_\nu(k))^2 \leq \nu(J'_\nu(\nu))^2.$$ 

We use the following asymptotic expansions [1] (Formulas 9.3.33) valid for all $\nu > 1/2$:

$$J'_\nu(\nu) = b\nu^{-2/3} - c\nu^{-4/3} + O(\nu^{-8/3}) \quad (3.13)$$

where $b$ and $c$ are positive constants. Using this result in the estimate above we obtain that for large enough $\nu$

$$\frac{k^2(J'_\nu(k))^2}{(\nu^2 + k^2)^{\frac{1}{2}}} \leq \tilde{C}\nu^{-1/3} \leq C. \quad (3.14)$$

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Using the identity [1] (Formulas 9.1.27)

\[ J'_\nu(k) = \frac{\nu}{k} J_\nu(k) - J_{\nu+1}(k), \]  

we have

\[ \frac{k^2}{(\nu^2 + k^2)^{\frac{3}{2}}} |J'_\nu(k)Y'_\nu(k)| \leq \frac{\nu k}{(\nu^2 + k^2)^{\frac{3}{2}}} |J_\nu(k)Y'_\nu(k)| + \frac{k^2}{(\nu^2 + k^2)^{\frac{3}{2}}} |J_{\nu+1}(k)Y'_\nu(k)|. \]  

(3.16)

It was established in Lemma 4.10 in [27] that

\[ 0 \leq J_{\nu+1}(k) \leq J_\nu(k), \quad k \in [0, \nu] \]  

(3.17)

and therefore it follows that

\[ \frac{k^2}{(\nu^2 + k^2)^{\frac{3}{2}}} |J'_\nu(k)Y'_\nu(k)| \leq \frac{\nu + k}{(\nu^2 + k^2)^{\frac{3}{2}}} |kJ_\nu(k)Y'_\nu(k)|. \]  

(3.18)

Using the identity for the Wronskian [1] (Formula 9.1.16)

\[ J_\nu(k)Y'_\nu(k) - J'_\nu(k)Y_\nu(k) = \frac{2}{\pi k} \]  

(3.19)

together with the estimate (3.11) in equation (3.16) we get the following estimate

\[ \frac{k^2}{(\nu^2 + k^2)^{\frac{3}{2}}} |J'_\nu(k)Y'_\nu(k)| \leq C \frac{\nu + k}{(\nu^2 + k^2)^{\frac{3}{2}}} \leq \sqrt{2}C. \]  

(3.20)

Combining estimates (3.14) and (3.20) we obtain that there exists a constant \( C' \) independent of \( k \) such that for large enough \( \nu \) we have the following estimate

\[ \frac{k^2}{(\nu^2 + k^2)^{\frac{3}{2}}} |J'_\nu(k)(H^{(1)}_\nu(k))'| \leq C' \text{ for } k < \nu. \]  

(3.21)

Case 2: \( \nu \leq k \). We define

\[ N_\nu(k) := |(H^{(1)}_\nu)'(k)| = \sqrt{(J'_\nu(k))^2 + (Y'_\nu(k))^2} \]  

(3.22)

and use the result established in Proposition 4.7 in [27], that is there exists \( k_1 \) such as for all \( \nu \) and \( k \) with \( k_1 \leq \nu \leq k \) we have

\[ kN^2_\nu(k) \leq \frac{4}{\pi}. \]  

(3.23)

It follows then that

\[ \frac{k^2}{(\nu^2 + k^2)^{\frac{3}{2}}} (J'_\nu(k))^2 \leq \frac{k}{(\nu^2 + k^2)^{\frac{3}{2}}} (kN^2_\nu(k)) \leq \frac{4}{\pi}. \]  

(3.24)

We also immediately obtain

\[ \frac{k^2}{(\nu^2 + k^2)^{\frac{3}{2}}} |J'_\nu(k)Y'_\nu(k)| \leq \frac{k}{2(\nu^2 + k^2)^{\frac{3}{2}}} (kN^2_\nu(k)) \leq \frac{2}{\pi}. \]  

(3.25)
Combining the estimates (3.24) and (3.25) we get that there exists $k_1 > 0$ and a constant $C''$ independent of $k$ such that

$$\frac{k^2}{(\nu^2 + k^2)^{\frac{3}{2}}} |J'_\nu(k)(H^{(1)}_\nu(k))'| \leq C'' \text{ for all } k \geq \nu \geq k_1.$$  

(3.26)

Combining estimates (3.11), (3.21), and (3.26), the result of the Theorem concerning $|p_m(k)|$ follows. Furthermore, using the result in Lemma 3.1 the similar bound on $|a_m(k)|$ also holds. 

\[\square\]

**Remark 3.3** It follows immediately from estimate (3.10) that for any $0 < p$, there exist $k_1 > 0$ and $C_p > 0$ independent of $k$ such that

$$\|A_{k,S} \|_{H^p(S^1) \rightarrow H^p(S^1)} \leq C_p(1 + |\eta|), \quad \|A_{k,PS} \|_{H^p(S^1) \rightarrow H^p(S^1)} \leq C_p(1 + |\eta|).$$  

(3.27)

The bounds derived in the case of circular geometries can be extended easily to the three-dimensional case. The operators $A_{k,S}$ defined in equation (2.16) can be extended to the three dimensional setting by replacing the two-dimensional Green’s function with the suitable three-dimensional Green’s function $G_k(z) = \frac{\varepsilon^{ik|z|}}{4\pi|z|}$, and by replacing the closed curve $\Gamma$ by a smooth and closed surface $S$ in three dimensions; we denote by $B_{k,S}$ the ensuing three dimensional ICFIE-R operators. These operators have been introduced in [9] where a variety of numerical results demonstrated the benefits of using integral equations based on the use of the operators $B_{k,S}$ operators in the high frequency regime in three dimensions. In the case when $S$ is the unit sphere in three dimensions, $S = S^2$, the operators $B_{k,S}$ are diagonalizable in the orthonormal basis of $L^2(S^2)$ given by the spherical harmonics $\{Y^\ell_m(\theta, \phi) : 0 \leq m, -m \leq \ell \leq m\}$ where $\theta$ and $\phi$ represent the spherical coordinates on $S^2$. Indeed, the eigenvalues of the boundary integral operators that make up the operators $B_{k,S}$ can be computed explicitly in terms of the spherical Bessel and Hankel functions [39]

$$S_k Y^\ell_m = ikj_m(k)h^{(1)}_m(k)Y^\ell_m$$

$$K^\ell_k Y^\ell_m = \left(-\frac{1}{2} + ik^2j'_m(k)h^{(1)}_m(k)\right)Y^\ell_m$$

$$N_k Y^\ell_m = ik^2j'_m(k)(h^{(1)}_m)'(k)Y^\ell_m.$$  

(3.28)

It follows that the eigenvalues of the operators $B_{k,S}$ are given by

$$B_{k,S} Y^\ell_m = b_m(k)Y^\ell_m, \quad b_m(k) = 1 - ik^2j'_m(k)h^{(1)}_m(k) - \eta k^4 j'_m(k)(h^{(1)}_m(k))'j_m(ik)(h^{(1)}_m(ik)).$$  

(3.29)

Given that $j_m(k) = \sqrt{\frac{2}{2k}}J_{m+\frac{1}{2}}(k)$ and $h^{(1)}_m(k) = \sqrt{\frac{2}{2k}}H^{(1)}_{m+\frac{1}{2}}(k)$ [1] we obtain

$$b_m(k) = a_{m+\frac{1}{2}}(k) + \frac{i\eta \pi^2}{4} J_{m+\frac{1}{2}}(k)H^{(1)}_{m+\frac{1}{2}}(k) - \frac{ik\eta \pi^2}{8} J_{m+\frac{1}{2}}(ik)H^{(1)}_{m+\frac{1}{2}}(ik) \times$$

$$\times \left[J_{m+\frac{1}{2}}(k)H^{(1)}_{m+\frac{1}{2}}(k) + J_{m+\frac{1}{2}}(k)(H^{(1)}_{m+\frac{1}{2}}(k))' - \frac{1}{2k} J_{m+\frac{1}{2}}(k)H^{(1)}_{m+\frac{1}{2}}(k)\right].$$  

(3.30)

We use the following two estimates which were established in [6] (Proposition 3.10): there exists a constant $C > 0$ independent of $k$ such that $|J_\nu(k)H^{(1)}_\nu(k)| \leq Ck^{-\frac{3}{2}}$ and $|J'_\nu(k)H^{(1)}_\nu(k)| \leq C$ for
Figure 1: Plots of $\min_m(\Re(a_m(k)))$ for 5041 values of $k$, namely $k = 8, 8.1, 8.2, \ldots, 512$ with $\eta = 4$ (left) and $\eta = k^{1/3}$ (right). The plots of $\min_m(\Re(p_m(k)))$ are extremely similar.

sufficiently large $k$. From the second estimate above and the Wronskian identity (3.19) we get that $|kJ_\nu(k)(H_\nu^{(1)})'(k)| \leq C$ as well. Using these estimates together with those established in Lemma 3.1 and Theorem 3.4, we obtain that there exist constants $C > 0$, $C_1 > 0$ and a constant $k'_1 > 0$ such that

$$|b_m(k)| \leq C(1 + |\eta|) + C|\eta|k^{-1} + C|\eta|k^{-\frac{5}{3}} \leq C_1(1 + |\eta|) \text{ for all } k'_1 \leq k.$$ 

It follows from the estimate above that

**Theorem 3.4** If $S$ is the unit sphere in $\mathbb{R}^3$, $S = \mathbb{S}^2$, then for all $0 \leq p$, there exist $k'_1 > 0$ and $C'_p > 0$ independent of $k$ such that

$$\|B_{k,S_{ik}}\|_{H^p(\mathbb{S}^2) \to H^p(\mathbb{S}^2)} \leq C'_p(1 + |\eta|) \text{ for all } k'_1 \leq k.$$ 

### 3.2 Coercivity of the operators ICFIE-R and ICFIE-RPS in the case of circular and spherical geometries

In this section we study the $L^2$ coercivity of the operators $A_{k,S_{ik}}$ and $A_{k,PS_{ik}}$ in the case when $\Gamma = \mathbb{S}^1$. We will show that, and just as in the Dirichlet case [27], for certain choices of the coupling parameter $\eta$, the regularized combined field integral operators $A_{k,S_{ik}}$ and $A_{k,PS_{ik}}$ are coercive in $L^2(\Gamma)$ in the case when $\Gamma = \mathbb{S}^1$. We illustrate this fact in Figure 1 where we plot $\min_m(\Re(a_m(k)))$ for values $k = 8, 8.1, 8.2, \ldots, 512$ for two values of the coupling parameter $\eta = 4$ and $\eta = k^{1/3}$ respectively.

In order to probe the coercivity of the operators $A_{k,S_{ik}}$ and $A_{k,PS_{ik}}$ in the case when $\Gamma$ is the unit circle in two dimensions, we start with the following representation which follows immediately from the definition of $p_m(k)$ given in equation (3.3)

$$\Re(p_m(k)) = \frac{\pi k}{2} \left( \frac{2}{\pi k} + J'_m(k)Y_m(k) + \frac{k\eta}{2(m^2 + k^2)^{\frac{3}{2}}} (J'_m(k))^2 \right). \quad (3.31)$$

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In addition to the equation (3.31) we use an alternative representation of \( \Re(p_m(k)) \) which follows immediately from the formula for the Wronskian (3.19)

\[
\Re(p_m(k)) = \frac{\pi k}{2} \left( J_m(k)Y'_m(k) + \frac{k\eta}{2(m^2 + k^2)^{\frac{3}{2}}} (J'_m(k))^2 \right).
\]

(3.32)

Using the asymptotic expansions (3.12), we obtain for a fixed \( m \) there exists \( k_m \) large enough such that for all \( k > k_m \) the following estimate holds true

\[
\Re(p_m(k)) = \cos^2(k - \frac{m\pi}{2} - \frac{\pi}{4}) + \frac{k\eta}{2(m^2 + k^2)^{\frac{3}{2}}} \sin^2(k - \frac{m\pi}{2} - \frac{\pi}{4}) + O(k^{-1/2})
\]

\[
\geq \min(1, \frac{\eta}{2\sqrt{2}}) + O(k^{-1/2}) > 0.
\]

(3.33)

The estimate established in equation (3.33) can be viewed as an asymptotic confirmation of the coercivity of the operators \( A_{k,PS_i} \). In order to establish rigorously the coercivity property of the operators \( A_{k,S_i} \) and \( A_{k,PS_i} \) in the case when \( \Gamma \) is the unit circle, we will examine the positivity of the real part of the eigenvalues \( a_m(k) \) and \( p_m(k) \) for large enough \( k \) and for all modes \( 0 \leq m \). To this end, we first state a useful result whose technical proof follows closely the lines of the proof of Lemma 4.5 in [27]

**Lemma 3.5** There exist \( \varepsilon_1 > 0 \) and \( m_1 > 1 \) such that for all \( m \geq m_1 \) and \( k \in (0, m] \) the function \( f_m(k) = kJ_m(k)Y'_m(k) \) has a unique local extremum point that lies in the interval \( (m - \varepsilon_1 m^{\frac{1}{2}}, m) \). This local extremum point is a global minimum of the function \( f_m \) in the interval \( (0, m] \).

**Proof.** Given that the functions \( C_m \) are solutions of the differential equation \( k^2 C'_m(k) + kC'_m(k) + (k^2 - m^2)C_m(k) = 0 \) where \( C_m \) denote either \( J_m \) or \( Y_m \) we obtain

\[
f'_m(k) = J_m(k)Y'_m(k) + kJ'_m(k)Y'_m(k) + kJ_m(k)Y''_m(k)
\]

\[
= J_m(k)Y'_m(k) + kJ'_m(k)Y'_m(k) + kJ_m(k)\left(-\frac{1}{k}Y'_m(k) + \left(\frac{m^2}{k^2} - 1\right)Y_m(k)\right)
\]

\[
= kJ'_m(k)Y'_m(k) + \left(\frac{m^2}{k} - k\right) J_m(k)Y'_m(k)
\]

(3.34)
\[ f''_m(k) = J'_m(k)Y'_m(k) + kJ''_m(k)Y'_m(k) + kJ'_m(k)Y''_m(k) - \left( \frac{m^2}{k^2} + 1 \right) J_m(k)Y_m(k) \]
\[
+ \left( \frac{m^2}{k} - k \right) (J_m(k)Y_m(k))' \\
= J'_m(k)Y'_m(k) + kY'_m(k) \left( -\frac{1}{k} J'_m(k) + \left( \frac{m^2}{k^2} - 1 \right) J_m(k) \right) \\
+ kJ'_m(k) \left( -\frac{1}{k} Y'_m(k) + \left( \frac{m^2}{k^2} - 1 \right) Y_m(k) \right) \\
- \left( \frac{m^2}{k^2} + 1 \right) J_m(k)Y_m(k) + \left( \frac{m^2}{k} - k \right) (J_m(k)Y_m(k))' \\
= -J'_m(k)Y'_m(k) - \left( \frac{m^2}{k^2} + 1 \right) J_m(k)Y_m(k) + 2 \left( \frac{m^2}{k} - k \right) (J_m(k)Y_m(k))' \\
(3.35) \]

and
\[ f'''_m(k) = -J'''_m(k)Y'_m(k) - J'_m(k)Y'''_m(k) + \frac{2m^2}{k^3} J_m(k)Y_m(k) - \left( \frac{m^2}{k^2} + 1 \right) (J_m(k)Y_m(k))' \\
- 2 \left( \frac{m^2}{k^2} + 1 \right) (J_m(k)Y_m(k))' + 2 \left( \frac{m^2}{k} - k \right) (J_m(k)Y_m(k) + J_m(k)Y'_m(k) + 2J'_m(k)Y'_m(k)) \\
= \left( 1 - \frac{m^2}{k^2} \right) J_m(k)Y'_m(k) + \frac{1}{k} J'_m(k)Y'_m(k) + \left( 1 - \frac{m^2}{k^2} \right) J'_m(k)Y'_m(k) + \frac{1}{k} J''_m(k)Y'_m(k) \\
+ \frac{2m^2}{k^3} J_m(k)Y_m(k) - 3 \left( \frac{m^2}{k^2} + 1 \right) (J_m(k)Y_m(k))' \\
+ 2 \left( \frac{m^2}{k} - k \right) \left( 2 \left( \frac{m^2}{k^2} - 1 \right) J_m(k)Y_m(k) - \frac{1}{k} J'_m(k)Y_m(k) - \frac{1}{k} J_m(k)Y'_m(k) + 2J'_m(k)Y'_m(k) \right) \\
= \frac{2m^2 + 4(m^2 - k^2)^2}{k^3} J_m(k)Y_m(k) + \left( \frac{2}{k} + 4 \frac{m^2}{k} - 4k \right) J'_m(k)Y'_m(k) - \frac{6m^2}{k^2} (J_m(k)Y_m(k))'. \\
(3.36) \]

If we take into account equation (3.15) and we make use again of the formulas [1] (Formulas 9.1.27)
\[ Y'_m(k) = Y_{m-1}(k) - \frac{m}{k} Y_m(k), \quad m \geq 1 \]

in the definition of the functions \( f_m \) and in equations (3.34) and (3.35), we obtain equivalent expressions for the functions \( f_m, f'_m \) and \( f'''_m \) in a form that allows us to evaluate these functions at
We will study next the properties of the values \( k \) in formulas (3.37) we obtain that for all \( m \) where the function \( f \) obtain since

\[
\begin{align*}
J_m(z) &= \frac{1}{m!} \left( \frac{z}{2} \right)^m + O(z^{m+1}), \\
Y_m(z) &= -\frac{(m-1)!}{\pi} \left( \frac{z}{2} \right)^{-m} + O(z^{-m+1}), \quad |z| \to 0
\end{align*}
\]

in formulas (3.37) we obtain that for all \( m \geq 2 \)

\[
f_m(0) = \frac{1}{\pi}, \quad f'_m(0) = 0, \quad f''_m(0) = -\frac{1}{\pi m(m^2 - 1)} < 0.
\]

It follows then that there exists \( \eta_m > 0 \) (which may depend on \( m \)) such that \( f'_m(k) < 0 \) for \( k \in [0, \eta_m] \).

