Wavenumber estimates for regularized combined field boundary integral operators in acoustic scattering problems with Neumann boundary conditions

Yassine Boubendir, Catalin Turc
boubendir@njit.edu, catalin.c.turc@njit.edu

Abstract

We study the coercivity properties and the norm dependence on the wavenumber k of certain regularized combined field boundary integral operators that we recently introduced for the solution of two and three-dimensional acoustic scattering problems with Neumann boundary conditions. We show that in the case of circular and spherical boundaries, our regularized combined field boundary integral operators are L^2 coercive for large enough values of the coupling parameter, and that the norms of these operators are bounded by constant multiples of the coupling parameter. We establish that the norms of the regularized combined field boundary integral operators grow modestly with the wavenumber k for smooth boundaries and we provide numerical evidence that these operators are L^2 coercive for two-dimensional starlike boundaries. We present and analyze a fully discrete collocation (Nyström) method for the solution of two-dimensional acoustic scattering problems with Neumann boundary conditions based on regularized combined field integral equations. In particular, for analytic boundaries and boundary data, we establish pointwise superalgebraic convergence rates of the discrete solutions.

Keywords: Helmholtz equations, Regularized Combined Field Integral Equations, Coercivity, Numerical range, Trigonometric interpolation, Collocation methods.

1 Introduction

Numerical methods based on integral equation formulations have certain advantages over those that use volumetric formulations for the solution of acoustic homogeneous scattering problems, largely owing to 1) the reduction in dimensionality that is achieved from posing the scattering problems on the lower-dimensional boundary of the scatterers and 2) their explicit enforcement of the radiation conditions through choices of outgoing Green's functions in the boundary layer potentials that enter the integral formulations. In addition, well-conditioned integral equations are available for acoustic scattering problems with various boundary conditions. For instance, in the case of Dirichlet boundary conditions the scattered field can be represented as a suitable linear combination of single and double layer potentials leading to classical Combined Field Integral Equations (CFIE) [7]. However, in the case of Neumann boundary conditions, the same approach leads to CFIE that contain hypersingular operators., e.g. the normal derivatives of double layer potentials [12]. These hypersingular operators are pseudodifferential operators of order one (they behave like a derivative) and thus the spectra of the CFIE operators accumulate at infinity. These unfavorable spectral distributions lead upon discretizations to linear systems that require increasingly larger numbers of Krylov-subspace iterations for refined discretizations. The derivative-like effect of the hypersingular operators can be mitigated by the use of the regularizing operators that are pseudo-inverses of the hypersingular operators [3–5, 7, 9, 10, 13, 35, 40]; the resulting combined field regularized integral equations (CFIER) are uniquely solvable across the frequency spectrum and behave in practice as integral equations of the second kind.

The aforementioned CFIER formulations constitute important alternatives to the classical CFIE, especially in the very low and high-frequency regimes where solvers based on certain CFIER consistently outperform those based on CFIE in terms of computational cost with no undue compromise in accuracy [9]. However, besides evidence that solvers based on CFIER formulations work very well in practice for high-frequency problems, no significant attempts have been made at understanding the dependence on the wavenumber of the norms and condition numbers of the boundary integral operators that enter those formulations. This situation is in contrast to the case of Dirichlet boundary conditions wherein several important recent papers have investigated the explicit wavenumber dependence in the high-frequency regime of the norms of the classical CFIE operators introduced in [7] together with that of the norm of their inverses [16, 17, 27, 38]. An extremely interesting and important fact that was emphasized in these works is the L^2 coercivity of the CFIE operators which was first rigorously established for the case of circular and spherical geometries in [27], then conjectured to be true based on supporting numerical evidence for other smooth and non-smooth two-dimensional geometries [11]. Recently, the availability of frequency-uniform coercive boundary integral equations was rigorously established [45] in the case of Dirichlet boundary conditions. The new integral equations introduced in [45], although different from the classical CFIE in [7], are still integral equations of the second kind. The availability of boundary integral equations whose underlying operators are coercive is important as the coercivity property facilitates error analysis of Galerkin discretizations of boundary integral equations in non-smooth domains.

The combined field regularized integral equations (CFIER) formulations that we use in this paper consist of suitable combinations of single layer potentials and double layer potentials that act on certain regularizing operators. The regularizing operators that we use are pseudoinverses of Neumann-to-Dirichlet maps that in addition are coercive. More specifically, these regularizing operators are either single layer operators with purely imaginary wavenumbers or the principal symbols of such operators in the sense of pseudodifferential operators. We note that a similar approach was advocated in [4, 5, 35], yet the choice of regularizing operators in those references is different from ours: the choice of the regularizing operators advocated in the former reference relies on use of the principal symbol of the Neumann-to-Dirichlet map for circular geometries combined with certain quadratic partitions of unities while the choice presented in the latter reference consists of (local) Padé approximations of complexified versions of the principal symbol of the Neumann-to-Dirichlet map. Although various regularizing operators leading to various types of CFIER have been proposed in the literature, they all rely on the Calderón's identities. Regularizing operators that correspond to single layer potentials with zero wavenumbers were introduced in [3, 40], whereas regularizing operators that are boundary layer potentials corresponding to complex wavenumbers were introduced in [2, 22]. The latter operators lead to CFIER with superior spectral properties that outperform their counterparts that rely on boundary layer potential with zero wavenumbers, especially in the challenging high frequency regime [9]. In this paper we aim to justify the excellent spectral properties of complexified CFIER operators in the high-frequency regime.

After introducing the complexified CFIER operators and having established their invertibility in appropriate functional spaces, we study the wavenumber dependence of the L^2 norms of the CFIER operators in the case of two and three-dimensional smooth geometries. In particular, we establish that these norms can grow at most as $Ck^{1/2}$ in terms of the wavenumber k, where C are constants that depend only on the geometry and the coupling constants. A spectral analysis allows us to obtain sharper versions of these results in the case of circular and spherical geometries. We show in this case that the norms of the CFIER operators are bounded from above by constants C that depend only on the coupling parameters. Furthermore, we rigorously establish that for high enough frequencies and sufficiently large coupling constants—that is for coupling constants larger than $Ck^{1/3}$, the CFIER operators are coercive in $L^2(\Gamma)$ in the case of circular and spherical geometries Γ, and that the coercivity constants do not depend on the wavenumbers k. We recall that if H is a Hilbert space, an operator $\mathcal{A} : H \to H'$ (where H' is the dual of H) is coercive if there exists a constant $\gamma > 0$ such that $\gamma \|u\|_H^2 \leq \Re\langle \mathcal{A}u, u\rangle$ for all $u \in H$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between H and H'. The question of the L^2 coercivity of the CFIER operators for general boundaries remains an open issue. We provide numerical evidence that the coercivity property holds for two-dimensional starlike boundaries and in the case of strictly-convex boundaries the coercivity property is uniform in the wavenumber. We have recently extended the CFIER formulations to the case of non-smooth boundaries [10]; we will study the coercivity of the CFIER operators in non-smooth domains in a future contribution. We note that integral equation formulations whose operators are compact perturbations of coercive operators are available for the solution of acoustic and electromagnetic scattering problems [28, 29], yet those integral equations are not of the second kind.

We present a collocation (Nyström) method for the discretization of our CFIER operators that follows the discretization method introduced in [30]. This collocation algorithm is based on global trigonometric approximations, splitting of the kernels of integral operators into singular and smooth components, and explicit quadratures of products of singular parts (logarithms) and trigonometric polynomials [34, 36]. We establish the convergence of the fully discrete method in appropriate Sobolev spaces which implies pointwise convergence of the discrete solutions. In the case of analytic boundaries, we establish superalgebraic convergence of the collocation method. The main ingredients in the error analysis proof are the mapping properties of the boundary integral operators that enter CFIER and Sobolev spaces bounds of the error in trigonometric interpolation. We note that unlike the classical CFIE operators that map Sobolev spaces H^{p+1} into H^p, the CFIER operators map H^p into H^p, which simplifies the error analysis; however, one has to deal with compositions of discrete operators. Using these high-order discretizations of the CFIER operators, we provide numerical evidence of the L^2 coercivity of the CFIER operators for smooth, starlike boundaries. Specifically, just as in [11], we compute numerical ranges of the discretizations of the CFIER operators corresponding to very fine meshes, and we highlight the fact that the discrete numerical ranges do not contain the origin, which means that the continuous CFIER operators are coercive.

The paper is organized as follows: in Section 2 we introduce our Regularized Combined Field Integral Equations; in Section 3 we establish upper bounds on the norms of the CFIER operators and their coercivity in the case of circular and spherical boundaries; in Sections 4 we describe the Nyström method and we derive error estimates for the discrete solutions; in Section 5 we present numerical results that confirm the high-order nature of our solvers and we provide numerical evidence of the coercivity of the CFIER operators for smooth, starlike boundaries.