We will study next the properties of the values \( f_m(k) \) in the case of large enough orders \( m \) and arguments \( k \) that are close to \( m \). We employ the expansions [1] (Formulas 9.3.33) valid for \( m \) large:

\[
\begin{align*}
J_m(m) &= am^{-1/3} + O(m^{-5/3}) \\
Y'_m(m) &= \sqrt{3}(bm^{-2/3} + cm^{-4/3}) + O(m^{-8/3})
\end{align*}
\]

(3.38)

for positive constants \( a, b, \) and \( c \). We obtain

\[
f_m(m) = ab\sqrt{3} + O(m^{-1/3}) = \frac{1}{\pi} + O(m^{-1/3}), \quad m \to \infty
\]

since \( ab = \frac{2}{\sqrt{3\pi(1/3)\Gamma(2/3)}} = \frac{1}{\sqrt{3\pi}} \) according to Formulas 9.3.31 and 9.3.34 in [1]. Furthermore, we obtain \( f'_m(m) = 2^4 m^{3/2} / (3^{1/6}\Gamma(2/3)^2) + O(m^{-4/3}) \), \( m \to \infty \), and thus \( f''_m(m) > 0 \) for large enough \( m \). For any fixed \( \varepsilon > 0 \) we use the transitional asymptotics in Formulas 9.3.23–9.3.28 in [1] to obtain

\[
f'_m(m - \varepsilon m^{1/3}) = -2^4 \varepsilon^{1/3} \varphi(\varepsilon) + O(m^{-1/3})
\]

(3.39)

where the function \( \varphi \) is defined in terms of Airy functions

\[
\varphi(\varepsilon) = Ai'(2^{1/3}\varepsilon)Bi'(2^{1/3}\varepsilon) + 2^{1/3}\varepsilon Ai(2^{1/3}\varepsilon)Bi(2^{1/3}\varepsilon).
\]

(3.40)
From Formulas 10.4.4-10.4.5 in [1] we get that \( \varphi(0) = -1/(3^{1/6}\Gamma(2/3)^2) < 0. \) Since \( \varphi(0) < 0, \) we get that there exists \( \varepsilon_2 > 0 \) such that \( \varphi(x) < 0 \) for all \( 0 \leq x \leq \varepsilon_2 \) and thus \( f_m''(m-xm^{\frac{1}{2}}) > 0 \) for large enough values of \( m \) and for all \( 0 \leq x \leq \varepsilon_2. \) We obtain consequently that \( m - \varepsilon_2m^{\frac{1}{2}} \leq \sqrt{m^2 - 1/4} \) for large enough values of \( m, \) and hence \( f_m'(k) > 0 \) for all \( k \in [\sqrt{m^2-1/4}, m] \) (the role of the expression \( \sqrt{m^2-1/4} \) will be apparent shortly).

If in addition to the fact that \( f_m'(k) > 0 \) for \( k \in [\sqrt{m^2-1/4}, m] \) we use the fact that \( f_m(k) < 0 \) for \( k \in [0, \eta_m], \) it follows that \( f_m \) has at least one critical point in the interval \((0, m]\) at which \( f_m' \) changes sign (i.e. a point of local extremum). Furthermore, the first such point must be a local minimum for \( f_m, \) and we denote it by \( \eta_1. \) We will show that the point of local extremum is unique in the interval \((0, m]\), and there exists \( \epsilon_1 > 0, \) independent of \( m, \) such that this local minimum lies in the interval \((m - \epsilon_1m^{\frac{1}{2}}, \sqrt{m^2-1/4}). \) To this end, we define the function

\[
g_m(k) = k^3 f_m'(k)
\]

and we obtain that the derivative of the function \( g_m \) can be written as

\[
g_m'(k) = 3k^2 f_m''(k) + k^3 f_m'''(k)
\]

\[
= -3k^2 J_m'(k)Y_m'(k) - 3(m^2 + k^2)J_m(k)Y_m(k) + 6(km^2 - k^3)(J_m(k)Y_m(k))' + (2m^2 + 4(m^2 - k^2)^2)J_m(k)Y_m(k) + (2k^2 + 4m^2k^2 - 4k^4)J_m'(k)Y_m'(k) - 6k^3m^2(J_m(k)Y_m(k))' + (4m^2k^2 - 4k^4 - k^2)J_m'(k)Y_m'(k) + (4(m^2 - k^2)^2 - m^2 - 3(2k^2)J_m(k)Y_m(k) - 6k^3(J_m(k)Y_m(k))').
\]

(3.41)

if we use formulas (3.35) and (3.36). On the other hand, using formula (3.34) we get

\[
-k(1 + 4k^2 - 4m^2) f_m'(k) = 6k^3(J_m(k)Y_m(k))' - 4k^2 J_m(k)Y_m(k)
\]

\[
= -(1 + 4k^2 - 4m^2)k^2 J_m'(k)Y_m'(k) + (1 + 4k^2 - 4m^2)(m^2 - k^2)J_m(k)Y_m(k) - 4k^2 J_m(k)Y_m(k) - 6k^3(J_m(k)Y_m(k))' + (4m^2k^2 - 4k^4 - k^2)J_m'(k)Y_m'(k) + (4(m^2 - k^2)^2 - m^2 - 3(2k^2)J_m(k)Y_m(k) - 6k^3(J_m(k)Y_m(k))').
\]

(3.42)

From formulas (3.41) and (3.42) it follows that the function \( g_m \) satisfies the following equation:

\[
g_m'(k) = -k(1 + 4k^2 - 4m^2) f_m'(k) - 6k^3(J_m(k)Y_m(k))' - 4k^2 J_m(k)Y_m(k).
\]

(3.43)

If we suppose that \( f_m \) has at least one more point of local extremum in the interval \((0, \sqrt{m^2-1/4}), \) we denote by \( \eta_2 \in (0, \sqrt{m^2-1/4}) \) the next point of local extremum for \( f_m. \) Since \( \eta_1 \) is a local minimum and \( \eta_1 < \eta_2, \) it must be that \( \eta_2 \) is a local maximum for \( f_m. \) Then, for \( k \in [\eta_1, \eta_2], \) we have \( f_m'(k) \geq 0 \) and since \( k < \sqrt{m^2-1/4} \) we have \( 1 + 4k^2 - 4m^2 < 0. \) Using these two facts together with the relations \( J_m(k) > 0 \) and \( Y_m(k) < 0 \) when \( k \in (0, m] \) and the fact that \( (J_m Y_m)'(k) < 0 \) when \( k \in (0, m] \) (which was established in Lemma 4.5 in [27]) we conclude from equation (3.43) that

\[
g_m'(k) > 0, \quad \text{for all} \ k \in [\eta_1, \eta_2].
\]
Since $\eta_1$ is a local minimum for $f_m$, we have that $f''_m(\eta_1) \geq 0$ and thus $g_m(\eta_1) \geq 0$. In addition, since $g'_m(k) > 0$ for all $k \in [\eta_1, \eta_2]$, it follows that $g_m$ is increasing in the intervals $[\eta_1, \eta_2]$. If we take into account that $g_m(\eta_1) \geq 0$, we obtain that $g_m(k) > 0$ for all $k \in [\eta_1, \eta_2]$ and hence $f''_m(k) > 0$ for all $k \in (\eta_1, \eta_2)$. This contradicts the fact that $\eta_2$ is a local maximum of $f_m$. Thus, $f_m$ has at most a single local extremum point in $(0, \sqrt{m^2 - 1}/4)$. Since we established that $f_m$ has a local minimum point in the interval $(0, m]$ and that $f''_m(k) > 0$ for all $k \in (\sqrt{m^2 - 1}/4, m]$, we get that $f_m$ has a unique local minimum point in the interval $(0, m]$.

We finally establish that there exists $\varepsilon_1 > 0$ such that the unique local minimum point of $f_m$ lies in the interval $(m - \varepsilon_1 m^{3/2}, \sqrt{m^2 - 1}/4)$. To this end, we use the fact that $\varphi(2^{-1/3}) \approx 0.015$, where $\varphi$ was defined in equation (3.40). Given that we previously established that $\varphi(0) < 0$, it follows from formula (3.39) that for large enough values of $m$ the function $f''_m/(m - \varepsilon m^{3/2})$ has a zero for values of $\varepsilon$ in the interval $(0, 2^{-1/3})$. Consequently, there exists $\varepsilon_1 > 0$ such that the function $f_m$ has a critical point in the interval $(m - \varepsilon_1 m^{3/4}, \sqrt{m^2 - 1}/4)$ at which $f'_m$ changes sign. Given that we established that $f_m(0) = 1/\pi$ and $f_m(m) \approx 1/\pi$, it follows that there exists $m_1 > 1$ such that $f_m$ has a unique minimal point in the interval $(0, m]$, and this minimal point is actually in the interval $(m - \varepsilon_1 m^{3/4}, m)$ for all $m \geq m_1$, which completes the proof of the Lemma. 

We next state and prove the main result concerning the $L^2(\Gamma)$ coercivity of the operators $A_{k, S_{ik}}$ and $A_{k, PS_{ik}}$ in the case when $\Gamma$ is the unit circle:

**Theorem 3.6** There exist positive constants $C_0, C_S$, and $C_{PS}$ independent of $k$ and a wavenumber $k_2$ such that for all $k > k_2$ and coupling constants $\eta$ with the property that $\eta > C_0 k^{1/3}$ the following estimates hold:

$$\inf_{m \in \mathbb{N}} \Re(a_m(k)) \geq C_S > 0, \quad \inf_{m \in \mathbb{N}} \Re(p_m(k)) \geq C_{PS} > 0, \text{ for all } k > k_2. \quad (3.44)$$

Consequently, the operators $A_{k, S_{ik}}$ and $A_{k, PS_{ik}}$ are $L^2$ coercive in the case when $\Gamma$ is the unit circle in two dimensions.

**Proof.** We first study the properties of $\Re(p_m(k))$. We will use throughout the two representations of $\Re(p_m(k))$ expressed in equations (3.31) and (3.32). Just as before, let us fix $k$ large enough and consider several cases. In each case we will prove first that $\inf_{m \in \mathbb{N}} \Re(p_m(k)) > 0$, then we will establish the stronger result that $\inf_{m \in \mathbb{N}} \Re(p_m(k)) \geq C_{PS}$.

**Case 1:** $m > k > 0$. We use the results in Watson [48] (Section 15.3) which established that $J_m(k) > 0$ and $J'_{m}(k) > 0$ to get that

$$\Re(p_m(k)) > 0 \text{ for } m > k > k_0.$$ 

Using the asymptotic expansions of the Bessel and Neumann functions for fixed $k$ for large orders $\nu \to \infty$, namely [1] (Formula 9.3.1)

$$J_{\nu}(k) \sim \frac{1}{\sqrt{2 \pi \nu}} \left( \frac{ek}{2\nu} \right)^\nu$$

$$J'_{\nu}(k) \sim \sqrt{\frac{\nu}{2\pi}} \left( \frac{ek}{2\nu} \right)^\nu k^{-1}$$

$$Y_{\nu}(k) \sim \sqrt{\frac{2\nu}{\pi}} \left( \frac{e}{2\nu} \right)^{-\nu} k^{-\nu - 1} \quad (3.45)$$
we get that
\[ \Re (p_\nu(k)) \sim \frac{1}{2} + \frac{\eta}{8} \left( \frac{ck^2}{2\nu} \right) ^\nu, \quad \nu \to \infty \] (3.46)
from which we conclude that for a fixed \( k \) large enough there exists a constant \( m_k \) such that
\[ \Re (p_m(k)) \geq 0.4 \quad \text{for all} \quad m_k \leq m. \]

Given that for all other indices \( m > k \) we have that \( \Re (p_m(k)) > 0 \) it follows that for all large enough \( k \) we have the following estimate
\[ \inf_{m > k} \Re (p_m(k)) > 0. \] (3.47)

We derive in what follows a sharper lower bound on \( \inf_{k \in (0,m]} \Re (p_m(k)) \). To this end, we use the result established in Lemma 3.5 that there exist \( \varepsilon_1 > 0 \) and \( m_1 > 1 \) such that for all \( m \geq m_1 \) the function \( f_m(k) = kJ_m(k)Y'_m(k) \) has a unique minimum point in the interval \((0,m)\), that actually lies in the interval \((m - \varepsilon_1 m^{1/3}, m)\). Given the transition asymptotic Formulas 9.3.23 and 9.3.28 in [1] we have that for a fixed \( x \in [0, \varepsilon_1] \)
\[ (m - xm^{1/3})J_m(m - xm^{1/3})Y'_m(m - xm^{1/3}) \sim 2Ai(2^{1/3}x)B'(2^{1/3}x) + \mathcal{O}(m^{-\frac{4}{3}}), \quad m \to \infty. \] (3.48)

It can easily be checked that the function
\[ G(x) = Ai(x)B'(x), \quad x \geq 0 \]
has a unique critical point \( x_\ast \approx 0.72 \) in \((0, \infty)\), and that \( x_\ast \) is a point of global minimum for \( G \). Since \( G(x_\ast) \approx 0.1238 \) it follows that \( G(x) \geq 0.12 \) for all \( x \geq 0 \). Using this in equations (3.48) we obtain that there exists \( m_1 \) large enough such that
\[ \Re (p_m(k)) \geq \frac{\pi}{2} f_m(k) \geq \frac{\pi}{2} \min_{k \in (m - \varepsilon_1 m^{1/3}, m)} f_m(k) \geq 0.12\pi \approx 0.377, \quad \text{for all} \quad m \geq m_1, \quad k \in (0,m). \] (3.49)

Thus, it follows from equation (3.49) that there exists a constant \( C_{2,1} \) and a sufficiently large \( k_{2,1} \) such that
\[ \Re (p_m(k)) \geq C_{2,1}, \quad \text{for all} \quad k \geq k_{2,1}, \quad \text{and for all} \quad m > k. \] (3.50)

Case 2: \( m \leq \frac{\sqrt{3}}{2} k \). In this case we use the representation of \( \Re (p_m(k)) \) given in formula (3.31). We have that
\[ |J'_m(k)Y_m(k)| \leq \frac{1}{2} (J'_m(k))^2 + \frac{1}{2} (Y_m(k))^2 \leq \frac{1}{2} (J'_m(k))^2 + \frac{1}{2} (M_m(k))^2 \] (3.51)
where the expression \( M_m(k) \) is defined as \( M_m(k) = \sqrt{J'_m(k)^2 + Y_m(k)^2} \). Using the estimate \( M_m^2(k) \leq \frac{2}{\pi \sqrt{k^2 - m^2}} \) [48] (Section 13.74), which is valid for all \( k \geq m \) we obtain
\[ \frac{\pi k}{2} |J'_m(k)Y_m(k)| \leq \frac{\pi k}{4} (J'_m(k))^2 + \frac{k}{2\sqrt{k^2 - m^2}}. \]

Since \( m \leq \frac{\sqrt{3}}{2} k \) it follows that \( \frac{k}{2\sqrt{k^2 - m^2}} \leq 1 \) and thus
\[ \frac{\pi k}{2} |J'_m(k)Y_m(k)| \leq \frac{\pi \eta k^2}{4\sqrt{m^2 + k^2}} (J'_m(k))^2 + 1 \] (3.52)
as long as \( \frac{\sqrt{7}}{2} \leq \eta \). It follows then that if \( \frac{\sqrt{7}}{2} \leq \eta \)

\[
-\frac{\pi k}{2} J'_m(k) Y_m(k) \leq \frac{\pi k}{2} |J'_m(k) Y_m(k)| \leq \frac{\pi \eta k^2}{4\sqrt{m^2 + k^2}} (J'_m(k))^2 + 1
\]
or equivalently

\[
\Re(p_m(k)) \geq 0. \tag{3.53}
\]

If \( \frac{\sqrt{7}}{2} < \eta \), equality in the estimate (3.53) would be obtained if we had equality in equation (3.51), that is if \( J'_m(k) = Y_m(k) \) and \( J_m(k) = 0 \). Furthermore, \( \Re(p_m(k)) = 0 \) and \( J_m(k) = 0 \) would imply via equation (3.67) that \( J'_m(k) = 0 \). Thus we get that \( \Re(p_m(k)) = 0 \) iff all the values \( J_m(k), Y_m(k), J'_m(k) \) were equal to zero, which can never be the case since the real zeros of \( J_m \) and \( Y_m \) interlace [48](Section 15.3). In conclusion, if \( \frac{\sqrt{7}}{2} < \eta \) we get that for any \( k > k_0 \)

\[
\inf_{m \leq \frac{\sqrt{3}k}{2}} \Re(p_m(k)) > 0. \tag{3.54}
\]

In order to get a sharper estimate on the quantity \( \inf_{m \leq \frac{\sqrt{3}k}{2}} \Re(p_m(k)) \) we use again the representation (3.31) and we study the behavior of the quadratic function for values \( m \leq \frac{\sqrt{3}k}{2} \):

\[
g(x) = \frac{2}{\pi k} + Y_m(k) x + \frac{k\eta}{2(m^2 + k^2)} x^2. \tag{3.55}
\]

We have that

\[
g(x) \geq -\Delta_g (m^2 + k^2)^{\frac{1}{2}} \]

where \( \Delta_g \) denotes the discriminant of the quadratic function \( g \) whose formula is given by

\[
\Delta_g = Y_m^2(k) - \frac{4\eta}{\pi(m^2 + k^2)^{\frac{1}{2}}}. \tag{3.56}
\]

We first show that \( \Delta_g < 0 \). To this end we use again the estimate \( M_m^2(k) \leq \frac{2}{\pi \sqrt{k^2 - m^2}} \) and we get

\[
-\Delta_g = \frac{4\eta}{\pi(m^2 + k^2)^{\frac{1}{2}}} - Y_m^2(k) \geq \frac{4\eta}{\pi(m^2 + k^2)^{\frac{1}{2}}} - M_m^2(k)
\]

\[
\geq \frac{4\eta}{\pi(m^2 + k^2)^{\frac{1}{2}}} \left( 1 - \frac{2}{\pi \sqrt{k^2 - m^2}} \right)
\]

\[
\geq \frac{4\eta}{\pi(m^2 + k^2)^{\frac{1}{2}}} \left( 1 - \frac{\sqrt{7}}{2\eta} \right) \geq 0. \tag{3.57}
\]

It follows then that for all \( x \in \mathbb{R} \) we have

\[
g(x) \geq \frac{-\Delta_g (m^2 + k^2)^{\frac{1}{2}}}{2k\eta} \geq \frac{2}{\pi k} \left( 1 - \frac{\sqrt{7}}{2\eta} \right)
\]

which implies via representation (3.31) that there exists \( k_{2.2} \) large enough such that

\[
\Re(p_m(k)) \geq 1 - \frac{\sqrt{7}}{2\eta} > \frac{1}{2}, \text{ for all } k > k_{2.2}, \ 0 \leq m \leq \frac{\sqrt{3}}{2} k. \tag{3.58}
\]
We have used the fact that \( \eta > C_0 k^{\frac{1}{2}} \) to establish estimate (3.58).

**Case 3:** \( \sqrt{3} k < m \leq k \). We start with the case when \( m \approx k \) and \( k \) is large. We use formulas (3.13) and (3.38) and we get that

\[
\Re(p_\nu(\nu)) = \frac{\pi}{2} ab\sqrt{3} + O(\nu^{-1/3}), \; \nu \to \infty, \tag{3.59}
\]

which in particular implies that \( \Re(p_m(\nu)) > 0 \) for \( m \) large enough. We proceed with the analysis of the case when \( k - m \) is small enough with respect to \( k \). More precisely, we consider first the following subcase

**Case 3a:** \( k - c_0 k^{\frac{1}{2}} \leq m \leq k \) and \( k \) large enough, where \( c_0 \) is a constant that will be specified later. In this case we consider the representation of \( \Re(p_m(\nu)) \) given by equations (3.32), and we can write \( k = m + zm^{1/3} \) where \( 0 \leq z \leq z_0 \) where we can take \( z_0 = c_0(2/\sqrt{3})^{\frac{1}{2}} \). We consider a fixed \( z \) such that \( 0 \leq z \leq z_0 \) and large values of the orders \( m \), and we use Formulas 9.3.23 and 9.3.28 in [1] to obtain that

\[
J_m(m + zm^{1/3})Y'_m(m + zm^{1/3}) = \frac{2}{m} \text{Ai}(-2^{1/3}z) \text{Bi}'(-2^{1/3}z) + O(m^{-4/3}) \tag{3.60}
\]

where \( \text{Ai} \) and \( \text{Bi} \) denote Airy functions. Denoting by

\[
f(z) := \text{Ai}(-2^{1/3}z) \text{Bi}'(-2^{1/3}z), \; 0 \leq z \tag{3.61}
\]

we get according to Formulas 10.4.4 and 10.4.5 in [1] that

\[
f(0) = \frac{1}{2\pi} > 0, \; f'(0) = \frac{2^{1/3}}{3^{1/6}(\Gamma(1/3))^2} > 0.
\]

Let \( z_0 \approx 0.8712 \) denote the first positive zero of the derivative \( f' \) of the function \( f \) defined in equation (3.61). It follows that \( f \) is increasing on the interval \( [0, z_0] \) and thus \( f(z) \geq f(0) = \frac{1}{2\pi} \) for all \( 0 \leq z \leq z_0 \). It follows via the asymptotic formula (3.60) that there exists a \( m_0 \) large enough such that for all \( 0 \leq z \leq z_0 \) and all \( m > m_0 \) we have

\[
J_m(m + zm^{1/3})Y'_m(m + zm^{1/3}) \geq \frac{0.9}{\pi m}
\]

from which it follows that

\[
\Re(p_m(m + zm^{1/3})) \geq \frac{\pi(m + zm^{1/2})}{2} J_m(m + zm^{1/3})Y'_m(m + zm^{1/3}) \geq 0.45. \tag{3.62}
\]

The last equation implies that there exists \( c_0 = z_0(2/\sqrt{3})^{-\frac{1}{2}} \approx 0.8304 \) and \( k_{2,3} \) such that

\[
\Re(p_m(\nu)) \geq 0.45, \; \text{for all} \; k_{2,3} \leq k, \; k - c_0 k^{\frac{1}{2}} \leq m \leq k \tag{3.63}
\]

**Case 3b:** In the remaining case of orders \( m \) such that \( \sqrt{3} k \leq m \leq k - c_0 k^{\frac{1}{2}} \), we investigate again the discriminant \( \Delta_g \) of the quadratic function \( g \) defined in equation (3.55). We use one more time
the estimate $M^2_m(k) \leq \frac{2}{\pi \sqrt{k^2 - m^2}}$ and we get if we repeat the arguments that led to estimate (3.57)

\[-\Delta_g \geq \frac{4\eta}{\pi(m^2 + k^2)^{\frac{1}{2}}} \left(1 - \frac{(m^2 + k^2)^{\frac{1}{2}}}{2\eta(k^2 - m^2)^{\frac{1}{2}}} \right)\]

Clearly, taking $\eta > \tilde{C}_0 k^{\frac{4}{3}}$ with $\tilde{C}_0$ large enough such that $\tilde{C}_0 > \frac{1}{\sqrt{2c_0}}$, it follows from equation (3.64) that $\Delta_g < 0$ and thus $g(x) > 0$ for all $x \in \mathbb{R}$. Furthermore, we have that for all $x \in \mathbb{R}$

$$g(x) \geq \frac{-\Delta_g (m^2 + k^2)^{\frac{1}{2}}}{2k\eta} \geq \frac{2}{\pi k} \left(1 - \frac{1}{\sqrt{2c_0\tilde{C}_0}}\right)$$

which implies that

$$\Re(p_m(k)) \geq 1 - \frac{1}{\sqrt{2c_0\tilde{C}_0}}$$ for all $k_{2,3} \leq k$, $\frac{\sqrt{3}}{2} k \leq m \leq k - c_0 k^{\frac{4}{3}}$. \hspace{1cm} (3.65)

Taking $k_2$ to be the maximum of $k_{2,1}, k_{2,2}$, and $k_{2,3}$ and $C_{PS}$ to be the minimum of the constants that appear in the right hand sides of equations (3.50), (3.58), (3.63), and (3.65) (with the choice $\tilde{C}_0 = 1$ in the condition $\eta > \tilde{C}_0 k^{1/3}$, this would amount to an estimate $C_{PS} \approx 0.377$ which is fairly close to the value in the right panel of Figure 1), we obtain that for all $k \geq k_2$

$$\Re(p_m(k)) \geq C_{PS}, \text{ for all } m. \hspace{1cm} (3.66)$$

Finally, the real part of the eigenvalues $a_m(k)$ can be expressed as

$$\Re(a_m(k)) = \frac{\pi k}{2} \left(J_m(k)Y'_m(k) + \frac{k\pi\eta}{2} (J'_m(k))^2 [iJ_m(ik)H^{(1)}_m(ik)]\right), \hspace{1cm} (3.67)$$

where we used the fact that $iJ_m(ik)H^{(1)}_m(ik) \in \mathbb{R}$. Using the result in Lemma 3.1 we obtain that for all $k \geq k_0$ and all $m$ we have that

$$\frac{\pi k}{2} \left(J_m(k)Y'_m(k) + \frac{k\pi\eta C_1}{2(m^2 + k^2)^{\frac{1}{2}}} (J'_m(k))^2\right) \leq \Re(a_m(k)).$$

Based on the result that we established above about uniform lower bounds on $\Re(p_m(k))$, it follows that if $\eta \geq \frac{C_1}{C_0} k^{\frac{4}{3}}$, there exists a constant $k''$ and a constant $C_S$ such that for all $k > k''$

$$\Re(a_m(k)) \geq C_S, \text{ for all } m. \hspace{1cm} (3.68)$$

Clearly, combining equations (3.66) and (3.68), the result of the Theorem follows immediately. \[\blacksquare\]
Remark 3.7 A weaker version of the result in the Theorem 3.6 holds, that is if \( \eta > \frac{\sqrt{\pi}}{2} \), then there exists \( k_3 > 0 \) such that

\[
\inf_{m \in \mathbb{N}} \Re(a_m(k)) > 0, \quad \inf_{m \in \mathbb{N}} \Re(p_m(k)) > 0, \text{ for all } k > k_2.
\]

Furthermore, \( \inf_{m \in \mathbb{N}} \Re(a_m(k)) \sim C |\eta|k^{-\frac{1}{3}} \), and a similar estimate holds for \( \inf_{m \in \mathbb{N}} \Re(p_m(k)) \).

Remark 3.8 An immediate corollary of Theorem 3.6 is that if \( \eta > C_0 k^\frac{1}{3} \), there exist \( k_2 > 0 \) and two constants \( C_{1,p} \) and \( C_{2,p} \) such that

\[
\| A_{k,S_{ik}}^{-1} \|_{\mathcal{L}(\mathcal{H}^p(\mathbb{S}^3))} \leq C_{1,p}, \quad \| A_{k,PS_{ik}}^{-1} \|_{\mathcal{L}(\mathcal{H}^p(\mathbb{S}^3))} \leq C_{2,p}, \quad 0 \leq p.
\]

In the three dimensional case, a direct calculation gives the following expression of the real part of the eigenvalues \( b_m(k) \) of the operators \( B_{k,S_{ik}} \):

\[
\Re(b_m(k)) = \Re(a_{m+\frac{1}{2}}(k)) - \frac{\pi}{4} J_{m+\frac{1}{2}}(k)Y_{m+\frac{1}{2}}(k) + \frac{\eta \pi}{16} J_{m+\frac{1}{2}}^2(k)J_{m+\frac{1}{2}}(ik)H_{m+\frac{1}{2}}^\prime(ik)
\]

\[\quad \quad - \frac{\eta \pi}{4} J_{m+\frac{1}{2}}^2(k)J_{m+\frac{1}{2}}'(k)J_{m+\frac{1}{2}}(ik)H_{m+\frac{1}{2}}^\prime(ik). \tag{3.69}\]

We use the following results established in [27]: there exists a constant \( C > 0 \) such that for all \( m \) and for all \( k \) large enough we have

\[
|J_m(k)Y_m(k)| \leq C k^{-\frac{3}{4}}, \quad |J_m(k)| + |J'_m(k)| \leq C k^{-\frac{3}{4}}.
\]

The second inequality above implies that \( |J_m(k)J'_m(k)| \leq \frac{1}{4} (|J_m(k)| + |J'_m(k)|)^2 \leq C k^{-\frac{3}{2}} \). Using these estimates together with the results in Lemma 3.1 we get that

\[
\Re(b_m(k)) \geq \Re(a_{m+\frac{1}{2}}(k)) - C k^{-\frac{3}{4}} - C |\eta|k^{-\frac{3}{2}}
\]

for all \( m \geq 0 \) and all \( k \) large enough. The last estimate and the coercivity estimates in Theorem 3.6 imply the following result:

**Theorem 3.9** There exist positive constants \( C_0 \) and \( C \) independent of \( k \) and a wavenumber \( k_3 \) such that for all \( k > k_3 \) and coupling constants \( \eta \) with the property that \( \eta \approx C_0 k^{1/3} \) the following estimates hold:

\[
\inf_{m \in \mathbb{N}} \Re(b_m(k)) \geq C > 0, \text{ for all } k > k_2. \tag{3.70}
\]

Consequently, the operators \( B_{k,S_{ik}} \) are coercive in \( L^2(\mathbb{S}^3) \).

### 3.3 Wavenumber dependent bounds of the norms of the operators ICFIE-R and ICFIE-RPS in the case of general smooth boundaries \( \Gamma \)

The next set of results concerns establishing upper bounds on the 2-norm of the operators \( A_{k,S_{ik}} \) and \( A_{k,PS_{ik}} \) for general smooth curves \( \Gamma \). For this we will need the following technical result:
Lemma 3.10 For each \( m = 0, 1, \ldots \), there exists \( C_m > 0 \) independent of \( k \) such that
\[
\int_0^{2\pi} K_m(2k|\sin t/2|)|\sin t/2|^m dt \leq C_m k^{-1-m}
\]
for \( k \) sufficiently large, where \( K_m \) stands for the modified Bessel function of the third kind.