2 Regularized Combined Field Integral Equations

Let us consider the time-harmonic acoustic scattering problem for a two-dimensional sound-hard bounded obstacle Ω. In this case, the scattered field u^s satisfies the Helmholtz equation together with Neumann conditions on the boundary Γ of Ω and Sommerfeld radiation conditions at infinity,

$$\Delta u^s + k^2 u^s = 0 \text{ in } \mathbb{R}^2 \setminus \Omega \tag{2.1}$$

$$\frac{\partial u^s}{\partial n} = -\frac{\partial u^{\text{inc}}}{\partial n} \text{ on } \Gamma \tag{2.2}$$

$$\lim_{|r|\to\infty} r^{1/2}(\partial u^s/\partial r - iku^s) = 0, \tag{2.3}$$

where u^{inc} denotes the incident acoustic field. We assume that the boundary Γ is a smooth and closed curve in two-dimensions. The classical Indirect Combined Field Integral Equation formulation (ICFIE), which was originally introduced by Brakhage and Werner [7] and Panich [40], assumes a representation of the scattered field as a combination of single and double layer potentials of the form

$$\begin{aligned} u^s(\mathbf{z}) &= (u_{\text{SL}}\phi)(\mathbf{z}) + i\eta(u_{\text{DL}}\phi)(\mathbf{z}) \\ &= \int_\Gamma G_k(\mathbf{z}-\mathbf{y})\phi(\mathbf{y})ds(\mathbf{y}) + i\eta \int_\Gamma \frac{\partial G_k(\mathbf{z}-\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})}\phi(\mathbf{y})ds(\mathbf{y}) \end{aligned} \tag{2.4}$$

where $\mathbf{z} \in \mathbb{R}^2 \setminus \Omega$, ϕ denotes a (smooth) function defined on Γ which we will refer to as the density, η is a coupling parameter $\eta \neq 0$, and G_k denotes the outgoing free-space Green's function

$$G_k(\mathbf{z}-\mathbf{y}) = \frac{i}{4}H_0^{(1)}(k|\mathbf{z}-\mathbf{y}|). \tag{2.5}$$

Taking the normal derivative of both sides of (2.4), using the trace formulas for the normal derivative of layer potentials for smooth domains together with the sound-hard boundary condition the classical ICFIE is given by

$$-\frac{\phi(\mathbf{x})}{2} + (K_k^{'}\phi)(\mathbf{x}) + i\eta(N_k\phi)(\mathbf{x}) = -\frac{\partial u^{\text{inc}}(\mathbf{x})}{\partial \mathbf{n}(\mathbf{x})},\ \mathbf{x} \text{ on } \Gamma \tag{2.6}$$

where $K_k^{'}$ denotes the semi-sum of the interior and exterior Neumann traces of the single layer potential u_{SL} on Γ. Specifically, the operator $K_k^{'}$ is defined as

$$(K_k^{'}\phi)(\mathbf{x}) = \int_\Gamma \frac{\partial G(\mathbf{x}-\mathbf{y})}{\partial \mathbf{n}(\mathbf{x})}\phi(\mathbf{y})ds(\mathbf{y}),\ \mathbf{x} \text{ on } \Gamma. \tag{2.7}$$

The operator N_k in equations (2.6) denotes the Neumann trace of the double layer potential u_{DL} on Γ given in terms of finite part integrals [20]

$$\begin{aligned} (N_k\phi)(\mathbf{x}) &= \int_\Gamma \frac{\partial^2 G_k(\mathbf{x}-\mathbf{y})}{\partial \mathbf{n}(\mathbf{x})\partial \mathbf{n}(\mathbf{y})}\phi(\mathbf{y})ds(\mathbf{y}) \\ &= k^2 \int_\Gamma G_k(\mathbf{x}-\mathbf{y})(\mathbf{n}(\mathbf{x})\cdot\mathbf{n}(\mathbf{y}))\phi(\mathbf{y})ds(\mathbf{y}) + \int_\Gamma \partial_s G_k(\mathbf{x}-\mathbf{y})\partial_s\phi(\mathbf{y})ds(\mathbf{y}) \end{aligned} \tag{2.8}$$

where ∂_s denotes the tangential derivative on Γ.

In what follows we recall the definition of Sobolev spaces $H^p(\Gamma)$ according to [31, 42], as we make frequent use of these spaces. Our presentation follows very closely that in [31]. We start with the space $L^2[0, 2\pi]$ of 2π periodic functions that are square integrable. A function $\varphi \in L^2[0, 2\pi]$ can be expressed in terms of its Fourier series $\varphi(t) = \sum_{m=-\infty}^{\infty} a_m e^{imt}$ where $a_m = \frac{1}{2\pi}\int_0^{2\pi} \varphi(t)e^{-imt}dt$. Then, for a given $0 \leq p < \infty$, we denote by $H^p[0,2\pi]$ the space of functions $\varphi \in L^2[0,2\pi]$ with the property

$$\sum_{m=-\infty}^{\infty} (1+m^2)^p |a_m|^2 < \infty$$

in terms of the Fourier coefficients a_m of φ. The space $H^p[0,2\pi]$ is called a Sobolev space and it is equipped with the scalar product defined by

$$(\varphi, \psi)_p = \sum_{m=-\infty}^{\infty} (1+m^2)^p a_m \bar{b}_m$$

in terms of the Fourier coefficients a_m and b_m of φ and ψ respectively. For $0 \leq p < \infty$ we denote by $H^{-p}[0,2\pi]$ the dual space of $H^p[0,2\pi]$, that is the space of bounded linear functionals on $H^p[0,2\pi]$. If the curve Γ is represented by a smooth (infinitely differentiable) 2π periodic parametrization $\Gamma = \{x(t) : t \in [0,2\pi)\}$, then for $0 \leq p < \infty$, the space $H^p(\Gamma)$ is defined as the space of functions $\varphi \in L^2(\Gamma)$ such that $\varphi \circ x \in H^p[0,2\pi]$. It was shown in [31] that this definition of $H^p(\Gamma)$ is invariant with respect to the parametrization.

Having reviewed the definition of Sobolev spaces $H^p(\Gamma)$, we return to the issue of the unique solvability of (2.6) in the appropriate Sobolev spaces $H^{1/2}(\Gamma)$. This is settled in a classical manner via Fredholm theory. Indeed, the operator N_k is strongly elliptic as it satisfies a Gårding type inequality in $H^{\frac{1}{2}}(\Gamma)$ — its principal part corresponding to N_0 is coercive in the space of functions in $H^{\frac{1}{2}}(\Gamma)$ with zero mean [23, 39]—and the operator $K_k^{'}$ is compact in the same space for smooth surfaces, and therefore the operators in (2.6) are Fredholm of index 0. The injectivity of these operators is established in connection with the fact that solutions of the Helmholtz equation in Ω with zero impedance boundary conditions on Γ are trivial [20] from which their invertibility follows from the Fredholm theory.

Although the integral equations (2.6) are uniquely solvable for all values of the wavenumber k, they are not integral equations of the second kind. Indeed, the operator N_k is a pseudodifferential operator of order 1 [39] and thus the eigenvalues of the integral operators that enter the formulation (2.6) accumulate at infinity. In order to mitigate the "derivative" effect of N_k we use recently introduced Regularized Combined Field Integral equations [9]. The derivation of these integral equation formulations starts with the Green's representation formula

$$u^s(\mathbf{z}) = \int_\Gamma \left(-G_k(\mathbf{z}-\mathbf{y}) \frac{\partial u^s(\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} + \frac{\partial G_k(\mathbf{z}-\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} u^s(\mathbf{y}) \right) ds(\mathbf{y}), \quad \mathbf{z} \in \mathbb{R}^2 \setminus \Omega \tag{2.9}$$

and it is based on the use of the Neumann-to-Dirichlet operator (NtD) Y_k that maps the Neumann trace of u^s on Γ to the Dirichlet trace of u^s on Γ, so that $Y_k(\partial u^s/\partial \mathbf{n}|_\Gamma) = u^s|_\Gamma$ and $Y_k : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$. With these notations equations (2.9) becomes

$$u^s(\mathbf{z}) = \int_\Gamma \left(-G_k(\mathbf{z}-\mathbf{y})(\partial u^s/\partial \mathbf{n})(\mathbf{y}) + \frac{\partial G_k(\mathbf{z}-\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} Y_k(\partial u^s/\partial \mathbf{n})(\mathbf{y}) \right) ds(\mathbf{y}), \quad \mathbf{z} \in \mathbb{R}^2 \setminus \Omega. \tag{2.10}$$

Applying the exterior Neumann trace to equations (2.10) we obtain

$$\frac{1}{2}\partial u^s/\partial \mathbf{n} = -K_k'(\partial u^s/\partial \mathbf{n}) + (N_k Y_k)(\partial u^s/\partial \mathbf{n}) \tag{2.11}$$

from which it follows that Y_k is a right pseudo-inverse (parametrix) of N_k in the sense that $N_kY_k = I/2+\mathcal{K}$ where $\mathcal{K}$ is a compact operator in $H^{-1/2}(\Gamma)$. For a general domain Ω the NtD operator Y_k is not known, yet operators $\mathcal{R}$ that approximate the NtD operator in the sense that $\mathcal{R}$ differ from Y_k by smoother operators can be obtained with relative ease. For example, the single layer operator S_k is such an operator in the light of Calderón's identities [15]. For a given density $\psi \in H^{-1/2}(\Gamma)$ the operator $S_k : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is defined as the Dirichlet trace of the single layer potential $u_{\rm SL}\psi$

$$(S_k\psi)(\mathbf{x}) = \int_\Gamma G_k(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})ds(\mathbf{y}),\ \mathbf{x} \text{ on } \Gamma. \tag{2.12}$$