Proof. We use
\[
\int_0^{2\pi} K_m(2k|\sin t/2|)|\sin t/2|^m dt = 4 \int_0^{\pi/2} K_m(2k \sin t) \sin^m t dt
\]
and since \( K_m(r) = \left( \frac{r}{\pi} \right)^{1/2} e^{-r} (1 + O(r^{-1})) \) as \( r \to \infty \) [1] (Formula 9.7.2), we get
\[
\int_0^{\pi/6} K_m(2k \sin t) \sin^m t dt \leq C_m k^{-1/2} e^{-k}
\]
with \( C_m \) independent of \( k \). On the other hand, using the change of variables \( x = 2k \sin t \) we obtain
\[
\int_0^{\pi/6} K_m(2k \sin t) \sin^m t dt \leq \frac{2}{\sqrt{3}(2k)^{m+1}} \int_0^k K_m(x) x^m dx.
\]
The last integral can be split as
\[
\int_0^k K_m(x) x^m dx = \int_0^L K_m(x) x^m dx + \int_L^k K_m(x) x^m dx \leq C_L (1 + \int_L^k t^{-1/2} e^{-t} dt)
\]
for \( k \to \infty \) since \( K_m(x) x^m \) is locally integrable in \([0, \infty) \) [1]. The result follows now readily.

We also need the following technical result:

Lemma 3.11 There exists \( C > 0 \) such that for all \( t > 0 \) and all \( \epsilon > 0 \) the following estimate holds:
\[
\left| H_1^{(1)}(it) + \frac{2}{\pi t} \right| \leq C t^{-1+\epsilon}
\]

Proof. We use the fact that \( H_1^{(1)}(it) = -\frac{2}{\pi} K_1(t) \), where \( K_1 \) is the modified Bessel function of the third kind and order one. From the series representation of \( K_1 \) [1] (Formula 9.6.11) for \( t > 0 \)
\[
K_1(t) = \frac{1}{t} + t \ln t \sum_{j=0}^{\infty} b_j t^{2j} + t \sum_{j=0}^{\infty} a_j t^{2j}
\]
we conclude that \( \left| H_1^{(1)}(it) + \frac{2}{\pi t} \right| \leq C t^{-1+\epsilon} \) for small enough \( t \). Using the asymptotic expansion [1] (Formula 9.7.2) \( H_1^{(1)}(it) = \left( \frac{\pi}{2} \right)^{1/2} e^{-t} (1 + O(t^{-1})) \), \( t \to \infty \) we obtain that \( \left| H_1^{(1)}(it) + \frac{2}{\pi t} \right| \leq C t^{-1+\epsilon} \)
for all large enough \( t \). Finally, since the function \( H_1^{(1)}(it) + \frac{2}{\pi t} \) is continuous on any compact interval strictly included in \((0, \infty)\), the result of the Lemma follows.

Using the results of the previous Lemma 3.10 and Lemma 3.11 we will derive wavenumber dependent upper bounds on the norms of the operators \( A_{k,S} \) and \( A_{k,PS} \):
Theorem 3.12 In the case when $\Gamma$ is of class $C^\infty$, there exist $k_0 > 0$ and $C > 0$ independent of $k$ such that

$$\|A_{k,S_{ik}}\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq C (1 + |\eta|) k^{1/2+\epsilon}, \quad \|A_{k,PS_{ik}}\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq C (1 + |\eta|) k^{1/2}$$

for any $\epsilon > 0$.

**Proof.** In the proof of the Theorem we will use generic constants $C$ that do not depend on the wavenumber $k$. We start by noting that it was established in Theorem 4.14 in [27] that $\|K''\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq C k^{1/2}$. We derive next a wavenumber $k$ dependent bound on $\|S_{ik}\|_{L^2(\Gamma) \to L^2(\Gamma)}$. To this end, we write the integral operator $S_{ik}$ in the form

$$(S_{ik}\psi)(s) = \int_0^{2\pi} S^k(s,t)\psi(t)dt$$

where

$$S^k(s,t) = \frac{i}{4} H_0^{(1)}(ik|x(s) - x(t)|)|x'(t)|.$$ 

With these notations we have that

$$\int_0^{2\pi} \left| \int_0^{2\pi} S^k(s,t)\psi(t) dt \right|^2 ds = \frac{1}{4} \int_0^{2\pi} \left| \int_0^{2\pi} W_{ik}(s,t)H_0^{(1)}(2ik|\sin((s - t)/2)|)\psi(t) dt \right|^2 ds$$

with

$$W_{ik}(s,t) = \frac{S^k(s,t)}{H_0^{(1)}(2ik|\sin((s - t)/2)|)}.$$

We note that the denominator of the kernel $W_{ik}$ corresponds to the evaluation of its numerator on the unit circle $x(s) = (\cos s, \sin s)$. Since the singularity as $s \to t$ on the numerator and denominator of $W_{ik}$ are the same, and for $s \neq t$, the asymptotic behavior of the top and the bottom of the expression $W_{ik}(s,t)$ as $k \to \infty$ are the same, it follows that $|W_{ik}(s,t)|$ can be bounded uniformly for all $k$ large enough and $s, t \in [0, 2\pi]^2$. Thus, we obtain using Young’s inequality $\|f \ast g\|_2 \leq \|f\|_1 \|g\|_2$, and Lemma 3.10

$$\int_0^{2\pi} \left| \int_0^{2\pi} S^k(s,t)\psi(t) dt \right|^2 ds \leq C \left( \int_0^{2\pi} K_0(2k|\sin t/2|)|dt\right)^2 \|\psi\|_2^2 \leq Ck^{-2}\|\psi\|_2^2,$$

and consequently

$$\|S_{ik}\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq Ck^{-1}.$$  \hfill (3.72)

We derive next a bound on $\|N_{ik} \circ S_{ik}\|_{L^2(\Gamma) \to L^2(\Gamma)}$. Based on equations (2.8), we write

$$([N_{ik} \circ S_{ik}]\phi)(x) = k^2 \int_\Gamma G_k(x - y)(n(x) \cdot n(y))(S_{ik}\phi)(y)dy$$

$$+ \int_\Gamma \partial_s[G_k(x - y) - G_0(x - y)]\partial_s(S_{ik}\phi)(y)dy$$

$$+ \frac{1}{4} \int_\Gamma \partial_s G_0(x - y) \partial_s(S_{ik}\phi)(y)dy$$

$$= (A_k^{(1)} \circ S_{ik}\phi)(x) + [A_k^{(2)}(\partial_s S_{ik}\phi)](x) + [A_k^{(3)}(\partial_s S_{ik}\phi)](x)$$  \hfill (3.73)
follows that \( \kappa \) where we have used the fact that the kernel \( \tilde{k} \) defined by equations (3.75) with kernels \( \tilde{k} \) to derive that
\[
|\tau| \text{ in terms of the tangent vectors } \tau \text{ in } 2D. \text{ We use the fact that } \|A\| \text{ from which it follows immediately that } \|A^{(1)}\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq Ck^{3/2}, \text{ and thus}
\[
\|A^{(1)} \circ S_{ik}\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq Ck^{1/2}. \tag{3.74}
\]

We derive next bounds on \( \|A^{(2)}\|_{L^2(\Gamma) \to L^2(\Gamma)} \) based on the Riesz-Thorin interpolation techniques used in [16]. The main idea is to consider integral operators of the form
\[
A\phi(x) = \int_{\Gamma} \kappa(x, y) \phi(y) ds(y) \tag{3.75}
\]
and to obtain estimates of the operator norms associated with such operators \( A \) as mappings from \( L^2(\Gamma) \to L^2(\Gamma) \) via estimates on the operator norms of these operators as mappings from \( L^1(\Gamma) \to L^1(\Gamma) \) and \( L^\infty(\Gamma) \to L^\infty(\Gamma) \) respectively. Once the latter bounds are established, the bounds on the \( L^2 \) norms of operators \( A \) are obtained through the classical Riesz-Thorin interpolation theorem [25]
\[
\|A\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq \|A\|_{L^1(\Gamma) \to L^1(\Gamma)}^{1/2} \|A\|_{L^\infty(\Gamma) \to L^\infty(\Gamma)}^{1/2}.
\]

The two norms that appear on the right-hand side of the previous estimate can be computed explicitly
\[
\|A\|_{L^1(\Gamma) \to L^1(\Gamma)} = \text{ess sup}_{y \in \Gamma} \int_{\Gamma} |\kappa(x, y)| ds(x), \quad \|A\|_{L^\infty(\Gamma) \to L^\infty(\Gamma)} = \text{ess sup}_{x \in \Gamma} \int_{\Gamma} |\kappa(x, y)| ds(y).
\]

The integral operator \( A^{(2)} \) is an integral operator of the kind described in equations (3.75) with a kernel
\[
\kappa_k(x, y) = -\left( \frac{i}{4} k H_1^{(1)}(k|x - y|) - \frac{1}{2\pi |x - y|} \right) \frac{(x - y) \cdot \tau(x)}{|x - y||\tau(x)|}
\]
in terms of the tangent vectors \( \tau(x) \) to the curve \( \Gamma \) at the points \( x \). We use the following estimate established in Equations 3.9 in [16]
\[
|H_1^{(1)}(t) + \frac{2i}{\pi t}| \leq \frac{C}{t^{1/2}}, \quad t > 0
\]
to derive that \( |\kappa_k(x, y)| \leq C\tilde{k}(x, y) \) where \( \tilde{k}(x, y) = \frac{k^{1/2}}{|x - y|^{1/2}} \). We consider integral operators \( \tilde{A} \) defined by equations (3.75) with kernels \( \tilde{k}(x, y) \) defined above. On account of the symmetry of the kernels \( \tilde{k} \) we have that
\[
\|\tilde{A}\|_{L^1(\Gamma) \to L^1(\Gamma)} = \|\tilde{A}\|_{L^\infty(\Gamma) \to L^\infty(\Gamma)} \leq Ck^{1/2} \text{ess sup}_{x \in \Gamma} \int_{\Gamma} \frac{1}{|x - y|^{1/2}} ds(y) \leq Ck^{1/2},
\]
where we have used the fact that the kernel \( \tilde{k} \) is weakly singular. Since \( |\kappa_k(x, y)| \leq C\tilde{k}(x, y) \), it follows that \( \|A^{(2)}\|_{L^1(\Gamma) \to L^1(\Gamma)} \leq Ck^{1/2}, \quad \|A^{(2)}\|_{L^\infty(\Gamma) \to L^\infty(\Gamma)} \leq Ck^{1/2} \), from which we get based on the Riesz-Thorin interpolation theorem that
\[
\|A^{(2)}\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq Ck^{1/2}. \tag{3.76}
\]
The operator $A^{(3)}$ can be expressed as

$$(A^{(3)}\psi)(s) = \int_0^{2\pi} T(s,t)\psi(t)dt$$

in terms of the kernel $T_k(s,t)$ defined as follows

$$T(s,t) = T^{(1)}(s,t) + T^{(2)}(s,t)$$

$$T^{(1)}(s,t) = \left(2\frac{(x(s) - x(t)) \cdot x'(s)}{|x(t) - x(s)|^2} - \cot \frac{s - t}{2}\right) \frac{|x'(t)|}{4\pi |x'(s)|}$$

$$T^{(2)}(s,t) = \cot \frac{s - t}{2} \frac{|x'(t)|}{4\pi |x'(s)|}.$$  \(3.77\)

Since the kernel $T^{(1)}(s,t)$ is continuous on $[0, 2\pi] \times [0, 2\pi]$ it follows that

$$\left|\int_0^{2\pi} \int_0^{2\pi} T^{(1)}(s,t)\psi(t) dt\right|^2 ds \leq C\|\psi\|_2^2.$$  \(3.78\)

The integral operators with kernels $T^{(2)}(s,t)$ are related to the classical operator $H_0$ with the Hilbert kernel

$$(T_0\psi)(s) = \frac{1}{2\pi i} \int_0^{2\pi} \left( \cot \frac{t - s}{2} + i \right) \psi(t)dt$$

which satisfies $H_0^2 = I$ on $L^2(\Gamma)$ [31]. From this, we immediately obtain

$$\left|\int_0^{2\pi} \int_0^{2\pi} T^{(2)}(s,t)\psi(t) dt\right|^2 ds \leq C\|\psi\|_2^2.$$  \(3.79\)

We conclude from equations (3.77)-(3.79) that

$$\|A^{(3)}\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq C.$$  \(3.80\)

The expressions $\partial_s S_{ik}\psi$, in turn, can be written as

$$\partial_s S_{ik}\psi = A^{(2)}_{ik}\psi + A^{(3)}\psi$$  \(3.81\)

where the integral operator $A^{(2)}_{ik}$ is an integral operator of the type described in equations (3.75) with kernel

$$\kappa_{ik}(x, y) = \left(\frac{k}{4} H_1^{(1)}(ik|x-y|) + \frac{1}{2\pi |x-y|} \right) \frac{(x - y) \cdot \tau(x)}{|x-y||\tau(x)|}.$$  

The application of the same techniques that delivered the estimate (3.76) in conjunction with the result in Lemma 3.11 leads us via an estimate of the form $|\kappa_{ik}(x, y)| \leq C\frac{k^\epsilon}{|x-y|^{1-\epsilon}}$ to the estimate

$$\|A^{(2)}_{ik}\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq Ck^\epsilon$$  \(3.82\)

for all $\epsilon > 0$ and hence the following estimate also holds

$$\|\partial_s S_{ik}\psi\|_2 \leq Ck^\epsilon\|\psi\|_2, \ \epsilon > 0.$$  \(3.83\)

Consequently, combining the estimates (3.74), (3.76), (3.80), and (3.83) we derive the first estimate

$$\|A_{k,S_{ik}}\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq C(1 + |\eta|)k^{1/2+\epsilon}.$$
For the second operator $A_{PS_k}$ we use formula (3.73) with $S_{ik}$ replaced by $PS_{ik}$. Given the definition of the regularizing operator $PS_{ik}$ (2.17) we have

$$\|PS_{ik}\|_2^2 = \sum_{n \in \mathbb{Z}} \frac{1}{n^2 + k^2} |\hat{\phi}(x')(n)|^2 \leq Ck^{-2}\|\phi\|_2^2$$ (3.84)

and thus $\|PS_{ik}\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq Ck^{-1}$. Furthermore,

$$\|\partial_s(PS_{ik}\phi)\|_2^2 \leq C\sum_{n \in \mathbb{Z}} \frac{n^2}{n^2 + k^2} |\hat{\phi}(x')(n)|^2 \leq C\|\phi\|_2^2$$ (3.85)

and thus the second estimate $\|A_{k,PS_{ik}}\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq C(1 + |\eta|)k^{1/2}$ follows from (3.74), (3.76), (3.80), (3.84), and (3.85). □

Remark 3.13 The estimates in Theorem 3.12 can be extended to the three dimensional ICFIE-R operators $B_{k,S_{ik}}$ introduced in Section 3.1. More precisely, there exists a constant $C$ independent of the wavenumber $k$ such that $\|B_{k,S_{ik}}\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq C(1 + |\eta|)k^{1+\epsilon}$ for all $\epsilon > 0$.

Having discussed the various regularization techniques, we will outline next the high-order Nyström method based on trigonometric interpolation introduced in [30] to discretize the two-dimensional integral operators that appear in (2.16) and (2.18).

4 Numerical method

We present in this part Nyström discretizations of equations (2.16) and (2.18) that are based on extensions of the Nyström discretization introduced in [30] for the numerical solution of equations (2.6). We also derive error estimates for the solutions that are obtained through these discretizations.

4.1 Discretization of the ICFIE-R equations (2.16)

We first use Calderón’s identities [20] to recast equations (2.14) with the choice of the regularizing operators described in equations (2.16) in a form that does not include hypersingular operators

$$(A_{k,S_{ik}}\mu)(x) = \frac{\partial u_{inc}(x)}{\partial n(x)} x \in \Gamma, \quad A_{k,S_{ik}}\mu = \left(\frac{1}{2} + \frac{i\eta}{4}\right)\mu - K'_k\mu - i\eta[(N_k - N_{ik})S_{ik}]\mu + i\eta(K'_{ik})^2\mu.$$ (4.1)

We assume that the boundary curve $\Gamma$ is analytic and given by the parametrization $x(t) = (x_1(t), x_2(t))$, $t \in [0, 2\pi]$ where the functions $x_j : \mathbb{R} \rightarrow \mathbb{R}$ are analytic and $2\pi$ periodic such that
\[ |x'(t)| > 0 \] for all \( t \). We write equations (4.1) in the following parametric form

\[
\left( \frac{1}{2} + \frac{i \eta}{4} \right) \mu(t)|x'(t)| + \int_0^{2\pi} H_k(t, \tau) \mu(\tau)d\tau + \int_0^{2\pi} K_k(t, \tau)(S_{ik\mu})(\tau)d\tau
\]

\[
- \int_0^{2\pi} K_{ik}(t, \tau)(S_{ik\mu})(\tau)d\tau + i\eta \int_0^{2\pi} H_{ik}(t, \tau)(K'_{ik\mu})(\tau)d\tau
\]

\[ = f(t), \ 0 \leq t \leq 2\pi \]  

(4.2)

where \( \mu(t) \equiv \mu(x(t)) \) and \( f(t) = |x'(t)|g(x(t)) \) with \( g = \frac{2\mu_{inc}}{2\mu} |\Gamma| \). According to the derivations in [30], the kernel \( H_k(t, \tau) \) is given by

\[
H_k(t, \tau) = \frac{ik}{4} n(t) \cdot (x(t) - x(\tau)) \frac{H_1^{(1)}(k|x(t) - x(\tau)|)}{|x(t) - x(\tau)|} |x'(\tau)||x'(t)|, \]

(4.3)

where \( n(t) = \frac{(x'_2(t), -x'_1(t))}{|x'(t)|} \), and the kernel \( K_k(t, \tau) \) is given by

\[
K_k(t, \tau) = -i\eta k^2 M_k(t, \tau)x'(t) \cdot x'(\tau) + i\eta N_k(t, \tau) \]

(4.4)

with

\[
M_k(t, \tau) = \frac{i}{4} H_0^{(1)}(k|x(t) - x(\tau)|) \]

(4.5)

\[
N_k(t, \tau) = -\frac{\partial^2}{\partial t \partial \tau} \left\{ \frac{i}{4} H_0^{(1)}(k|x(t) - x(\tau)|) + \frac{1}{4\pi} \ln \left( \frac{4 \sin^2 \frac{t - \tau}{2}}{2} \right) \right\} . \]

(4.6)

Similarly, the kernel \( K_{ik}(t, \tau) \) is given by

\[
K_{ik}(t, \tau) = i\eta k^2 M_{ik}(t, \tau)x'(t) \cdot x'(\tau) + i\eta N_{ik}(t, \tau) \]

(4.7)

with

\[
M_{ik}(t, \tau) = \frac{i}{4} H_0^{(1)}(ik|x(t) - x(\tau)|) \]

(4.8)

\[
N_{ik}(t, \tau) = -\frac{\partial^2}{\partial t \partial \tau} \left\{ \frac{i}{4} H_0^{(1)}(ik|x(t) - x(\tau)|) + \frac{1}{4\pi} \ln \left( \frac{4 \sin^2 \frac{t - \tau}{2}}{2} \right) \right\} . \]

(4.9)

The operators \( S_{ik\mu} \) and \( K'_{ik\mu} \), in turn, can be expressed themselves in parametric form as

\[
(S_{ik\mu})(t) = \int_0^{2\pi} M_{ik}(t, \tau)\mu(\tau)|x'(\tau)|d\tau \]

(4.10)

and

\[
(K'_{ik\mu})(t) = \int_0^{2\pi} H_{ik}(t, \tau)\frac{\mu(\tau)}{|x'(\tau)|}d\tau \]

(4.11)

where the kernels \( M_{ik}(t, \tau) \) are defined in the same manner as the kernels \( M_k(t, \tau) \) in equations (4.8) with \( k \) replaced by \( ik \), and the kernels \( H_{ik}(t, \tau) \) are defined like the kernels \( H_k(t, \tau) \) in equations (4.3) with \( k \) replaced by \( ik \).
The kernels $K_k$ and $H_k$ are weakly singular and can be written in the form [30]

$$
K_k(t, \tau) = K_{k,1}(t, \tau) \ln \left( \frac{4 \sin^2 \frac{t - \tau}{2}}{2} \right) + K_{k,2}(t, \tau)
$$

$$
H_k(t, \tau) = H_{k,1}(t, \tau) \ln \left( \frac{4 \sin^2 \frac{t - \tau}{2}}{2} \right) + H_{k,2}(t, \tau)
$$

(4.12)

for $2\pi$-periodic analytic functions $K_{k,1}, H_{k,1}$ and $K_{k,2}, H_{k,2}$. The main idea in the derivation of equations (4.12) is to decompose the fundamental solution $H_0^{(1)}(z)$ in the form $H_0^{(1)}(z) = J_0(z) + iY_0(z)$ and to use the fact that $J_0(z)$ and $Y_0(z) - \frac{2}{\pi} \ln \frac{z}{2}$ are analytic functions of $z$; similar decompositions are available for $H_1^{(1)}(z)$. The same splitting strategy, unfortunately, does not work in the case when the kernels involve the Hankel function $H_1^{(1)}(iz)$. Indeed, the modified Bessel function $I_0(z)$ grows exponentially, while $M_{ik,1}$ actually decays exponentially, as $|x(t) - x(\tau)|$ increases, and thus the splitting strategy generally gives rise to significant cancellation errors if used throughout the integration domain. In order to avoid subtraction of exponentially large quantities, we evaluate the operator $S_{ik}$ by means of a slight modification of the approach used for the operator with kernels $K_k$ and $H_k$: we use the truncated decomposition

$$
M_{ik}(t, \tau) = \chi(k|x(t) - x(\tau)|^4) \left\{ M_{ik,1}(t, \tau) \ln \left( \frac{4 \sin^2 \frac{t - \tau}{2}}{2} \right) + M_{ik,2}(t, \tau) \right\}
$$

$$
+ (1 - \chi(k|x(t) - x(\tau)|^4))M_{ik,1}(t, \tau) = M_{ik,1}(t, \tau) \ln \left( \frac{4 \sin^2 \frac{t - \tau}{2}}{2} \right) + M_{ik,2}(t, \tau)
$$

(4.13)

where $\chi \in C^\infty_0(\mathbb{R})$ is a function such that $\chi(t) \equiv 1$, $|t| \leq 1/2$ and $\chi(t) \equiv 0$, $|t| \geq 1/2$. It can be checked easily that decomposition (4.13) does not suffer from cancellation errors and that the kernels $M_{ik,1}(t, \tau), j = 1, 2$ in equation (4.13) are smooth (but not analytic) functions of $t$ and $\tau$.

Applying the strategy outlined above for the case of the kernel $M_{ik}$ to the kernels $K_{ik}$ and $H_{ik}$ leads to splittings of the form

$$
K_{ik}(t, \tau) = K_{ik,1}(t, \tau) \ln \left( \frac{4 \sin^2 \frac{t - \tau}{2}}{2} \right) + K_{ik,2}(t, \tau)
$$

(4.14)

$$
H_{ik}(t, \tau) = H_{ik,1}(t, \tau) \ln \left( \frac{4 \sin^2 \frac{t - \tau}{2}}{2} \right) + H_{ik,2}(t, \tau)
$$

(4.15)

for $2\pi$-periodic smooth functions $K_{ik,1}, H_{ik,1}$ and $K_{ik,2}, H_{ik,2}$. For a given $2\pi$ periodic function $\psi$
we introduce the following operators $A_j^k, j = 1, \ldots, 4, A_j^{ik}, j = 1, \ldots, 6$, and $A_7$ by

\[
(A_1^k \psi)(t) = \int_0^{2\pi} H_{k,1}(t, \tau) \ln \left(4 \sin^2 \frac{t - \tau}{2}\right) \psi(\tau) d\tau
\]

\[
(A_2^k \psi)(t) = \int_0^{2\pi} H_{k,2}(t, \tau) \psi(\tau) d\tau
\]

\[
(A_1^{ik} \psi)(t) = \int_0^{2\pi} H_{ik,1}(t, \tau) \ln \left(4 \sin^2 \frac{t - \tau}{2}\right) \psi(\tau) d\tau
\]

\[
(A_2^{ik} \psi)(t) = \int_0^{2\pi} H_{ik,2}(t, \tau) \psi(\tau) d\tau
\]

\[
(A_3^k \psi)(t) = \int_0^{2\pi} K_{k,1}(t, \tau) \ln \left(4 \sin^2 \frac{t - \tau}{2}\right) \psi(\tau) d\tau
\]

\[
(A_4^k \psi)(t) = \int_0^{2\pi} K_{k,2}(t, \tau) \psi(\tau) d\tau
\]

\[
(A_3^{ik} \psi)(t) = \int_0^{2\pi} K_{ik,1}(t, \tau) \ln \left(4 \sin^2 \frac{t - \tau}{2}\right) \psi(\tau) d\tau
\]

\[
(A_4^{ik} \psi)(t) = \int_0^{2\pi} K_{ik,2}(t, \tau) \psi(\tau) d\tau
\]

\[
(A_5^k \psi)(t) = \int_0^{2\pi} M_{k,1}(t, \tau) \ln \left(4 \sin^2 \frac{t - \tau}{2}\right) \psi(\tau) d\tau
\]

\[
(A_6^k \psi)(t) = \int_0^{2\pi} M_{k,2}(t, \tau) \psi(\tau) d\tau
\]

\[
(A_7 \psi)(t) = \left(\frac{1}{2} + \frac{i\eta}{4}\right) |x(t)| \psi(t)
\]

so that the integral equation (4.2) can be written in the form

\[
A_7 \mu + (A_1^k + A_2^k) \mu + [(A_3^k + A_4^k)(A_5^k + A_6^k)] \mu - [(A_1^{ik} + A_2^{ik})(A_3^{ik} + A_4^{ik})] \mu - i\eta [(A_1^k + A_2^k)(A_3^k + A_4^k)] \mu = f.
\]

Owing to the weakly singular nature of their kernels, the operators $A_j^k, j = 1, \ldots, 4$ and the operators $A_j^{ik}, j = 1, \ldots, 6$ map the Sobolev spaces $H^p[0, 2\pi]$ to the Sobolev spaces $H^{p+1}[0, 2\pi]$ [31], where $H^p[0, 2\pi]$ is the Sobolev space of periodic functions on the interval $[0, 2\pi]$ defined in [31].

We describe next a Nyström method based on trigonometric interpolation that follows closely the quadrature method introduced by Kress in [30], which in turn relies on the logarithmic quadrature methods introduced by Kussmaul [34] and Martensen [36]. We choose $n \in \mathbb{N}$ and the equidistant mesh $t_j^{(n)} = \frac{j\pi}{n}, j = 0, 1, \ldots, 2n - 1$. With respect to these nodal points the interpolation problem in the space $T_n$ of trigonometric polynomials of the form

\[
v(t) = \sum_{m=0}^{n} a_m \cos mt + \sum_{m=1}^{n-1} b_m \sin mt
\]

is uniquely solvable [31]. We denote by $P_n : C[0, 2\pi] \rightarrow T_n$ the corresponding interpolation operator and we will use in the error analysis the estimate [31]

\[
\|P_n g - g\|_q \leq C n^{q-p} \|g\|_p, \quad 0 \leq q \leq p, \quad \frac{1}{2} < p
\]

(4.18)
which is valid for all \( g \in H^p[0,2\pi] \) and a constant \( C \) depending on \( p \) and \( q \), where we denoted by \( \|g\|_p \) the norm of \( g \) in the Sobolev space \( H^p[0,2\pi] \). We use the quadrature rules [30]

\[
\int_0^{2\pi} \ln \left( 4 \sin^2 \frac{t - \tau}{2} \right) f(\tau) d\tau \approx \int_0^{2\pi} \ln \left( 4 \sin^2 \frac{t - \tau}{2} \right) (P_n f)(\tau) d\tau = \sum_{j=0}^{2n-1} R_j^{(n)}(t) f(t_j^{(n)})
\]

(4.19)

where the expressions \( R_j^{(n)}(t) \) are given by

\[
R_j^{(n)}(t) = -\frac{2\pi}{n} \sum_{m=1}^{n-1} \frac{1}{m} \cos m(t - t_j^{(n)}) - \frac{\pi}{n^2} \cos n(t - t_j^{(n)}).
\]

We also use the trapezoidal rule

\[
\int_0^{2\pi} f(\tau) d\tau \approx \int_0^{2\pi} (P_n f)(\tau) d\tau = \frac{\pi}{n} \sum_{j=0}^{2n-1} f(t_j^{(n)}).
\]

(4.20)

We apply the quadrature rules (4.19) and (4.20) to derive an approximation of the integral equations (4.17). To this end, we define the numerical quadrature operators

\[
(A_{1,n}^k \psi)(t) = \int_0^{2\pi} \ln \left( 4 \sin^2 \frac{t - \tau}{2} \right) (P_n H_{k,1}(t,\cdot) \psi)(\tau) d\tau = \sum_{j=0}^{2n-1} R_j^{(n)}(t) H_{k,1}(t,t_j^{(n)}) \psi(t_j^{(n)})
\]

\[
(A_{2,n}^k \psi)(t) = \int_0^{2\pi} (P_n H_{2,k}(t,\cdot) \psi)(\tau) d\tau = \frac{\pi}{n} \sum_{j=0}^{2n-1} H_{2,k}(t,t_j^{(n)}) \psi(t_j^{(n)})
\]

\[
(A_{3,n}^k \varphi)(t) = \int_0^{2\pi} \ln \left( 4 \sin^2 \frac{t - \tau}{2} \right) (P_n K_{k,1}(t,\cdot) \varphi)(\tau) d\tau = \sum_{j=0}^{2n-1} R_j^{(n)}(t) K_{1,k}(t,t_j^{(n)}) \varphi(t_j^{(n)})
\]

\[
(A_{4,n}^k \varphi)(t) = \int_0^{2\pi} (P_n K_{2,k}(t,\cdot) \varphi)(\tau) d\tau = \frac{\pi}{n} \sum_{j=0}^{2n-1} K_{2,k}(t,t_j^{(n)}) \varphi(t_j^{(n)})
\]

and similarly we define the numerical quadrature operators \( A_{j,n}^{ik}, j = 1, \ldots, 6 \). With these notations, we derive the approximating equation to equation (4.17)

\[
A_7 \mu_n + (A_{1,n}^k + A_{2,n}^k) \mu_n + [(A_{3,n}^k + A_{4,n}^k) P_n (A_{5,n}^{ik} + A_{6,n}^{ik})] \mu_n - [(A_{3,n}^{ik} + A_{4,n}^{ik}) P_n (A_{5,n}^{ik} + A_{6,n}^{ik})] \mu_n - i \eta [(A_{1,n}^{ik} + A_{2,n}^{ik}) P_n (A_{1,n}^{ik} + A_{2,n}^{ik})] \mu_n = f
\]

(4.21)

which we solve for \( \mu_n \in T_n \); we have used the fact that \( P_n(fg) = P_n(f) P_n(g) \) in the derivation of equation (4.21). Finally, we collocate equation (4.21) with the interpolation operator \( P_n \) in order to arrive at an approximating equation which is reduced to solving a finite dimensional linear system. We obtain the following linear system

\[
A_7 \mu_n + P_n(A_{1,n}^k + A_{2,n}^k) \mu_n + P_n[(A_{3,n}^{ik} + A_{4,n}^{ik}) P_n(A_{5,n}^{ik} + A_{6,n}^{ik})] \mu_n - P_n[(A_{3,n}^{ik} + A_{4,n}^{ik}) P_n(A_{5,n}^{ik} + A_{6,n}^{ik})] \mu_n = P_n f
\]