Using such operators $\mathcal{R}$ that approximate the NtD operator Y_k in the sense defined above, and taking a cue from the form of equations (2.10), we look for scattered fields u^s in the form

$$u^s(\mathbf{z}) = -\int_\Gamma \left(G_k(\mathbf{z}-\mathbf{y})\mu(\mathbf{y}) + i\eta\frac{\partial G_k(\mathbf{z}-\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})}(\mathcal{R}\mu)(\mathbf{y})\right)ds(\mathbf{y}) \tag{2.13}$$

for a density function μ defined on Γ. Representations (2.13) lead to the (Indirect) Regularized Combined Field Integral Equation formulations:

$$(\mathcal{A}_{k,\mathcal{R}}\mu)(\mathbf{x}) = \frac{\mu(\mathbf{x})}{2} - (K_k'\mu)(\mathbf{x}) - i\eta(N_k\circ\mathcal{R})\mu(\mathbf{x}) = \frac{\partial u^{\rm inc}(\mathbf{x})}{\partial\mathbf{n}(\mathbf{x})},\ \mathbf{x} \text{ on } \Gamma. \tag{2.14}$$

Since the operators $\mathcal{R}$ are chosen to be pseudo-inverses of the operator N_k, i.e. $N_k\mathcal{R} = \alpha\ I +$ Compact in the natural trace space $H^{m-\frac{1}{2}}(\Gamma),\ m \geq 0$ of boundary valued problems for the Helmholtz equation, then (a) the operators $\mathcal{R}$ have the following mapping property $\mathcal{R} : H^{m-\frac{1}{2}}(\Gamma) \to H^{m+\frac{1}{2}}(\Gamma)$ and (b) the operator in the left hand side of equations (2.14) is itself a compact perturbation of a multiple of identity, and thus the integral equations (2.14) are of the second kind. On account of the Fredholm theory, the operators $\mathcal{A}_{k,\mathcal{R}}$ are invertible if and only if they are injective. If in addition to being pseudo-inverses of N_k, the operators $\mathcal{R}$ are coercive with respect to the duality pairing $\langle H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)\rangle$, then the unique solvability property of the operators $\mathcal{A}_{k,\mathcal{R}}$ in equations (2.14) is ensured:

Theorem 2.1 *Assume that the operators $\mathcal{R}$ the left-hand side of (2.14) (1) are right pseudo-inverses of the operators N_k in $H^{m-\frac{1}{2}}(\Gamma),\ m \geq 0$ and (2) satisfy the following (coercivity) property*

$$\langle\mathcal{R}\psi,\psi\rangle \geq \alpha\|\psi\|^2_{H^{-1/2}(\Gamma)},\ \alpha > 0,\ \forall\psi \in H^{-1/2}(\Gamma).$$

Then, if $\eta \neq 0$, the operators $\mathcal{A}_{k,\mathcal{R}}$ are invertible in $H^{m-\frac{1}{2}}(\Gamma),\ m \geq 0$.

Proof. Give the assumption (1) and the fact that the operators K_k' are compact, it follows that the operators $\mathcal{A}_{k,\mathcal{R}}$ are multiples of identity plus compact operators. Using the Fredholm property of the operators $\mathcal{A}_{k,\mathcal{R}}$, the invertibility of $\mathcal{A}_{k,\mathcal{R}}$ is equivalent to their injectivity. We prove next that the coercivity condition (2) in the Theorem implies the injectivity of the operators $\mathcal{A}_{k,\mathcal{R}}$. To this end, let μ be a solution of the homogeneous equation (2.14), then use the representation (2.13) to define a field u everywhere in $\mathbb{R}^2\setminus\Gamma$; we denote by u^+ the values of the field u in the exterior of Ω and by u^- the values of the field u in the interior of Ω. It follows then that u^+ is a radiative solution to the Helmholtz problem in the exterior of Ω with $\partial u^+/\partial\mathbf{n} = 0$ on Γ whence $u^+ = 0$ in the exterior of Ω. The jump relations for the traces of layer potentials classically give

$$-u_- = i\eta\ \mathcal{R}\mu,\quad \frac{\partial u_-}{\partial\mathbf{n}} = \mu \text{ on } \Gamma$$

from which we get using Green's identities

$$i\eta\int_\Gamma (\mathcal{R}\mu)\bar{\mu}ds = -\int_\Gamma u_-\frac{\partial \bar{u}_-}{\partial \mathbf{n}}ds = \int_D (|\nabla u_-|^2 - k^2|u_-|^2)dx. \tag{2.15}$$

Using the assumption (2) and the fact that the quantity in the right-hand side of equations (2.15) is real, we obtain that $\mu = 0$, and thus the proof is complete. ■

One class of operators $\mathcal{R}$ that satisfy conditions (1) and (2) in Theorem 2.1 consists of operators of the form $\mathcal{R} = S_{ik_1},\ k_1 > 0$, where S_{ik_1} is the single layer potential defined in equations (2.12) corresponding to the purely imaginary wavenumber ik_1. This choice was introduced in our previous effort [9], where we proved that $\mathcal{R} = S_{ik_1},\ k_1 > 0$ satisfies conditions (1) and (2) in Theorem 2.1. Given this choice of regularizing operators $\mathcal{R}$, we denote the corresponding regularized integral operators by $\mathcal{A}_{k,S_{ik_1}}$ whose definition is given by:

$$(\mathcal{A}_{k,S_{ik_1}}\mu)(\mathbf{x}) = \frac{\mu(\mathbf{x})}{2} - (K_k'\mu)(\mathbf{x}) - i\eta(N_k \circ S_{ik_1})\mu(\mathbf{x}). \tag{2.16}$$

Besides the operators $\mathcal{R} = S_{ik_1}$, another natural choice of operators $\mathcal{R}$ that satisfy the properties (1) and (2) in Theorem 2.1 is given by $\mathcal{R} = PS_{ik_1}$ where PS_{ik_1} represents the principal symbol of the operator S_{ik_1} in the sense of pseudodifferential operators [47]. Before giving the explicit definition of these operators, we briefly review the concept of principal symbols of pseudodifferential operators on a smooth and periodic curve Γ. Given that we can assume that the smooth and closed curve Γ can be represented via a smooth and 2π periodic parametrization, we restrict our exposition to periodic pseudodifferential operators, following the presentation in [42]. In this context, linear and bounded operators A between Sobolev spaces of 2π periodic functions, that is $A : H^s[0,2\pi] \to H^{s'}[0,2\pi]$, have the following spectral representation [42]

$$(Au)(t) = \sum_{n\in\mathbb{Z}} \sigma(t,n)\hat{u}(n)e^{int},\ \ \sigma(t,n) = e_{-n}(t)(Ae_n)(t),$$

where $e_n(t) = e^{int}$ and $\hat{u}(n)$ are the Fourier coefficients of the function u, that is

$$\hat{u}(n) = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-inx}dx.$$

A linear operator A of the kind described above is a periodic pseudodifferential operator of order α, denoted by $A \in OPS(\alpha)$, if its symbol $\sigma(t,n)$ is the restriction to $\mathbb{R}\times\mathbb{Z}$ of a function $\sigma(t,\xi)$ which is smooth in $\mathbb{R}\times\mathbb{R}$, 2π periodic in t and satisfies the inequalities

$$|\left(\frac{\partial}{\partial t}\right)^k\left(\frac{\partial}{\partial \xi}\right)^l\sigma(t,\xi)| \le c_{k,l}(1+|\xi|)^{\alpha-l},\ \ k,l\in\mathbb{N},$$

in which case we say that $\sigma(t,n) \in \Sigma^\alpha$. It follows easily form their definition that operators $A \in OPS(\alpha)$ are bounded from $H^s([0,2\pi])$ to $H^{s-\alpha}([0,2\pi])$ [42, 47]. If the symbol σ of an operator $A \in OPS(\alpha)$ is such that there exist $\alpha = \alpha_0 > \alpha_1 > \ldots > \alpha_j \to -\infty$ with the property that $\sigma - \sum_{j=0}^{N-1}\sigma_j \in \Sigma^{\alpha_N}$ for all $N \in \mathbb{N}$, then σ_0 is called the principal symbol of σ. Using a global smooth and periodic parametrization of the closed curve Γ in terms of the arclength, i.e. a diffeomorphism $x : [0,2\pi] \to \Gamma$, the operator A can be lifted to an operator acting on periodic functions $\tilde{u}$ defined on Γ by $A^x\tilde{u} = x^{-1}\circ A(u\circ x)$. Using the change of variables formula for the principal symbol of A [47], the principal symbol of the pseudodifferential operator A^x can be also defined.