(4.22)
which we solve for $\mu_n \in H^p[0, 2\pi]$, $p > \frac{1}{2}$. Further, we can write the linear system (4.22) in explicit form

$$
\sum_{j=0}^{2n-1} \mu_n(t_j^{(n)}) \left\{ R_{[\ell-j]}^{(n)} H_{k,1}(t_j^{(n)}, t_j^{(n)}) + \frac{\pi}{n} H_{k,2}(t_j^{(n)}, t_j^{(n)}) \right\}
$$

$$+
\sum_{j=0}^{2n-1} \left( \sum_{m=0}^{2n-1} \mu_n(t_j^{(n)}) \left\{ R_{[\ell-m]}^{(n)} M_{ik,1}(t_j^{(n)}, t_m^{(n)}) + \frac{\pi}{n} M_{ik,2}(t_j^{(n)}, t_m^{(n)}) \right\} \right) \times
$$

$$\left\{ R_{[\ell-j]}^{(n)} K_{k,1}(t_j^{(n)}, t_j^{(n)}) + \frac{\pi}{n} K_{k,2}(t_j^{(n)}, t_j^{(n)}) \right\}
$$

$$-
\sum_{j=0}^{2n-1} \left( \sum_{m=0}^{2n-1} \mu_n(t_j^{(n)}) \left\{ R_{[\ell-m]}^{(n)} M_{ik,1}(t_j^{(n)}, t_m^{(n)}) + \frac{\pi}{n} M_{ik,2}(t_j^{(n)}, t_m^{(n)}) \right\} \right) \times
$$

$$\left\{ R_{[\ell-j]}^{(n)} K_{ik,1}(t_j^{(n)}, t_j^{(n)}) + \frac{\pi}{n} K_{ik,2}(t_j^{(n)}, t_j^{(n)}) \right\}
$$

$$+i\eta \sum_{j=0}^{2n-1} \left( \sum_{m=0}^{2n-1} \mu_n(t_j^{(n)}) \left\{ R_{[\ell-m]}^{(n)} H_{ik,1}(t_j^{(n)}, t_m^{(n)}) + \frac{\pi}{n} H_{ik,2}(t_j^{(n)}, t_m^{(n)}) \right\} \right) \times
$$

$$\left\{ R_{[\ell-j]}^{(n)} H_{ik,1}(t_j^{(n)}, t_j^{(n)}) + \frac{\pi}{n} H_{ik,2}(t_j^{(n)}, t_j^{(n)}) \right\}
$$

$$+ \left( \frac{1}{2} + \frac{i\eta}{4} \right) |x'(t_j^{(n)})| \psi(t_j^{(n)}) = |x'(t_j^{(n)})|g(x(t_j^{(n)})), \quad \ell = 0, 1, \ldots, 2n - 1, \quad (4.23)
$$

with $R_k^{(n)}(t_j) = R_{[k-j]}^{(n)}$ where

$$R_j^{(n)} = -\frac{1}{n} \left\{ \sum_{m=1}^{n-1} \frac{1}{m} \cos \frac{m \pi}{n} + \frac{(-1)^j}{2n} \right\}, \quad j = 0, 1, \ldots, 2n - 1.
$$

We will prove the convergence $\|\mu_n - \mu\|_p \to 0$, $n \to \infty$ for all $\frac{1}{2} < p \leq 1$ where $\|\psi\|_p$ denotes the norm of $\psi$ in $H^p[0, 2\pi]$. To this end, we use several results established in [31] concerning operators of the kind

$$(A\varphi)(t) = \int_0^{2\pi} \ln \left( 4 \sin^2 \frac{t - \tau}{2} \right) K(t, \tau)\varphi(\tau) d\tau
$$

$$(B\varphi)(t) = \int_0^{2\pi} H(t, \tau)\varphi(\tau) d\tau
$$

and their collocated versions

$$P_n A_n \varphi = P_n \int_0^{2\pi} \ln \left( 4 \sin^2 \frac{t - \tau}{2} \right) P_n(K(t, \cdot)\varphi)(\tau) d\tau
$$

$$P_n B_n \varphi = P_n \int_0^{2\pi} P_n(H(t, \cdot)\varphi)(\tau) d\tau.
$$

Under the assumption that $K$ and $H$ are infinitely differentiable and $2\pi$-periodic with respect to both variables, the following results holds

**Lemma 4.1** Under the assumption above on the kernels $K$ and $H$, for all $\frac{1}{2} < p \leq 1$ and all $\varphi \in H^p[0, 2\pi]$ we have the estimate

$$\|P_n A_n \varphi - A \varphi\|_p \leq Cn^{-p}\|\varphi\|_p, \quad \|P_n B_n \varphi - B \varphi\|_p \leq Cn^{-p}\|\varphi\|_p.
$$

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Proof. It follows from the result in Theorem 12.18 in [31] (page 215) that for all \( p > \frac{1}{2} \) and all \( \varphi \in H^p[0, 2\pi] \) we have the estimate

\[
\|A_n \varphi - A \varphi\|_1 \leq C_1 n^{-p}\|\varphi\|_p. \tag{4.24}
\]

We have for all \( p \) such that \( \frac{1}{2} < p \leq 1 \)

\[

\|P_n A_n \varphi - A \varphi\|_p \leq \|P_n A_n \varphi - P_n A \varphi\|_p + \|P_n A \varphi - A \varphi\|_p \leq C_2 \|A_n \varphi - A \varphi\|_p + \|P_n A \varphi - A \varphi\|_p
\]

Using estimate (4.24) to estimate the first term in the right-hand side of the equation above and estimate (4.18) for \( q = p \) and \( p = p + 1 \) to estimate the second term in the right-hand side of the equation above, we obtain

\[
\|P_n A_n \varphi - A \varphi\|_p \leq C_1 C_2 n^{-p}\|\varphi\|_p + C_3 n^{-1}\|A \varphi\|_{p+1} \leq C n^{-p}\|\varphi\|_p \tag{4.25}
\]

where we used the fact that \( A : H^p[0, 2\pi] \to H^{p+1}[0, 2\pi] \) is bounded [31]. The result for the operators \( B \) follows similarly. \( \square \)

Based on this result we establish the following

**Lemma 4.2** Let \( A_1 \) and \( A_2 \) be operators of either of the type \( A \) or \( B \) described above. Then for all \( \frac{1}{2} < p \leq 1 \) and all \( \varphi \in H^p[0, 2\pi] \) we have the estimate

\[
\|P_n A_{1,n} P_n A_{2,n} \varphi - A_1 A_2 \varphi\|_p \leq C n^{-p}\|\varphi\|_p.
\]

**Proof.** We start with the estimate

\[
\|P_n A_{1,n} P_n A_{2,n} \varphi - A_1 A_2 \varphi\|_p \leq \|P_n A_{1,n} P_n A_{2,n} \varphi - A_1 P_n A_{2,n} \varphi\|_p + \|A_1 P_n A_{2,n} \varphi - A_1 A_2 \varphi\|_p.
\]

We estimate the second term in the right-hand side of the equation above using the result in Lemma 4.1.

\[
\|A_1 P_n A_{2,n} \varphi - A_1 A_2 \varphi\|_p \leq \|A_1\|_p \|P_n A_{2,n} \varphi - A_2 \varphi\|_p \leq C_1 n^{-p}\|\varphi\|_p.
\]

A direct consequence of the result in Lemma 4.1 is that

\[
\|P_n A_{2,n} \varphi\|_p \leq C\|\varphi\|_p.
\]

From this estimate and one more application of Lemma 4.1 we obtain

\[
\|P_n A_{1,n} P_n A_{2,n} \varphi - A_1 P_n A_{2,n} \varphi\|_p \leq C_2 n^{-p}\|P_n A_{2,n} \varphi\|_p \leq C n^{-p}\|\varphi\|_p
\]

from which the result of the Lemma follows. \( \square \)

We denote by \( \mathcal{A} \) the continuous operator on the left-hand side of equation (4.1) and by \( \mathcal{A}_n \) the discrete operator on the left-hand side of equation (4.22). We view both operators as operators from \( H^p[0, 2\pi] \) to \( H^p[0, 2\pi] \) for \( \frac{1}{2} < p \leq 1 \). We note that in the case of analytic curves \( \Gamma \), the solution \( \mu \) of equation (4.1) is itself analytic and thus it belongs to all of the Sobolev spaces \( H^p[0, 2\pi] \) for all
$p > \frac{1}{2}$. It follows from the results in Lemma 4.1 and Lemma 4.2 that there exists a constant $C > 0$ such that

$$
\|A\psi - A_n\psi\|_p \leq cn^{-p}\|\psi\|_p
$$

(4.26)

for all $\psi \in H^p[0, 2\pi]$, $\frac{1}{2} < p \leq 1$. We formulate the convergence result of the solutions $\mu_n$ of equation (4.22) to the solution $\mu$ of equation (4.1).

**Theorem 4.3** For $\frac{1}{2} < p \leq 1$, if $f \in H^{2p}[0, 2\pi]$, then the approximating equation (4.22) has a unique solution $\mu_n$ for sufficiently large $n$ and we have the following error estimate

$$
\|\mu_n - \mu\|_p \leq Cn^{-p}(\|f\|_{2p} + \|\mu\|_p)
$$

for some constant $C = C(p)$. In the case when the boundary $\Gamma$ is analytic and $f$ is analytic, we have convergence of $\mu_n$ to $\mu$ in $H^p[0, 2\pi]$ of the order $O(n^{-m})$ for all $m \in \mathbb{N}$.

**Proof.** We have that $A : H^p[0, 2\pi] \to H^p[0, 2\pi]$ has a bounded inverse, and since by the estimates established in equation (4.26) we also have that the operators $A_n$ converge in the $p$-norm to the operator $A$. It follows from Neumann series considerations (see Theorem 10.1 in [31]) that the operators $A_n$ are invertible for large enough $n$ and the inverse operators $A_n^{-1}$ are uniformly bounded. We write

$$
\mu_n - \mu = A_n^{-1}\{(P_n f - f) + (A - A_n)\mu\}
$$

from which it follows that

$$
\|\mu_n - \mu\|_p \leq C(\|P_n f - f\|_p + \|A\mu - A_n\mu\|_p).
$$

Using estimates (4.18) and (4.26) together with the estimate above, we obtain the first result of the Theorem.

In the case when $\Gamma$ is analytic and the incident field is analytic (and thus $f$ is analytic), the solution $\mu$ of the integral equation (4.1) is itself analytic, and in particular infinitely differentiable. In the case of an infinitely differentiable function $f$, the estimate $\|P_n f - f\|_p$ can be improved to order $O(n^{-m})$ for all $m \in \mathbb{N}$. This estimate leads in turn to the same improved order of convergence in the estimate in Theorem 12.18 in [31]. Based on this fact, and since the kernels $K(\cdot, \cdot)$ and $H(\cdot, \cdot)$ of the integral operators of the type $A$ and $B$ described above that enter the integral equation (4.17) are infinitely differentiable (however the integral operators which involve purely imaginary wavenumbers are not analytic because of the use of the cutoff function $\chi$), we obtain the second result of the Theorem.

*Remark 4.4* The constants $C$ that appear in Theorem 4.3 depend on $\|A\|_{H^p(\Gamma) \to H^p(\Gamma)}$ and $A^{-1}\|_{H^p(\Gamma) \to H^p(\Gamma)}$. According to the estimates established in Sections 3.1 and 3.2, in the case when $\Gamma = S^1$, the latter quantities grow no faster than $Ck^{\frac{1}{2}}$ if the coupling coupling constant $\eta$ is such that $\eta \sim k^{\frac{1}{2}}$. Furthermore, our numerical evidence suggests that in the case of starlike boundaries $\Gamma$ the condition numbers of the operators $A$ can grow at most linearly with $k^{1/2}$ for a wide range of values of $\eta$.

*Remark 4.5* We note that since $p > \frac{1}{2}$, the result established in Theorem 4.3 together with the observations above imply uniform convergence of $\mu_n$ to $\mu$ with superalgebraic order of convergence.
4.2 Discretization of the ICFIER-RPS equations (2.18)

Assuming that the boundary curve $\Gamma$ is analytic and given by the parametrization, equation (2.18) can be written in parametric form as

$$-\frac{i\eta}{4\pi} \int_{0}^{2\pi} \cot \frac{\tau - t}{2} (PS_{ik} \psi)'(\tau) d\tau + \int_{0}^{2\pi} K_k(t, \tau)(PS_{ik} \psi)(\tau) d\tau + \int_{0}^{2\pi} H_k(t, \tau) \psi(\tau) d\tau + \frac{\psi(t) |x'(t)|}{2} = f(t), \quad 0 \leq t \leq 2\pi$$

(4.27)

where $\psi(t) \equiv \phi(x(t))$ and $f(t) = |x'(t)|g(x(t))$ with $g = \frac{\partial n^{inc}}{\partial n} |_\Gamma$, and the operators $H_k$ and $K_k$ were defined in equations (4.3) and (4.4) respectively. We introduce the following operators $T_0$ and $A_5$ by

$$T_0 = -\frac{1}{4\pi} \int_{0}^{2\pi} \cot \frac{\tau - t}{2} \psi'(\tau) d\tau$$

$$A_5 \psi(t) = \frac{|x'(t)|}{2} \psi(t)$$

(4.28)

so that the integral equation (4.27) can be written in the form

$$i\eta A_0 \psi + (A_{k,1} + A_{k,2}) \psi + [(A_{k,3} + A_{k,4}) PS_{ik}] \psi + A_5 \psi = f,$$

(4.29)

where $A_0 = T_0 \circ PS_{ik}$. The operator $PS_{ik}$ has the following mapping property: $PS_{ik} : H^p[0, 2\pi] \rightarrow H^{p+1}[0, 2\pi]$ which follows immediately from its spectral representation (2.17). The mapping properties of the operator $A_0$ can be also derived based on its spectral representation. Indeed, given that $\psi$ is a periodic function on the interval $[0, 2\pi]$ and hence the function $\psi(t)|x'(t)|$ can be expanded in Fourier series $\psi(t)|x'(t)| = \sum_{n \in \mathbb{Z}} \psi_n e^{int}$, the spectral representation of the operator $A_0$ is given by

$$(A_0 \psi)(t) = \sum_{n \in \mathbb{Z}} \frac{|n|}{4(n^2 + k^2)^{1/2}} \psi_n e^{int}$$

(4.30)

where we have used the fact that $T_0 u_n = \frac{|n|}{2} u_n$, $n \in \mathbb{Z}$ with $u_n(t) = e^{int}$ [31]. Equation (4.30) can be written as

$$(A_0 \psi)(t) = \frac{\psi(t)|x'(t)|}{4} + (\tilde{A}_0 \psi)(t)$$

where the operator $\tilde{A}_0$ is defined as

$$(\tilde{A}_0 \psi)(t) = -\frac{k^2}{4} \sum_{n \in \mathbb{Z}} \frac{\psi_n}{n^2 + k^2 + |n|(n^2 + k^2)^{1/2}} e^{int} = -\frac{k^2}{4} \sum_{n \in \mathbb{Z}} \zeta_n(k) \psi_n e^{int}$$

(4.31)

and hence $\tilde{A}_0 : H^p[0, 2\pi] \rightarrow H^{p+2}[0, 2\pi]$. With these notations, the integral equation (4.29) becomes

$$i\eta \tilde{A}_0 \psi + (A_{k,1} + A_{k,2}) \psi + [(A_{k,3} + A_{k,4}) PS_{ik}] \psi + A_7 \psi = f.$$  

(4.32)
In order to discretize the operators $PS_{ik}$ and $\tilde{A}_0$ we use the following interpolatory approximations

$$(PS_{ik}\psi)(t) \approx (PS_{ik}P_n\psi)(t) = (PS)_n\psi)(t) = \sum_{m=0}^{n} \frac{\psi_{m}^{(1)}}{(m^2 + k^2)^{1/2}} \cos mt + \sum_{m=1}^{n} \frac{\psi_{m}^{(2)}}{(m^2 + k^2)^{1/2}} \sin mt$$

$$(\tilde{A}_0\psi)(t) \approx (\tilde{A}_0 P_n\psi)(t) = (\tilde{A}_{0,n}\psi)(t) = \sum_{m=0}^{n} \psi_{m}^{(1)} \zeta_m(k) \cos mt + \sum_{m=1}^{n} \psi_{m}^{(2)} \zeta_m(k) \sin mt \quad (4.33)$$

where $(P_n\psi)(t) = \sum_{m=0}^{n} \psi_{m}^{(1)} \cos mt + \sum_{m=1}^{n-1} \psi_{m}^{(2)} \sin mt$.