The connection between boundary integral operators of potential theory and pseudodifferential operators is well known [14, 46, 47]. For instance, it can be shown that the single layer operator S_{ik_1} is a periodic pseudodifferential operator of order -1 on Γ whose principal symbol equals $\sigma^0_{S_{ik_1}} = \frac{1}{2}(k_1^2 + |\xi|^2)^{-\frac{1}{2}}$ (see Appendix for a proof of this fact), that is:

$$(\mathcal{R}\phi)(x(t)) = (PS_{ik_1}\phi)(x(t)) = \sum_{n\in\mathbb{Z}} \sigma^0_{k_1}(n)\hat{\psi}(n)e^{int},\ \sigma^0_{k_1}(n) = \frac{1}{2}(k_1^2 + n^2)^{-\frac{1}{2}}, \tag{2.17}$$

where $\hat{\psi}(n)$ are the Fourier coefficients of the function $\phi(x(t))|x'(t)|$. The difference between S_{ik_1} and PS_{ik_1} is a pseudodifferential operator of order -3 (since the next term in the asymptotic expansion of the symbol $\sigma_{S_{ik_1}}$ is in Σ^{-3} [35]) and hence $S_{ik_1} - PS_{ik_1} : H^{m-\frac{1}{2}}(\Gamma) \to H^{m+\frac{5}{2}}(\Gamma)$ (we prove this fact in an Appendix). Thus, we have $N_k \circ PS_{ik_1} = N_k \circ S_{ik_1} - N_k \circ (S_{ik_1} - PS_{ik_1})$. The last operator in this identity maps $H^{m-\frac{1}{2}}(\Gamma)$ to $H^{m+\frac{3}{2}}(\Gamma)$ and thus is compact as an operator from $H^{m-\frac{1}{2}}(\Gamma)$ to itself. Consequently, since condition (1) in Theorem 2.1 is satisfied by $\mathcal{R} = S_{ik_1}$, it is also satisfied given the choice $\mathcal{R} = PS_{ik_1}$. Furthermore, the operator PS_{ik_1} is strongly elliptic since its symbol satisfies $\frac{1}{2}(k_1^2+|\xi|^2)^{-\frac{1}{2}} \geq \frac{1}{4|\xi|}$ for large enough $|\xi|$, and thus it is coercive on $H^{-\frac{1}{2}}(\Gamma)$ based on Gårding inequality [42, 47] (another proof of properties (1) and (2) can be obtained based on representations of the operators in parametric form that we present in Section 4). Thus, this second choice of regularizing operator $\mathcal{R} = PS_{ik_1}$ also satisfies conditions (1) and (2) in Theorem 2.1. Given the choice $\mathcal{R} = PS_{ik_1}$, we denote the corresponding regularized integral operators by $\mathcal{A}_{k,PS_{ik_1}}$; their definition is given by:

$$(\mathcal{A}_{k,PS_{ik_1}}\mu)(\mathbf{x}) = \frac{\mu(\mathbf{x})}{2} - (K'_k\mu)(\mathbf{x}) - i\eta(N_k \circ PS_{ik_1})\mu(\mathbf{x}). \tag{2.18}$$

We denote the equations (2.14) when the choice of the regularizing operators is $\mathcal{R} = S_{ik_1}$ by the label ICFIE-R —which corresponds to the operators $\mathcal{A}_{k,S_{ik_1}}$ defined in equation (2.16), and by the label ICFIE-RPS when the choice of regularizing operators is $\mathcal{R} = PS_{ik_1}$ —which corresponds to the operators $\mathcal{A}_{k,PS_{ik_1}}$ defined in equation (2.18). We mention that the idea of using regularizing operators based on principal symbols of pseudodifferential operators was first introduced in [35]. The regularizing operators in [35] consist of suitable modifications via quadratic partitions of unity on Γ of the principal symbol of the Neumann-to-Dirichlet operator for circular geometries, which is different from our choice. In the next Section we investigate certain norm properties of the ICFIE-R and ICFIE-RPS operators and their inverses in connection with the wavenumbers k and k_1 and the coupling parameter η.

3 Wavenumber dependence of the norms of the regularized combined field integral operators

The selection of the coupling parameter η and the wavenumber k_1 in the definition of the regularizing operators $\mathcal{R} = S_{ik_1}$ and $\mathcal{R} = PS_{ik_1}$ in the ICFIE-R and ICFIE-RPS operators (2.16) and (2.18) is guided by considerations the spectral properties of these operators. In our previous effort [9], we presented numerical evidence that the choice $\eta = 2$ and $k_1 = k$ leads to small numbers of

Krylov subspace iterations. In general, the choice of these parameters is driven by the spectral properties of the ensuing combined field integral operators in the case of circular (or spherical) geometries [3, 32]. In this case, all of the acoustic boundary integral operators that enter the formulations (2.16) and (2.18) are diagonalizable in the (complete) basis given by the Fourier harmonics $\{e^{im\theta},\ m \in \mathbb{Z}\}$ and the eigenvalues of these operators can be computed explicitly. In this section we investigate the spectral properties of the ICFIE-R and ICFIE-RPS operators, with an emphasis on explicit wavenumber estimates on the condition numbers of these operators in the case of circular and spherical geometries. In this case, we show that in the high-frequency regime, and for certain choices of the coupling parameter η and the wavenumber k_1 in the definition of the regularizing operators, the ICFIE-R and ICFIE-RPS operators defined in equations (2.16) and (2.18) enjoy similar condition number properties to their CFIE sound-soft counterparts. The latter properties are as follows: the norms of the sound-soft CFIE operators can be uniformly bounded in k with constants that depend on the coupling constant [6], and for certain choices of the coupling constants, the sound-soft CFIE operators are L^2 coercive with coercivity constants that do not depend on k [27].

First, our numerical evidence suggests that the choice $k_1 = k$ in the definition of the regularizing operators $\mathcal{R} = S_{ik_1}$ and $\mathcal{R} = PS_{ik_1}$ leads to operators ICFIE-R and ICFIE-RPS with superior spectral properties, and this is the choice that we use in the analysis that follows. We note that this choice is not essential, as the results that we will establish hold for the more general case when $k_1 = ck$, where $c > 0$ is a constant that does not depend on k. Our analysis is based on the spectral properties of the boundary integral operators $S_k,\ K_k',\ N_k$ [32], whose eigenvalues/eigenvectors can be computed explicitly in the case when $\Gamma = \mathbb{S}^1$, where $\mathbb{S}^1$ denotes the unit circle in two-dimensions:

$$\begin{aligned} S_k\, e^{im\theta} &= \frac{i\pi}{2} J_{|m|}(k) H^{(1)}_{|m|}(k) e^{im\theta}, \\ K_k'\, e^{im\theta} &= \left(-\frac{1}{2} + \frac{i\pi}{2} k J'_{|m|}(k) H^{(1)}_{|m|}(k)\right) e^{im\theta} \\ N_k\, e^{im\theta} &= \frac{i\pi}{2} k^2 J'_{|m|}(k) (H^{(1)}_{|m|}(k))' e^{im\theta} \end{aligned} \tag{3.1}$$

in terms of the Bessel and Hankel functions of order $|m|$ and argument k and their derivatives with respect to k. The integral operators ICFIE-R and ICFIE-RPS defined in equations(2.16) and (2.18) are diagonalizable in the complete basis of Fourier harmonics and we get that the eigenvalues of the ICFIE-R operators are given by

$$\mathcal{A}_{k,S_{ik}}\, e^{im\theta} = a_m(k)\, e^{im\theta},\ a_m(k) = 1 - \frac{i\pi k}{2} J'_{|m|}(k) H^{(1)}_{|m|}(k) + \frac{i\eta\pi^2 k^2}{4} J'_{|m|}(k) (H^{(1)}_{|m|}(k))' J_{|m|}(ik) H^{(1)}_{|m|}(ik) \tag{3.2}$$

while the eigenvalues of the ICFIE-RPS operators are given by

$$\mathcal{A}_{k,PS_{ik}}\, e^{im\theta} = p_m(k)\, e^{im\theta},\ p_m(k) = 1 - \frac{i\pi k}{2} J'_{|m|}(k) H^{(1)}_{|m|}(k) + \frac{\eta\pi k^2}{4} J'_{|m|}(k) (H^{(1)}_{|m|}(k))' (m^2 + k^2)^{-\frac{1}{2}}. \tag{3.3}$$

3.1 Uniform bounds on the norms of the operators ICFIE-R and ICFIE-RPS in the case of circular and spherical geometries

In this subsection we investigate the uniform boundedness in wavenumber k of the operators ICFIE-R and ICFIE-RPS in the cased when $\Gamma = \mathbb{S}^1$. We start with a useful result

Lemma 3.1 *There exist two constants $C_1 > 0$ and $C_2 > 0$ and a number $k_0 > 0$ such that*

$$C_1(m^2 + k^2)^{-1/2} \le iJ_m(ik)H_m^{(1)}(ik) \le C_2(m^2 + k^2)^{-1/2}$$

for all $k > k_0$ and all $m \ge 0$.

Proof. We begin by using the representation of the functions $J_m(ik)$ and $H_m^{(1)}(ik)$ in terms of the Bessel functions of the third kind $J_m(ik)H_m^{(1)}(ik) = \frac{2}{\pi i} I_m(k)K_m(k)$ where both $I_m(k) > 0$ and $K_m(k) > 0$. Using the uniform asymptotic expansions as $\nu \to \infty$ (Formulas 9.7.7 and 9.7.8 in [1])

$$\begin{aligned} I_\nu(\nu z) &\sim \frac{1}{\sqrt{2\pi\nu}} \frac{e^{\nu\mu}}{(1+z^2)^{\frac{1}{4}}}(1 + \mathcal{O}(\nu^{-1})) \\ K_\nu(\nu z) &\sim \frac{\sqrt{\pi}}{\sqrt{2\nu}} \frac{e^{-\nu\mu}}{(1+z^2)^{\frac{1}{4}}}(1 + \mathcal{O}(\nu^{-1})) \end{aligned} \tag{3.4}$$

where $\mu = \sqrt{1+z^2} + \ln \frac{z}{1+\sqrt{1+z^2}}$, we get that there exists a constant $M_0 > 0$ such that

$$\frac{1}{4}(m^2 + k^2)^{-1/2} \le I_m(k)K_m(k) \le (m^2 + k^2)^{-1/2}, \text{ for all } m > M_0,\ k > 0. \tag{3.5}$$

Using the asymptotic expansion (Formula 9.7.5 in [1]) which is valid for ν fixed and $z \to \infty$

$$I_\nu(z)K_\nu(z) \sim \frac{1}{2z}\left(1 - \frac{1}{2}\frac{\mu - 1}{(2z)^2} + \mathcal{O}\left(\frac{\mu^2}{z^4}\right)\right),\ \mu = 4\nu^2 \tag{3.6}$$

we get that for a fixed m there exists k_m such that

$$\frac{1}{4k} \le I_m(k)K_m(k) \le \frac{1}{k}, \text{ for all } k \ge k_m. \tag{3.7}$$

It follows from the estimate (3.7) that there exist $k_0 = \max_{m \le M_0} k_m$ and $c = (1 + M_0^2/k_0^2)^{\frac{1}{2}}$ such that

$$\frac{1}{4}(m^2 + k^2)^{-1/2} \le \frac{1}{4k} \le I_m(k)K_m(k) \le \frac{1}{k} \le c(m^2 + k^2)^{-1/2}, \text{ for all } 0 \le m \le M_0,\ k \ge k_0. \tag{3.8}$$

We obtain by combining estimates (3.5) and (3.8) that there exist constants $C_1 = \frac{1}{2\pi}$, $C_2 = \frac{2}{\pi}\max(1, (1 + M_0^2/k_0^2)^{\frac{1}{2}})$, and $k_0 > 0$ such that

$$C_1(m^2 + k^2)^{-1/2} \le iJ_m(ik)H_m^{(1)}(ik) = \frac{2}{\pi} I_m(k)K_m(k) \le C_2(m^2 + k^2)^{-1/2} \tag{3.9}$$

for all $m \ge 0$ and all $k \ge k_0$. ∎

We prove the following result which can be viewed as the sound-hard counterpart of a result established in [6] in the sound-soft case (all the norms in the subsequent results are the 2 norm unless specified otherwise):

Theorem 3.2 *When Γ is the unit circle in $\mathbb{R}^2$, then there exist $k_1 > 0$ and $C > 0$ independent of k such that*

$$\|\mathcal{A}_{k,S_{ik}}\|_{L^2(\mathbb{S}^1)\to L^2(\mathbb{S}^1)} \le C(1+|\eta|),\ \|\mathcal{A}_{k,PS_{ik}}\|_{L^2(\mathbb{S}^1)\to L^2(\mathbb{S}^1)} \le C(1+|\eta|) \text{ for all } k_1 \le k.$$

Proof. Given that in the case $\Gamma = \mathbb{S}^1$ the set $\{e^{im\theta} : m \in \mathbb{Z}\}$ constitutes an orthonormal basis of $L^2(\Gamma)$ and that both operators $\mathcal{A}_{k,S_{ik}}$ and $\mathcal{A}_{k,PS_{ik}}$ are diagonalizable in this basis, the result of the theorem follows immediately once we established that there exists $k_1 > 0$ such that

$$|a_m(k)| \le C(1+|\eta|), \qquad |p_m(k)| \le C(1+|\eta|), \text{ for all } m \in \mathbb{Z},\ k \ge k_1. \tag{3.10}$$

We note that it is enough to prove the estimates (3.10) in the case $m \ge 0$. We establish first the bound on $|p_m(k)|$. In order to show that there exists $k_1 > 0$ such that $|p_m(k)|$ is uniformly bounded in k for all $k > k_1$, we use several estimates established in Lemma 3.9, Proposition 3.10 in [6], Lemma 4.9, and Lemma 4.10 in [27]. First, we use the result in Proposition 3.10 in [6] which established that there exists a constant C independent of k such that for all $m \ge 0$ and for any $k \ge 1$

$$|kJ'_m(k)H^{(1)}_m(k)| \le C. \tag{3.11}$$

Using asymptotic expansions for fixed m as $k \to \infty$ of Bessel and Hankel functions [1] (Formulas 9.2.1, 9.2.11, 9.2.12)

$$\begin{aligned}
J_m(k) &= \sqrt{\frac{2}{\pi k}}\cos(k - \frac{m\pi}{2} - \frac{\pi}{4}) + \mathcal{O}(k^{-1})\\
J'_m(k) &= -\sqrt{\frac{2}{\pi k}}\sin(k - \frac{m\pi}{2} - \frac{\pi}{4}) + \mathcal{O}(k^{-1})\\
Y'_m(k) &= \sqrt{\frac{2}{\pi k}}\cos(k - \frac{m\pi}{2} - \frac{\pi}{4}) + \mathcal{O}(k^{-1})
\end{aligned} \tag{3.12}$$

we obtain that for a fixed m, the quantities $k^2 J'_m(k)(H^{(1)}_m(k))'(m^2+k^2)^{-\frac{1}{2}}$ are bounded for all k sufficiently large. In what follows, we will denote by ν the orders of the Bessel functions. Since for all orders $\nu \ge 0$ and all $k > 0$, the functions $J'_\nu(k)(H^{(1)}_\nu(k))'$ are continuous in k and ν, it follows that it suffices to investigate the uniform boundedness of the products $k^2 J'_\nu(k)(H^{(1)}_\nu(k))'(\nu^2+k^2)^{-\frac{1}{2}}$ for large enough orders ν. We proceed, just as in the proof of Proposition 3.10. in [6], by distinguishing two cases.

Case 1: $\nu > k \ge 1/2$. We use the result from Lemma 3.9. in [6] which established that $kJ'_\nu(k)$ is a positive increasing function of k for $0 < k < \nu$. It follows that

$$\frac{k^2(J'_\nu(k))^2}{(\nu^2+k^2)^{\frac{1}{2}}} \le \nu^{-1}k^2(J'_\nu(k))^2 \le \nu(J'_\nu(\nu))^2.$$

We use the following asymptotic expansions [1] (Formulas 9.3.33) valid for all $\nu > 1/2$:

$$J'_\nu(\nu) = b\nu^{-2/3} - c\nu^{-4/3} + \mathcal{O}(\nu^{-8/3}) \tag{3.13}$$

where b and c are positive constants. Using this result in the estimate above we obtain that for large enough ν

$$\frac{k^2(J'_\nu(k))^2}{(\nu^2+k^2)^{\frac{1}{2}}} \le \tilde{C}\nu^{-1/3} \le C. \tag{3.14}$$

Using the identity [1] (Formulas 9.1.27)

$$J'_\nu(k) = \frac{\nu}{k} J_\nu(k) - J_{\nu+1}(k), \tag{3.15}$$

we have

$$\frac{k^2}{(\nu^2+k^2)^{\frac{1}{2}}} |J'_\nu(k) Y'_\nu(k)| \le \frac{\nu k}{(\nu^2+k^2)^{\frac{1}{2}}} |J_\nu(k) Y'_\nu(k)| + \frac{k^2}{(\nu^2+k^2)^{\frac{1}{2}}} |J_{\nu+1}(k) Y'_\nu(k)|. \tag{3.16}$$

It was established in Lemma 4.10 in [27] that

$$0 \le J_{\nu+1}(k) \le J_\nu(k), \; k \in [0, \nu] \tag{3.17}$$

and therefore it follows that

$$\frac{k^2}{(\nu^2+k^2)^{\frac{1}{2}}} |J'_\nu(k) Y'_\nu(k)| \le \frac{\nu+k}{(\nu^2+k^2)^{\frac{1}{2}}} |k J_\nu(k) Y'_\nu(k)|. \tag{3.18}$$

Using the identity for the Wronskian [1] (Formula 9.1.16)

$$J_\nu(k) Y'_\nu(k) - J'_\nu(k) Y_\nu(k) = \frac{2}{\pi k} \tag{3.19}$$

together with the estimate (3.11) in equation (3.16) we get the following estimate

$$\frac{k^2}{(\nu^2+k^2)^{\frac{1}{2}}} |J'_\nu(k) Y'_\nu(k)| \le C \frac{\nu+k}{(\nu^2+k^2)^{\frac{1}{2}}} \le \sqrt{2} C. \tag{3.20}$$

Combining estimates (3.14) and (3.20) we obtain that there exists a constant C' independent of k such that for large enough ν we have the following estimate

$$\frac{k^2}{(\nu^2+k^2)^{\frac{1}{2}}} |J'_\nu(k) (H^{(1)}_\nu(k))'| \le C' \text{ for } k < \nu. \tag{3.21}$$

Case 2: $\nu \le k$. We define

$$N_\nu(k) := |(H^{(1)}_\nu)'(k)| = \sqrt{(J'_\nu(k))^2 + (Y'_\nu(k))^2} \tag{3.22}$$

and use the result established in Proposition 4.7 in [27], that is there exists k_1 such as for all ν and k with $k_1 \le \nu \le k$ we have

$$k N^2_\nu(k) \le \frac{4}{\pi}. \tag{3.23}$$

It follows then that

$$\frac{k^2}{(\nu^2+k^2)^{\frac{1}{2}}} (J'_\nu(k))^2 \le \frac{k}{(\nu^2+k^2)^{\frac{1}{2}}} (k N^2_\nu(k)) \le \frac{4}{\pi}. \tag{3.24}$$