We apply the quadrature rules (4.19) and (4.20) together with the approximations (4.33) to derive an approximation of the integral equations (4.32). We use the fact that $P_n(PS)_n = (PS)_n$ and we derive the approximating equation to equation (4.32)

$$i\eta \tilde{A}_{0,n}\psi_n + (A_{1,n}^k + A_{2,n}^k)\psi_n + (A_{3,n}^k + A_{4,n}^k)((PS)_n\psi_n) + A_\tau \psi_n = f \quad (4.34)$$

which we solve for $\psi_n \in T_n$. Finally, we collocate equation (4.34) with the interpolation operator $P_n$ in order to arrive at an approximating equation which is reduced to solving a finite dimensional linear system. We obtain

$$i\eta \tilde{A}_{0,n}\psi_n + P_n(A_{1,n}^k + A_{2,n}^k)\psi_n + P_n(A_{3,n}^k + A_{4,n}^k)((PS)_n\psi_n) + A_\tau \psi_n = P_n f \quad (4.35)$$

for $\psi_n \in H^p[0,2\pi]$. We used the fact that $P_n \tilde{A}_{0,n} = \tilde{A}_{0,n}$ in order to derive equations (4.35). We write the linear system (4.35) in the form $(PSA)_n\psi_n = f_n$ and we solve this linear system using Krylov subspace iterative solvers (e.g. GMRES). Thus, it suffices to describe the matrix-vector product $(PSA)_n\psi_n$. The components of the latter matrix-vector product are given explicitly by

$$((PSA)_n\psi_n)_\ell = \sum_{j=0}^{2n-1} \psi_n(t_j^{(n)}) \left\{ R_{\ell-j}^{(n)} H_1(t_j^{(n)}, t_\ell^{(n)}) + \frac{i\pi}{n} H_2(t_j^{(n)}, t_\ell^{(n)}) \right\}$$

$$+ \sum_{j=0}^{2n-1} \left\{ R_{\ell-j}^{(n)} K_1(t_j^{(n)}, t_\ell^{(n)}) + \frac{i\pi}{n} K_2(t_j^{(n)}, t_\ell^{(n)}) \right\} \times$$

$$F_{2n}^{-1} \{ \sigma(k,2n) \ast F_{2n} \{ \psi_n \} \} + i\eta F_{2n}^{-1} \{ \zeta(k,2n) \ast F_{2n} \{ \psi_n \} \} + \left( \frac{1}{2} + \frac{i\eta}{4} \right) \int_0^1 \sigma'(|t_\ell^{(n)}|) \psi_n(t_\ell^{(n)}), \ell = 0,1,\ldots,2n-1 \quad (4.36)$$

in terms of the discrete direct and indirect discrete Fourier transforms using $2n$ nodes, where we have used the MATLAB notation .* for multiplication in the Fourier space. We effect in practice the discrete Fourier transforms $F_{2n}$ and $F_{2n}^{-1}$ via the Fast Fourier Transform (FFT).

We denote by $B$ the continuous operator on the left-hand side of equation (4.27) and by $B_n$ the discrete operator on the left-hand side of equation (4.35). We view both operators as operators from $H^p[0,2\pi]$ to $H^p[0,2\pi]$ for $\frac{1}{2} < p \leq 1$. We note that in the case of analytic curves $\Gamma$, the solution $\psi$ of equation (4.27) is itself analytic and thus it belongs to all of the Sobolev spaces $H^p[0,2\pi]$ for all $p \geq 0$. It follows immediately from their definition that

$$\|(PS)_n\psi - PS_{ik}\psi\|_p \leq C_n^{-1} \|\psi\|_p, \|\tilde{A}_{0,n}\psi - \tilde{A}_0\psi\|_p \leq C_n^{-2} \|\psi\|_p.$$
and thus the results in Lemma 4.2 can be extended to derive similar bounds on the differences between the compositions of discrete and continuous operators that enter equation (4.35). Using this fact and the results in Lemma 4.1, we obtain that there exists a constant \( C > 0 \) such that

\[
\| \mathcal{B} \psi - \mathcal{B}_n \psi \|_p \leq C n^{-p} \| \psi \|_p
\]

for all \( \psi \in H^p[0,2\pi] \), \( \frac{1}{2} < p \leq 1 \). We formulate the convergence result of the solutions \( \psi_n \) of equation (4.35) to the solution \( \psi \) of equation (4.27). The proof of this result follows the lines of the proof of Theorem 4.3. In the case when the boundary \( \Gamma \) is analytic, the estimate in Theorem 12.18 in [31], which is the cornerstone of the error analysis presented above, can be improved to order \( O(e^{-ns}) \) for some positive constant \( s \). Since the kernels \( K(\cdot,\cdot) \) and \( H(\cdot,\cdot) \) of the integral operators of the type \( A \) and \( B \) described above that enter the integral equation (4.27) are analytic, we get the following

**Theorem 4.6** For \( \frac{1}{2} < p \leq 1 \), if \( f \in H^{2p}[0,2\pi] \), then the approximating equation (4.35) has a unique solution \( \psi_n \) for sufficiently large \( n \) and we have the following error estimate

\[
\| \psi_n - \psi \|_p \leq C n^{-p}(\| f \|_{2p} + \| \psi \|_p)
\]

for some constant \( C = C(p) \), where \( \psi \) is the solution of equation (4.27). In the case when the boundary \( \Gamma \) is analytic and the function \( f \) is analytic, we can deduce convergence of the order \( O(e^{-ns}) \) for some positive constant \( s \) in the estimate above.

**Remark 4.7** We note that since \( p > \frac{1}{2} \), the result established in Theorem 4.6 together with the observations above imply uniform convergence of \( \psi_n \) to \( \psi \) with exponential order of convergence.

## 5 Numerical Results

We present in this section a variety of numerical results that demonstrate the properties of the regularized combined field integral equations (2.16) and (2.18) constructed in the previous sections. Solutions of the linear systems arising from the Nyström discretizations of both types of Regularized Combined Field Integral Equations according to the description in Section 4 are obtained by means of the fully complex version of the iterative solver GMRES [41]. We used the coupling constant \( \eta = 2 \) in both equations (2.16) and (2.18); our extensive numerical experiments suggest that this value of the coupling constant \( \eta \) leads to nearly optimal numbers of GMRES iterations to reach desired (small) GMRES relative residuals. The results presented in the Tables 1–3 in this section were obtained by prescribing a GMRES relative residual tolerances equal to \( 10^{-16} \) in the case of circular geometries (for which exact solutions are available) or equal to either \( 10^{-8} \) or \( 10^{-12} \) otherwise.

We present scattering experiments concerning the following smooth geometries: (a) a unit circle, (b) an object with five petals whose parametrization is given in polar coordinates as \( x_1(t) = r(t) \cos t \), \( x_2(t) = r(t) \sin t \) with \( r(t) = 1 + 0.3 \cos 5t \), and (c) a cavity whose parametrization is given by \( x(t) = (x_1(t), x_2(t), x_3(t), x_4(t)) \), \( x_1(t) = (\cos t + 2 \cos 2t)/2.5 \), \( x_2(t) = Y(t)/2 - Y_s(t)/48 \), \( x_3(t) = -4 \sin t + 7 \sin 2t - 6 \sin 3t + 2 \sin 4t \), where \( Y(t) = \sin t + \sin 2t + 1/2 \sin 3t \), \( Y_s(t) = -4 \sin t + 7 \sin 2t - 6 \sin 3t + 2 \sin 4t \), see Figure 2. We note that each of these geometries has a diameter equals to 2.
For every scattering experiment we consider plane-wave incidence $u^{\text{inc}}$ and we present maximum far-field errors, that is we choose sufficiently many directions $\hat{x}$ and for each direction we compute the far-field amplitude $u_\infty^s(\hat{x})$ defined as

$$u^s(x) = \frac{e^{ik|x|}}{\sqrt{|x|}} \left( u_\infty^s(\hat{x}) + O\left(\frac{1}{|x|}\right) \right), \quad |x| \to \infty. \tag{5.1}$$

The maximum far-field errors were evaluated through comparisons of the numerical solutions $u_\infty^{s,\text{calc}}$ corresponding to either formulation (2.16) and (2.18) with reference solutions $u_\infty^{s,\text{ref}}$ by means of the relation

$$\varepsilon_\infty = \max|u_\infty^{s,\text{calc}}(\hat{x}) - u_\infty^{s,\text{ref}}(\hat{x})|. \tag{5.2}$$

The latter solutions $u_\infty^{s,\text{ref}}$ were produced using either exact Mie-series solutions in the case of circular geometries or solutions corresponding with refined discretizations (more precisely discretizations that use 12 points per wavelength) based on the formulation ICFIE-R (2.16) with GMRES residuals of $10^{-12}$ for all other geometries. Besides far field errors, Tables 1–3 display the numbers of iterations required by the GMRES solver to reach a relative residual of $10^{-16}$ in the case of circular geometries and of $10^{-8}$ for the other two geometries. We used discretizations ranging from 4 to 10 discretization points per wavelength, for frequencies in the medium to the high-frequency range, i.e. $k = 2^i, i = 3, \ldots, 9$ corresponding to acoustic scattering problems of sizes ranging from $2.5\lambda$ to $163.2\lambda$. The columns “Unknowns” in all Tables 1–3 display the numbers of unknowns used in each case, which equal to the value $2n$ defined in Section 4. In all of the scattering experiments we considered plane-wave incident fields; in the case of the cavity the direction of the incident field was taken to be equal to $d = (1, 0)$ in order to generate multiple reflections; in the other two cases we considered plane-wave incident fields of direction $d = (0, -1)$. As it can be seen in Tables 1–3, our solvers converge with high-order, as predicted by the error analysis in Section 4.

We conclude from the results displayed in Tables 1–3 that the formulations ICFIE-R (2.16) and ICFIE-RPS (2.18) lead to small numbers of GMRES residuals to reach desired tolerances, the iteration count being slightly more favorable in the case of the ICFIE-R formulation (2.16). However, for the same discretization, our implementation of a matrix-vector product corresponding to formulation ICFIE-RPS (2.18) is about 1.28 less expensive than a matrix-vector product corresponding to formulation ICFIE-R (2.16). For instance, using 512 unknowns in the case of the five-petal geometry, our MATLAB implementation of a matrix-vector product corresponding to the formulation ICFIE-R (2.18) took 43.68 seconds whereas a matrix-vector product corresponding to the formulation ICFIE-RPS (2.16) took 56.19 seconds on a MacPro machine with $2 \times 3\text{GHz}$ Quad-core Intel Xeon. We also remark that the number of GMRES iterations varies modestly with
Table 1: Scattering experiments for the unit circle, plane wave incidence, maximum errors in the far-field, and numbers of iterations needed to reach GMRES relative residuals equal to $10^{-16}$

<table>
<thead>
<tr>
<th>Size</th>
<th>Unknowns</th>
<th>ICFIE – RPS</th>
<th>ICFIE – R</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Iter. $\epsilon_\infty$</td>
<td>Iter. $\epsilon_\infty$</td>
</tr>
<tr>
<td>2.5$\lambda$</td>
<td>32</td>
<td>18</td>
<td>$3.5 \times 10^{-5}$</td>
</tr>
<tr>
<td>2.5$\lambda$</td>
<td>48</td>
<td>18</td>
<td>$3.4 \times 10^{-9}$</td>
</tr>
<tr>
<td>2.5$\lambda$</td>
<td>64</td>
<td>18</td>
<td>$5.0 \times 10^{-15}$</td>
</tr>
<tr>
<td>5.1$\lambda$</td>
<td>64</td>
<td>26</td>
<td>$5.1 \times 10^{-4}$</td>
</tr>
<tr>
<td>5.1$\lambda$</td>
<td>96</td>
<td>26</td>
<td>$2.7 \times 10^{-14}$</td>
</tr>
<tr>
<td>10.2$\lambda$</td>
<td>128</td>
<td>39</td>
<td>$2.0 \times 10^{-4}$</td>
</tr>
<tr>
<td>10.2$\lambda$</td>
<td>160</td>
<td>39</td>
<td>$1.2 \times 10^{-14}$</td>
</tr>
<tr>
<td>20.4$\lambda$</td>
<td>384</td>
<td>51</td>
<td>$6.9 \times 10^{-5}$</td>
</tr>
<tr>
<td>20.4$\lambda$</td>
<td>640</td>
<td>51</td>
<td>$1.4 \times 10^{-14}$</td>
</tr>
<tr>
<td>40.8$\lambda$</td>
<td>768</td>
<td>71</td>
<td>$2.5 \times 10^{-5}$</td>
</tr>
<tr>
<td>40.8$\lambda$</td>
<td>1280</td>
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</tr>
<tr>
<td>81.6$\lambda$</td>
<td>768</td>
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<td>$6.9 \times 10^{-6}$</td>
</tr>
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<td>1280</td>
<td>91</td>
<td>$7.2 \times 10^{-14}$</td>
</tr>
<tr>
<td>163.2$\lambda$</td>
<td>1536</td>
<td>105</td>
<td>$3.3 \times 10^{-6}$</td>
</tr>
<tr>
<td>163.2$\lambda$</td>
<td>2560</td>
<td>105</td>
<td>$2.8 \times 10^{-13}$</td>
</tr>
</tbody>
</table>

Table 2: Scattering experiments for the five petal object, plane wave incidence, maximum errors in the far-field, and numbers of iterations needed to reach GMRES relative residuals equal to $10^{-8}$

<table>
<thead>
<tr>
<th>Size</th>
<th>Unknowns</th>
<th>ICFIE – RPS</th>
<th>ICFIE – R</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Iter. $\epsilon_\infty$</td>
<td>Iter. $\epsilon_\infty$</td>
</tr>
<tr>
<td>2.5$\lambda$</td>
<td>48</td>
<td>19</td>
<td>$3.5 \times 10^{-5}$</td>
</tr>
<tr>
<td>2.5$\lambda$</td>
<td>64</td>
<td>19</td>
<td>$1.2 \times 10^{-8}$</td>
</tr>
<tr>
<td>5.1$\lambda$</td>
<td>96</td>
<td>21</td>
<td>$3.8 \times 10^{-5}$</td>
</tr>
<tr>
<td>5.1$\lambda$</td>
<td>128</td>
<td>21</td>
<td>$1.5 \times 10^{-7}$</td>
</tr>
<tr>
<td>10.2$\lambda$</td>
<td>192</td>
<td>24</td>
<td>$4.0 \times 10^{-6}$</td>
</tr>
<tr>
<td>10.2$\lambda$</td>
<td>256</td>
<td>24</td>
<td>$1.5 \times 10^{-8}$</td>
</tr>
<tr>
<td>20.4$\lambda$</td>
<td>384</td>
<td>27</td>
<td>$1.1 \times 10^{-3}$</td>
</tr>
<tr>
<td>20.4$\lambda$</td>
<td>512</td>
<td>27</td>
<td>$1.4 \times 10^{-8}$</td>
</tr>
<tr>
<td>40.8$\lambda$</td>
<td>768</td>
<td>30</td>
<td>$1.3 \times 10^{-3}$</td>
</tr>
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<td>1024</td>
<td>30</td>
<td>$1.4 \times 10^{-8}$</td>
</tr>
<tr>
<td>81.6$\lambda$</td>
<td>1536</td>
<td>37</td>
<td>$1.1 \times 10^{-3}$</td>
</tr>
<tr>
<td>81.6$\lambda$</td>
<td>2048</td>
<td>37</td>
<td>$1.6 \times 10^{-8}$</td>
</tr>
<tr>
<td>163.2$\lambda$</td>
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<td>45</td>
<td>$2.9 \times 10^{-3}$</td>
</tr>
<tr>
<td>163.2$\lambda$</td>
<td>4096</td>
<td>45</td>
<td>$1.8 \times 10^{-8}$</td>
</tr>
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</table>
Table 3: Scattering experiments for the cavity geometry, plane wave incidence, maximum errors in the far-field, and numbers of iterations needed to reach GMRES relative residuals equal to $10^{-8}$

<table>
<thead>
<tr>
<th>Size</th>
<th>Unknowns</th>
<th>ICFIE – RPS</th>
<th>ICFIE – R</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Iter.</td>
<td>$\epsilon_\infty$</td>
<td>Iter.</td>
</tr>
<tr>
<td>2.5$\lambda$</td>
<td>64</td>
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<td>$8.7 \times 10^{-4}$</td>
</tr>
<tr>
<td>2.5$\lambda$</td>
<td>96</td>
<td>21</td>
<td>$1.6 \times 10^{-7}$</td>
</tr>
<tr>
<td>5.1$\lambda$</td>
<td>128</td>
<td>25</td>
<td>$3.9 \times 10^{-4}$</td>
</tr>
<tr>
<td>5.1$\lambda$</td>
<td>160</td>
<td>25</td>
<td>$3.6 \times 10^{-7}$</td>
</tr>
<tr>
<td>10.2$\lambda$</td>
<td>256</td>
<td>35</td>
<td>$2.1 \times 10^{-4}$</td>
</tr>
<tr>
<td>10.2$\lambda$</td>
<td>320</td>
<td>35</td>
<td>$1.8 \times 10^{-8}$</td>
</tr>
<tr>
<td>20.4$\lambda$</td>
<td>512</td>
<td>52</td>
<td>$2.0 \times 10^{-4}$</td>
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<tr>
<td>20.4$\lambda$</td>
<td>640</td>
<td>52</td>
<td>$1.9 \times 10^{-8}$</td>
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<tr>
<td>40.8$\lambda$</td>
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<td>81.6$\lambda$</td>
<td>2034</td>
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<td>163.2$\lambda$</td>
<td>4096</td>
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<tr>
<td>163.2$\lambda$</td>
<td>5120</td>
<td>233</td>
<td>$3.8 \times 10^{-8}$</td>
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</table>

The frequency in the case of starlike geometries (Tables 1 and 2) but it tends to increase almost linearly with the frequency in the case of the cavity geometry in Table 3. We mention that the use of other coupling parameters such as $\eta = 4$ or $\eta = k^3$ leads to larger iteration counts; in the former case the numbers of GMRES iterations are only slightly larger, while in the latter case the numbers of GMRES iterations can be about two times larger than those presented in Tables 1–3 to reach the same GMRES residuals.