We also immediately obtain

$$\frac{k^2}{(\nu^2+k^2)^{\frac{1}{2}}} |J'_\nu(k) Y'_\nu(k)| \le \frac{k}{2(\nu^2+k^2)^{\frac{1}{2}}} (k N^2_\nu(k)) \le \frac{2}{\pi}. \tag{3.25}$$

Combining the estimates (3.24) and (3.25) we get that there exists $k_1 > 0$ and a constant C'' independent of k such that

$$\frac{k^2}{(\nu^2+k^2)^{\frac{1}{2}}}|J'_\nu(k)(H^{(1)}_\nu(k))'| \leq C'' \text{ for all } k \geq \nu \geq k_1. \tag{3.26}$$

Combining estimates (3.11), (3.21), and (3.26), the result of the Theorem concerning $|p_m(k)|$ follows. Furthermore, using the result in Lemma 3.1 the similar bound on $|a_m(k)|$ also holds. ■

Remark 3.3 *It follows immediately from estimate* (3.10) *that for any* $0 < p$, *there exist* $k_1 > 0$ *and* $C_p > 0$ *independent of* k *such that*

$$\|\mathcal{A}_{k,S_{ik}}\|_{H^p(\mathbb{S}^1)\to H^p(\mathbb{S}^1)} \leq C_p(1+|\eta|),\ \|\mathcal{A}_{k,PS_{ik}}\|_{H^p(\mathbb{S}^1)\to H^p(\mathbb{S}^1)} \leq C_p(1+|\eta|). \tag{3.27}$$

The bounds derived in the case of circular geometries can be extended easily to the three-dimensional case. The operators $\mathcal{A}_{k,S_{ik}}$ defined in equation (2.16) can be extended to the three dimensional setting by replacing the two-dimensional Green's function with the suitable three-dimensional Green's function $G_k(z) = \frac{e^{ik|z|}}{4\pi|z|}$, and by replacing the closed curve Γ by a smooth and closed surface S in three dimensions; we denote by $\mathcal{B}_{k,S_{ik}}$ the ensuing three dimensional ICFIE-R operators. These operators have been introduced in [9] where a variety of numerical results demonstrated the benefits of using integral equations based on the use of the operators $\mathcal{B}_{k,S_{ik}}$ operators in the high frequency regime in three dimensions. In the case when S is the unit sphere in three dimensions, $S = \mathbb{S}^2$, the operators $\mathcal{B}_{k,S_{ik}}$ are diagonalizable in the orthonormal basis of $L^2(\mathbb{S}^2)$ given by the spherical harmonics $\{Y^\ell_m(\theta,\phi) :\ 0 \leq m,\ -m \leq \ell \leq m\}$ where θ and ϕ represent the spherical coordinates on $\mathbb{S}^2$. Indeed, the eigenvalues of the boundary integral operators that make up the operators $\mathcal{B}_{k,S_{ik}}$ can be computed explicitly in terms of the spherical Bessel and Hankel functions [39]

$$\begin{aligned} S_k\ Y^\ell_m &= ikj_m(k)h^{(1)}_m(k)Y^\ell_m \\ K'_k\ Y^\ell_m &= \left(-\frac{1}{2} + ik^2 j'_m(k)h^{(1)}_m(k)\right)Y^\ell_m \\ N_k\ Y^\ell_m &= ik^3 j'_m(k)(h^{(1)}_m)'(k)Y^\ell_m. \end{aligned} \tag{3.28}$$

It follows that the eigenvalues of the operators $\mathcal{B}_{k,S_{ik}}$ are given by

$$\mathcal{B}_{k,S_{ik}}Y^\ell_m = b_m(k)Y^\ell_m, \quad b_m(k) = 1 - ik^2 j'_{|m|}(k)h^{(1)}_{|m|}(k) - \eta k^4 j'_{|m|}(k)(h^{(1)}_{|m|}(k))' j_{|m|}(ik)h^{(1)}_{|m|}(ik). \tag{3.29}$$

Given that $j_m(k) = \sqrt{\frac{\pi}{2k}}J_{m+\frac{1}{2}}(k)$ and $h^{(1)}_m(k) = \sqrt{\frac{\pi}{2k}}H^{(1)}_{m+\frac{1}{2}}(k)$ [1] we obtain

$$\begin{aligned} b_m(k) &= a_{m+\frac{1}{2}}(k) + \frac{i\pi}{4}J_{m+\frac{1}{2}}(k)H^{(1)}_{m+\frac{1}{2}}(k) - \frac{ik\eta\pi^2}{8}J_{m+\frac{1}{2}}(ik)H^{(1)}_{m+\frac{1}{2}}(ik) \times \\ &\times\ [J'_{m+\frac{1}{2}}(k)H^{(1)}_{m+\frac{1}{2}}(k) + J_{m+\frac{1}{2}}(k)(H^{(1)}_{m+\frac{1}{2}}(k))' - \frac{1}{2k}J_{m+\frac{1}{2}}(k)H^{(1)}_{m+\frac{1}{2}}(k)]. \end{aligned} \tag{3.30}$$

We use the following two estimates which were established in [6] (Proposition 3.10): there exists a constant $C > 0$ independent of k such that $|J_\nu(k)H^{(1)}_\nu(k)| \leq Ck^{-\frac{2}{3}}$ and $|kJ'_\nu(k)H^{(1)}_\nu(k)| \leq C$ for

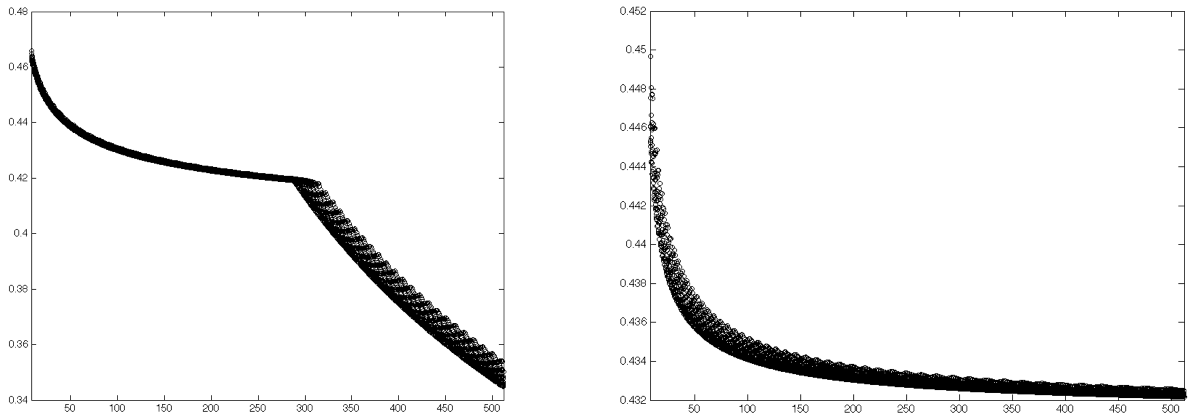

Figure 1: Plots of $\min_m(\Re(a_m(k)))$ for 5041 values of k, namely $k = 8, 8.1, 8.2, \ldots, 512$ with $\eta = 4$ (left) and $\eta = k^{1/3}$ (right). The plots of $\min_m(\Re(p_m(k)))$ are extremely similar.

sufficiently large k. From the second estimate above and the Wronskian identity (3.19) we get that $|kJ_\nu(k)(H_\nu^{(1)})'(k)| \leq C$ as well. Using these estimates together with those established in Lemma 3.1 and Theorem 3.4, we obtain that there exist constants $C > 0$, $C_1 > 0$ and a constant $k_1' > 0$ such that

$$|b_m(k)| \leq C(1 + |\eta|) + C|\eta|k^{-1} + C|\eta|k^{-\frac{5}{3}} \leq C_1(1 + |\eta|) \text{ for all } k_1' \leq k.$$

It follows from the estimate above that

Theorem 3.4 *If S is the unit sphere in $\mathbb{R}^3$, $S = \mathbb{S}^2$, then for all $0 \leq p$, there exist $k_1' > 0$ and $C_p' > 0$ independent of k such that*

$$\|\mathcal{B}_{k,S_{ik}}\|_{H^p(\mathbb{S}^2) \to H^p(\mathbb{S}^2)} \leq C_p'(1 + |\eta|) \text{ for all } k_1' \leq k.$$

3.2 Coercivity of the operators ICFIE-R and ICFIE-RPS in the case of circular and spherical geometries

In this section we study the L^2 coercivity of the operators $\mathcal{A}_{k,S_{ik}}$ and $\mathcal{A}_{k,PS_{ik}}$ in the case when $\Gamma = \mathbb{S}^1$. We will show that, and just as in the Dirichlet case [27], for certain choices of the coupling parameter η, the regularized combined field integral operators $\mathcal{A}_{k,S_{ik}}$ and $\mathcal{A}_{k,PS_{ik}}$ are coercive in $L^2(\Gamma)$ in the case when $\Gamma = \mathbb{S}^1$. We illustrate this fact in Figure 1 where we plot $\min_m(\Re(a_m(k)))$ for values $k = 8, 8.1, 8.2, \ldots, 512$ for two values of the coupling parameter $\eta = 4$ and $\eta = k^{\frac{1}{3}}$ respectively.

In order to probe the coercivity of the operators $\mathcal{A}_{k,S_{ik}}$ and $\mathcal{A}_{k,PS_{ik}}$ in the case when Γ is the unit circle in two dimensions, we start with the following representation which follows immediately from the definition of $p_m(k)$ given in equation (3.3)

$$\Re(p_m(k)) = \frac{\pi k}{2}\left(\frac{2}{\pi k} + J_m'(k)Y_m(k) + \frac{k\eta}{2(m^2 + k^2)^{\frac{1}{2}}}(J_m'(k))^2\right). \tag{3.31}$$

In addition to the equation (3.31) we use an alternative representation of $\Re(p_m(k))$ which follows immediately from the formula for the Wronskian (3.19)

$$\Re(p_m(k)) = \frac{\pi k}{2}\left(J_m(k)Y_m'(k) + \frac{k\eta}{2(m^2+k^2)^{\frac{1}{2}}}(J_m'(k))^2\right). \tag{3.32}$$

Using the asymptotic expansions (3.12), we obtain for a fixed m there exists k_m large enough such that for all $k > k_m$ the following estimate holds true

$$\begin{aligned}\Re(p_m(k)) &= \cos^2(k - \frac{m\pi}{2} - \frac{\pi}{4}) + \frac{k\eta}{2(m^2+k^2)^{\frac{1}{2}}}\sin^2(k - \frac{m\pi}{2} - \frac{\pi}{4}) + \mathcal{O}(k^{-1/2})\\ &\geq \min(1, \frac{\eta}{2\sqrt{2}}) + \mathcal{O}(k^{-1/2}) > 0. \end{aligned}\tag{3.33}$$

The estimate established in equation (3.33) can be viewed as an asymptotic confirmation of the coercivity of the operators $\mathcal{A}_{k,PS_{ik}}$. In order to establish rigorously the coercivity property of the operators $\mathcal{A}_{k,S_{ik}}$ and $\mathcal{A}_{k,PS_{ik}}$ in the case when Γ is the unit circle, we will examine the positivity of the real part of the eigenvalues $a_m(k)$ and $p_m(k)$ for large enough k and for all modes $0 \leq m$. To this end, we first state a useful result whose technical proof follows closely the lines of the proof of Lemma 4.5 in [27]

Lemma 3.5 *There exist $\varepsilon_1 > 0$ and $m_1 > 1$ such that for all $m \geq m_1$ and $k \in (0, m]$ the function $f_m(k) = kJ_m(k)Y_m'(k)$ has a unique local extremum point that lies in the interval $(m - \varepsilon_1 m^{\frac{1}{3}}, m)$. This local extremum point is a global minimum of the function f_m in the interval $(0, m]$.*

Proof. Given that the functions $\mathcal{C}_m$ are solutions of the differential equation $k^2\mathcal{C}_m''(k)+k\mathcal{C}_m'(k)+(k^2-m^2)\mathcal{C}_m(k) = 0$ where $\mathcal{C}_m$ denote either J_m or Y_m we obtain

$$\begin{aligned} f_m'(k) &= J_m(k)Y_m'(k) + kJ_m'(k)Y_m'(k) + kJ_m(k)Y_m''(k)\\ &= J_m(k)Y_m'(k) + kJ_m'(k)Y_m'(k) + kJ_m(k)\left(-\frac{1}{k}Y_m'(k) + \left(\frac{m^2}{k^2} - 1\right)Y_m(k)\right)\\ &= kJ_m'(k)Y_m'(k) + \left(\frac{m^2}{k} - k\right)J_m(k)Y_m(k) \end{aligned}\tag{3.34}$$

$$
\begin{aligned}
f_m''(k) &= J_m'(k)Y_m'(k) + kJ_m''(k)Y_m'(k) + kJ_m'(k)Y_m''(k) - \left(\frac{m^2}{k^2}+1\right) J_m(k)Y_m(k) \\
&+ \left(\frac{m^2}{k}-k\right)(J_m(k)Y_m(k))' \\
&= J_m'(k)Y_m'(k) + kY_m'(k)\left(-\frac{1}{k}J_m'(k) + \left(\frac{m^2}{k^2}-1\right)J_m(k)\right) \\
&+ kJ_m'(k)\left(-\frac{1}{k}Y_m'(k) + \left(\frac{m^2}{k^2}-1\right)Y_m(k)\right) \\
&- \left(\frac{m^2}{k^2}+1\right)J_m(k)Y_m(k) + \left(\frac{m^2}{k}-k\right)(J_m(k)Y_m(k))' \\
&= -J_m'(k)Y_m'(k) - \left(\frac{m^2}{k^2}+1\right)J_m(k)Y_m(k) + 2\left(\frac{m^2}{k}-k\right)(J_m(k)Y_m(k))'
\end{aligned} \tag{3.35}
$$

and

$$
\begin{aligned}
f_m'''(k) &= -J_m''(k)Y_m'(k) - J_m'(k)Y_m''(k) + \frac{2m^2}{k^3}J_m(k)Y_m(k) - \left(\frac{m^2}{k^2}+1\right)(J_m(k)Y_m(k))' \\
&- 2\left(\frac{m^2}{k^2}+1\right)(J_m(k)Y_m(k))' + 2\left(\frac{m^2}{k}-k\right)(J_m''(k)Y_m(k) + J_m(k)Y_m''(k) + 2J_m'(k)Y_m'(k)) \\
&= \left(1-\frac{m^2}{k^2}\right)J_m(k)Y_m'(k) + \frac{1}{k}J_m'(k)Y_m'(k) + \left(1-\frac{m^2}{k^2}\right)J_m'(k)Y_m(k) + \frac{1}{k}J_m'(k)Y_m'(k) \\
&+ \frac{2m^2}{k^3}J_m(k)Y_m(k) - 3\left(\frac{m^2}{k^2}+1\right)(J_m(k)Y_m(k))' \\
&+ 2\left(\frac{m^2}{k}-k\right)\left(2\left(\frac{m^2}{k^2}-1\right)J_m(k)Y_m(k) - \frac{1}{k}J_m'(k)Y_m(k) - \frac{1}{k}J_m(k)Y_m'(k) + 2J_m'(k)Y_m'(k)\right) \\
&= \frac{2m^2+4(m^2-k^2)^2}{k^3}J_m(k)Y_m(k) + \left(\frac{2}{k}+4\frac{m^2}{k}-4k\right)J_m'(k)Y_m'(k) - 6\frac{m^2}{k^2}(J_m(k)Y_m(k))'.
\end{aligned} \tag{3.36}
$$

If we take into account equation (3.15) and we make use again of the formulas [1] (Formulas 9.1.27)

$$
Y_m'(k) = Y_{m-1}(k) - \frac{m}{k}Y_m(k), \qquad m \geq 1
$$

in the definition of the functions f_m and in equations (3.34) and (3.35), we obtain equivalent expressions for the functions f_m, f_m' and f_m'' in a form that allows us to evaluate these functions at zero more easily:

$$
\begin{aligned}
f_m(k) &= kJ_m(k)Y_{m-1}(k) - mJ_m(k)Y_m(k) \\
f_m'(k) &= k\left(-J_{m+1}(k) + \frac{m}{k}J_m(k)\right)\left(Y_{m-1}(k) - \frac{m}{k}Y_m(k)\right) + \left(\frac{m^2}{k} - k\right)J_m(k)Y_m(k) \\
&= -kJ_m(k)Y_m(k) + mJ_m(k)Y_{m-1}(k) - kJ_{m+1}(k)Y_{m-1}(k) + mJ_{m+1}(k)Y_m(k) \\
f_m''(k) &= -\left(-J_{m+1}(k) + \frac{m}{k}J_m(k)\right)\left(Y_{m-1}(k) - \frac{m}{k}Y_m(k)\right) - \left(\frac{m^2}{k^2} + 1\right)J_m(k)Y_m(k) \\
&+ 2\left(\frac{m^2}{k} - k\right)\left[\left(-J_{m+1}(k) + \frac{m}{k}J_m(k)\right)Y_m(k) + \left(Y_{m-1}(k) - \frac{m}{k}Y_m(k)\right)J_m(k)\right] \\
&= J_{m+1}(k)Y_{m-1}(k) - J_m(k)Y_m(k) - (m - 2k^2 + 2m^2)\frac{J_{m+1}(k)Y_m(k)}{k} \\
&+ (2m^2 - 2k^2 - m)\frac{J_m(k)Y_{m-1}(k)}{k}.
\end{aligned} \tag{3.37}
$$

Using the series representations of the Bessel functions around zero (for $m > 0$) [1] (Formulas 9.1.7 and 9.1.9)

$$
J_m(z) = \frac{1}{m!}\left(\frac{z}{2}\right)^m + \mathcal{O}(z^{m+1}), \quad Y_m(z) = -\frac{(m-1)!}{\pi}\left(\frac{z}{2}\right)^{-m} + \mathcal{O}(z^{-m+1}), \quad |z| \to 0
$$

in formulas (3.37) we obtain that for all $m \geq 2$

$$
f_m(0) = \frac{1}{\pi}, \qquad f_m'(0) = 0, \qquad f_m''(0) = -\frac{1}{\pi m(m^2-1)} < 0.
$$

It follows then that there exists $\eta_m > 0$ (which may depend on m) such that $f_m'(k) < 0$ for $k \in [0, \eta_m]$.