We use next the high-order discretizations of the ICFIE-R operators defined in equations (2.16) to investigate numerically their coercivity in the case of starlike smooth domains.

### 5.1 Numerical evidence of coercivity of the ICFIE-R operators for starlike scatterers

We provide in this section numerical evidence of the coercivity of the ICFIE-R operators $A_{k,S_{ik}}$ defined in equation (2.16) for starlike geometries. For definiteness, we say that $\Gamma$ is starlike with respect to the origin $O$ if $\text{ess inf}_{x \in \Gamma} x \cdot n(x) > 0$, where $n(x)$ denotes the normal to the curve $\Gamma$ at the point $x$ [16]. Since the boundary integral operators $A_{k,S_{ik}}$ are not normal (a proof of this fact is missing in the literature) except for the case of circular geometries, their coercivity cannot be inferred from information on their spectra alone. Instead, and just as in [11], we investigate numerically the coercivity of the operators $A_{k,S_{ik}}$ through computation of their numerical ranges.

For an operator $T : H \rightarrow H$, where $H$ is a Hilbert space, the numerical range of $T$ is defined as the following (convex) set

$$W(T) = \left\{ \frac{\langle Tv, v \rangle}{\langle v, v \rangle}, \ v \in H, \ v \neq 0 \right\}. \ (5.3)$$

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If $0 \notin W(T)$, it follows that the operator $T$ is coercive with the coercivity constant $\gamma$ equal to the distance from the origin to the set $W(T)$. We investigate the coercivity of the operators $A_{k,S_{ik}}$ for various geometries and wavenumbers via their numerical ranges. Specifically, for each test geometry, we considered four wavenumbers $k = 16, 32, 64, 128$ and we computed approximations of the numerical ranges of the operators $A_{k,S_{ik}}$ by using the high-order Nyström discretization described in Section 4 using $N = [6k]$ discretization points for each wavenumber $k$. Specifically, we computed numerical ranges of the discrete operators by building explicitly the matrices corresponding to the discrete operators. Given the very high-order convergence of the Nyström discretization presented in Section 4, the numerical ranges of the discrete operators approximate very accurately the numerical ranges of the continuous operators. For the computation of the numerical ranges of the discrete operators we relied on a well-known algorithm [24], which for the sake of completeness we describe next (we note that the same algorithm has been used in [11]).

Given the matrix $A_N \in \mathbb{C}^{N \times N}$ corresponding to the Nyström discretization with $N$ points of the operators $A_{k,S_{ik}}$, and for each direction $\theta_j = \frac{j \pi}{P}$, $j = 0, 1, \ldots, P - 1$ we computed the Hermitian part of the matrix $e^{i \theta_j} A_N$, that is $A_{N,H}^{j} = \frac{1}{2} (e^{i \theta_j} A_N + e^{-i \theta_j} A_N^\ast)$ and the eigenvectors $v_{\min}^{j}$ and $v_{\max}^{j}$ corresponding to the smallest and respectively largest eigenvalues of the matrix $A_{N,H}^{j}$. For each of the vectors $v_{\min}^{j}$ and $v_{\max}^{j}$ we computed the complex quantities

$$w_{\min}^{j} = \frac{\langle A_N v_{\min}^{j}, v_{\min}^{j} \rangle}{\langle v_{\min}^{j}, v_{\min}^{j} \rangle}, \quad w_{\max}^{j} = \frac{\langle A_N v_{\max}^{j}, v_{\max}^{j} \rangle}{\langle v_{\max}^{j}, v_{\max}^{j} \rangle}$$

and then we computed the numerical range of $A_N$ as the convex hull of the pairs $(w_{\min}^{j}, w_{\max}^{j})$, $j = 0, 1, \ldots, P - 1$. For each geometry and wavenumber $k$ considered, we took $P = 100$ in the algorithm described above to compute the numerical ranges.

First, we present numerical ranges in the case of two strictly convex geometries: (a) a rounded triangle whose parametrization is given by $x(t) = (\cos t + 0.2 \cos 2t, \sin t - 0.2 \sin 2t)$ —see Figure 2, and (b) an ellipse with semi-axes $a = 1$ and $b = 0.25$. We present in Figure 3 and Figure 4 computed numerical ranges for these two geometries and four wavenumbers together with the spectra of the operators $A_{k,S_{ik}}$. For these geometries, and for certain choices of the coupling parameter $\eta$ the operators $A_{k,S_{ik}}$ are coercive, just as it was proved to be the case for circular geometries. Specifically, in the case $\eta = 4$ (which is used in Figure 3 and Figure 4) and $\eta = k^{1/3}$ the operators $A_{k,S_{ik}}$ are coercive since the origin $(0,0)$ is outside their numerical ranges. Furthermore, we computed approximations of the coercivity constants in each of the cases above. We computed numerical ranges for two discretizations: one corresponding to six discretization points per wavelength (used in Figure 3 and Figure 4, i.e. $N = [6k]$) and one corresponding to twelve discretization points per wavelength (i.e. $N = [12k]$) and the corresponding coercivity constants were computed as the distances from the origin to the numerical ranges. We note that the coercivity constants computed using the coarser discretization have three exact digits when compared against those corresponding to the finer discretizations. Specifically, in the case of the triangular geometry and coupling constant $\eta = 4$ we obtained the following values for the coercivity constants $\gamma \approx 0.450, 0.416, 0.387, 0.351$ for $k = 16, 32, 64$ and $128$ respectively; in the case of the coupling constant $\eta = k^{1/3}$ we obtained the following values for the coercivity constants $\gamma \approx 0.493, 0.486, 0.479, 0.475$ for $k = 16, 32, 64$ and $128$ respectively. In the case of the elliptical geometry and coupling constant $\eta = 4$ we obtained the following values for the coercivity constants $\gamma \approx 0.439, 0.393, 0.387, 0.367$ for $k = 16, 32, 64$ and $128$ respectively; in the case of coupling constant $\eta = k^{1/3}$ we obtained the following values for the coercivity constants $\gamma \approx 0.483, 0.478, 0.483, 0.487$ for $k = 16, 32, 64$ and $128$ respectively.
Figure 3: Computed numerical ranges and spectra of the operators $A_{k,Si_k}$ with $\eta = 4$ for the triangle geometry for four wavenumbers $k = 16, 32, 64, 128$ pictured top left, top right, bottom left and bottom right respectively. The numerical ranges are the bounded domains enclosed by the closed curves in the figures; for reference, we also displayed the origin which is not contained in any of these domains. The coercivity constants are $\gamma \approx 0.4507, 0.4163, 0.3877, 0.3510$ for $k = 16, 32, 64$ and 128 respectively.

We observe from these numerical experiments that for the choice of a coupling constant $\eta = k^{\frac{1}{3}}$ the coercivity constants associated with the operators $A_{k,Si_k}$ appear to be independent of the wavenumber $k$ for strictly convex geometries.

For certain choices of the coupling parameter $\eta$, the numerical evidence presented in Figure 5 suggests that the operators $A_{k,Si_k}$ are also coercive in the case of non-convex but starlike geometries such as the five petal geometry. For instance, in the case of the five-petal geometry, taking $\eta = 18$ for all four wavenumbers $k = 8, 16, 32, 128$ we see in Figure 5 that the origin is outside the computed numerical ranges. Furthermore, we get the following approximations for the coercivity constants $\gamma \approx 0.3775, 0.3114, 0.204, 0.0829$ for $k = 16, 32, 64$ and 128 respectively (again here, these approximations of the coercivity constant are accurate with three digits).

We conclude this section with two remarks. First, our numerical evidence suggests that the ICFIE-R operators $A_{k,Si_k}$ are not coercive in the case of the cavity geometry, regardless of the choice of the coupling constant $\eta$. Second, by constructing explicitly the matrices corresponding to the ICFIE-RPS operators $A_{k,PSi_k}$ defined in equation (2.18) (note that the discretization of these operators presented in Section 4 delivers only the matrix-vector product associated with these...
Figure 4: Computed numerical ranges and spectra of the operators $A_{k,S_{ik}}$ with $\eta = 4$ for an elliptical geometry of semi-axes $a = 1$ and $b = 0.25$ for four wavenumbers $k = 16, 32, 64, 128$ pictured top left, top right, bottom left and bottom right respectively. The numerical ranges are the bounded domains enclosed by the closed curves in the figures; for reference, we also displayed the origin which is not contained in any of these domains. The coercivity constants are $\gamma \approx 0.439, 0.393, 0.387, 0.367$ for $k = 16, 32, 64$ and 128 respectively.
Figure 5: Computed numerical ranges and spectra of the operators $A_{k,S_{ih}}$ with $\eta = 18$ for the five petal geometry and for four wavenumbers $k = 16, 32, 64, 128$ pictured top left, top right, bottom left and bottom right respectively. The numerical ranges are the bounded domains enclosed by the closed curves in the figures; for reference, we also displayed the origin which is not contained in any of these domains. The coercivity constants are $\gamma \approx 0.3775, 0.3114, 0.204, 0.0829$ for $k = 16, 32, 64$ and 128 respectively.
linear operators), we obtained similar numerical ranges to the ones pertaining to the ICFIE-R operators $A_{k,S_{ik}}$ defined in equation (2.16).

6 Conclusions

We analyzed the wavenumber dependence of the norms of the operators that enter a class of Regularized Combined Field Integral Equations (CFIER) for the solution of two and three dimensional scattering problems with Neumann boundary conditions. We established that these norms grow modestly with the wavenumber in the high frequency regime for smooth boundaries. Furthermore, we proved that for certain choices of the coupling parameters, the CFIER operators are coercive in the case of circular and spherical geometries with coercivity constants that are independent of the wavenumber. We presented discretizations of the two-dimensional CFIER integral operators that relies on collocation methods based on global trigonometric interpolation. The ensuing discrete operators converge with high-order to the continuous ones. Based on these very accurate discretizations, we compute the discrete numerical ranges of the two dimensional CFIER operators and we demonstrate that the two dimensional CFIER operators are coercive for (a) smooth and strictly convex domains with coercivity constants that are independent of the wavenumber, and (b) smooth, starlike non-convex domains with coercivity constants that appear to depend on the wavenumber.

Acknowledgments

Yassine Boubendir gratefully acknowledge support from NSF through contract DMS-1016405. Catalin Turc gratefully acknowledge support from NSF through contract DMS-1008076.

7 Appendix

We will prove that the principal symbol of the pseudodifferential operator $S_{ik}$ is equal to $\frac{1}{2\sqrt{k^2+\xi^2}}$ and that the next term in the asymptotic expansion of the symbol of $S_{ik}$ is of order $|\xi|^{-3}$. Our presentation uses techniques originally developed in [44]. Using a parametrization of the closed curve $\Gamma$, i.e. $\Gamma = \{x(t) : t \in I = [0, 2\pi]\}$, we get that $|x(t) - x(t_0)| = (|x'(t)| + L(t,t_0))|t - t_0|$ for all $t$ and $t_0$ in $I \times I$, where $L(t,t_0)$ is smooth on $I \times I$ and $L(t_0,t_0) = 0$. Using the fact [31] that

$$\frac{i}{4}H_0^{(1)}(kz) = -\frac{1}{2\pi} \log z + \frac{i}{4} - \frac{1}{2\pi} \left( \log \frac{k}{2} + C \right) + O(z^2 \log z)$$

where $C$ is Euler’s constant, we get that the kernel of $S_{iK}$ can be expressed as

$$\frac{i}{4}H_0^{(1)}(ik|x(t) - x(t_0)||x'(t)|) = \frac{i}{4}H_0^{(1)}(ik|t - t_0||x'(t)|)$$

$$+ \frac{i}{4}H_0^{(1)}(ik|t - t_0||r(t_0)(t - t_0)^2 + \Phi(t,t_0)$$

(7.1)
for two smooth functions $r(t)$ and $\Phi(t,t_0)$. Since any operator $A$ acting on functions $u \in C^\infty_0(I)$ can be lifted to an operator $A_{\Gamma}$ acting on $C^\infty_0(\Gamma)$ via $A = A_{\Gamma}(u \circ x^{-1}) \circ x$, we get that the operator $S_{ik} : C^\infty_0(\Gamma) \rightarrow C^\infty(\Gamma)$ can be represented using (7.1) as the $S_{\Gamma}$ of the following operator:

$$ (Su)(t_0) = \int_{\mathbb{R}} \frac{i}{4} H_0^{(1)}(ik|t-t_0|)|x'(t)|u(t)dt $$

$$ + \int_{\mathbb{R}} \frac{i}{4} H_0^{(1)}(ik|t-t_0|)r(t,t_0)(t-t_0)^2u(t)dt + \int_{\mathbb{R}} \eta(|t-t_0|)\Phi(t,t_0)u(t)dt \tag{7.2} $$

where $u \in C^\infty_0(I)$ and $\eta$ is a smooth cut-off function such that $\eta(|t-t_0|) = 1$ for all $t, t_0 \in I$. Since the integral operators in (7.2) are of convolution type, we get that

$$ (Su)(t_0) = \frac{1}{2\pi} \int_{\mathbb{R}} \sigma(t_0, \xi)e^{it_0 \xi} \hat{u}(\xi)d\xi $$

$$ = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{2\sqrt{k^2 + \xi^2}} e^{it_0 \xi} \hat{u}(\xi)d\xi + \frac{1}{2\pi} \int_{\mathbb{R}} \sigma_1(t_0, \xi)e^{it_0 \xi} \hat{u}(\xi)d\xi, \tag{7.3} $$

where we denoted by $\hat{u}$ the Fourier transform of the function $|x'(t)|u(t)$ and we have used the fact that the Fourier transform of $\frac{i}{4} H_0^{(1)}(i|z|) = \frac{1}{2\pi} K_0(|z|)$ is equal to $\frac{1}{2\sqrt{\xi^2}}$ with $\langle \xi \rangle = \sqrt{1 + \xi^2}$ [26]. Furthermore, it can be shown that the Fourier transform of $\frac{i}{4} H_0^{(1)}(ik|z|)z^2$ is of order $\langle \xi \rangle^{-3}$ (it can be shown to equal to $k^2/4\langle \xi \rangle^{-3}$ [35]) whereas the Fourier transform of $\eta(|z|)\Phi(z)$ decays exponentially on account of the smoothness of $\Phi$. Thus, the principal symbol of the pseudodifferential operator $S_{\Gamma}$ of order $-1$ is equal to $\frac{1}{2\sqrt{k^2 + \xi^2}}$ and the remainder is a pseudodifferential operator of order $-3$ as claimed in the text.

References


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