We will study next the properties of the values $f_m(k)$ in the case of large enough orders m and arguments k that are close to m. We employ the expansions [1] (Formulas 9.3.33) valid for m large:

$$
\begin{aligned}
J_m(m) &= am^{-1/3} + \mathcal{O}(m^{-5/3}) \\
Y_m'(m) &= \sqrt{3}(bm^{-2/3} + cm^{-4/3}) + \mathcal{O}(m^{-8/3})
\end{aligned} \tag{3.38}
$$

for positive constants a, b, and c. We obtain

$$
f_m(m) = ab\sqrt{3} + \mathcal{O}(m^{-\frac{1}{3}}) = \frac{1}{\pi} + \mathcal{O}(m^{-\frac{1}{3}}),\ m \to \infty
$$

since $ab = \frac{2}{3\Gamma(1/3)\Gamma(2/3)} = \frac{1}{\sqrt{3}\pi}$ according to Formulas 9.3.31 and 9.3.34 in [1]. Furthermore, we obtain $f_m'(m) = 2^{\frac{4}{3}}m^{\frac{1}{3}}/(3^{1/6}\Gamma(2/3)^2) + \mathcal{O}(m^{-\frac{1}{3}}),\ m \to \infty$, and thus $f_m'(m) > 0$ for large enough m. For any fixed $\varepsilon > 0$ we use the transitional asymptotics in Formulas 9.3.23–9.3.28 in [1] to obtain

$$
f_m'(m - \varepsilon m^{\frac{1}{3}}) = -2^{\frac{4}{3}}m^{\frac{1}{3}}\varphi(\varepsilon) + \mathcal{O}(m^{-\frac{1}{3}}) \tag{3.39}
$$

where the function φ is defined in terms of Airy functions

$$
\varphi(\varepsilon) = \mathrm{Ai}'(2^{1/3}\varepsilon)\mathrm{Bi}'(2^{1/3}\varepsilon) + 2^{1/3}\varepsilon\ \mathrm{Ai}(2^{1/3}\varepsilon)\mathrm{Bi}(2^{1/3}\varepsilon). \tag{3.40}
$$

From Formulas 10.4.4-10.4.5 in [1] we get that $\varphi(0) = -1/(3^{1/6}\Gamma(2/3)^2) < 0$. Since $\varphi(0) < 0$, we get that there exists $\varepsilon_2 > 0$ such that $\varphi(x) < 0$ for all $0 \le x \le \varepsilon_2$ and thus $f_m'(m - xm^{\frac{1}{3}}) > 0$ for large enough values of m and for all $0 \le x \le \varepsilon_2$. We obtain consequently that $m - \varepsilon_2 m^{\frac{1}{3}} \le \sqrt{m^2 - 1/4}$ for large enough values of m, and hence $f_m'(k) > 0$ for all $k \in [\sqrt{m^2 - 1/4}, m]$ (the role of the expression $\sqrt{m^2 - 1/4}$ will be apparent shortly).
If in addition to the fact that $f_m'(k) > 0$ for $k \in [\sqrt{m^2 - 1/4}, m]$ we use the fact that $f_m'(k) < 0$ for $k \in [0, \eta_m]$, it follows that f_m has at least one critical point in the interval $(0, m]$ at which f_m' changes sign (i.e. a point of local extremum). Furthermore, the first such point must be a local minimum for f_m, and we denote it by η_1. We will show that the point of local extremum is unique in the interval $(0, m]$, and there exists $\epsilon_1 > 0$, independent of m, such that this local minimum lies in the interval $(m - \epsilon_1 m^{\frac{1}{3}}, \sqrt{m^2 - 1/4})$. To this end, we define the function

$$g_m(k) = k^3 f_m''(k)$$

and we obtain that the derivative of the function g_m can be written as

$$\begin{aligned}
g_m'(k) &= 3k^2 f_m''(k) + k^3 f_m'''(k) \\
&= -3k^2 J_m'(k) Y_m'(k) - 3(m^2 + k^2) J_m(k) Y_m(k) + 6(km^2 - k^3)(J_m(k)Y_m(k))' \\
&+ (2m^2 + 4(m^2 - k^2)^2) J_m(k) Y_m(k) + (2k^2 + 4m^2k^2 - 4k^4) J_m'(k) Y_m'(k) \\
&- 6km^2 (J_m(k)Y_m(k))' \\
&= (4m^2k^2 - 4k^4 - k^2) J_m'(k) Y_m'(k) + (4(m^2 - k^2)^2 - m^2 - 3k^2) J_m(k) Y_m(k) \\
&- 6k^3 (J_m(k) Y_m(k))' \qquad (3.41)
\end{aligned}$$

if we use formulas (3.35) and (3.36). On the other hand, using formula (3.34) we get

$$\begin{aligned}
-k(1 + 4k^2 - 4m^2) f_m'(k) &- 6k^3 (J_m(k)Y_m(k))' - 4k^2 J_m(k) Y_m(k) \\
&= -(1 + 4k^2 - 4m^2) k^2 J_m'(k) Y_m'(k) \\
&- (1 + 4k^2 - 4m^2)(m^2 - k^2) J_m(k) Y_m(k) - 4k^2 J_m(k) Y_m(k) \\
&- 6k^3 (J_m(k)Y_m(k))' \\
&= (4m^2k^2 - 4k^4 - k^2) J_m'(k) Y_m'(k) \\
&+ (4(m^2 - k^2)^2 - m^2 - 3k^2) J_m(k) Y_m(k) - 6k^3 (J_m(k)Y_m(k))'. \\
&\qquad (3.42)
\end{aligned}$$

From formulas (3.41) and (3.42) it follows that the function g_m satisfies the following equation:

$$g_m'(k) = -k(1 + 4k^2 - 4m^2) f_m'(k) - 6k^3 (J_m(k)Y_m(k))' - 4k^2 J_m(k) Y_m(k). \qquad (3.43)$$

If we suppose that f_m has at least one more point of local extremum in the interval $(0, \sqrt{m^2 - 1/4})$, we denote by $\eta_2 \in (0, \sqrt{m^2 - 1/4})$ the next point of local extremum for f_m. Since η_1 is a local minimum and $\eta_1 < \eta_2$, it must be that η_2 is a local maximum for f_m. Then, for $k \in [\eta_1, \eta_2]$, we have $f_m'(k) \ge 0$ and since $k < \sqrt{m^2 - 1/4}$ we have $1 + 4k^2 - 4m^2 < 0$. Using these two facts together with the relations $J_m(k) > 0$ and $Y_m(k) < 0$ when $k \in (0, m]$ and the fact that $(J_m Y_m)'(k) < 0$ when $k \in (0, m]$ (which was established in Lemma 4.5 in [27]) we conclude from equation (3.43) that

$$g_m'(k) > 0, \qquad \text{for all } k \in [\eta_1, \eta_2].$$

Since η_1 is a local minimum for f_m, we have that $f''_m(\eta_1) \geq 0$ and thus $g_m(\eta_1) \geq 0$. In addition, since $g'_m(k) > 0$ for all $k \in [\eta_1, \eta_2]$, it follows that g_m is increasing in the intervals $[\eta_1, \eta_2]$. If we take into account that $g_m(\eta_1) \geq 0$, we obtain that $g_m(k) > 0$ for all $k \in [\eta_1, \eta_2]$ and hence $f''_m(k) > 0$ for all $k \in (\eta_1, \eta_2]$. This contradicts the fact that η_2 is a local maximum of f_m. Thus, f_m has at most a single local extremum point in $(0, \sqrt{m^2 - 1/4})$. Since we established that f_m has a local minimum point in the interval $(0, m]$ and that $f'_m(k) > 0$ for all $k \in (\sqrt{m^2 - 1/4}, m]$, we get that f_m has a unique local minimum point in the interval $(0, m]$.
We finally establish that there exists $\varepsilon_1 > 0$ such that the unique local minimum point of f_m lies in the interval $(m - \varepsilon_1 m^{\frac{1}{3}}, \sqrt{m^2 - 1/4})$. To this end, we use the fact that $\varphi(2^{-1/3}) \approx 0.015$, where φ was defined in equation (3.40). Given that we previously established that $\varphi(0) < 0$, it follows from formula (3.39) that for large enough values of m the function $f'_m(m - \varepsilon m^{\frac{1}{3}})$ has a zero for values of ε in the interval $(0, 2^{-1/3})$. Consequently, there exists $\varepsilon_1 > 0$ such that the function f_m has a critical point in the interval $(m - \varepsilon_1 m^{\frac{1}{3}}, \sqrt{m^2 - 1/4})$ at which f'_m changes sign. Given that we established that $f_m(0) = 1/\pi$ and $f_m(m) \approx 1/\pi$, it follows that there exists $m_1 > 1$ such that f_m has a unique minimal point in the interval $(0, m]$ and that minimal point is actually in the interval $(m - \varepsilon_1 m^{\frac{1}{3}}, m)$ for all $m \geq m_1$, which completes the proof of the Lemma. ∎

We next state and prove the main result concerning the $L^2(\Gamma)$ coercivity of the operators $\mathcal{A}_{k,S_{ik}}$ and $\mathcal{A}_{k,PS_{ik}}$ in the case when Γ is the unit circle:

Theorem 3.6 *There exist positive constants $\tilde{C}_0, C_S$, and C_{PS} independent of k and a wavenumber k_2 such that for all $k > k_2$ and coupling constants η with the property that $\eta > \tilde{C}_0\, k^{1/3}$ the following estimates hold:*

$$\inf_{m\in\mathbb{N}} \Re(a_m(k)) \geq C_S > 0, \quad \inf_{m\in\mathbb{N}} \Re(p_m(k)) \geq C_{PS} > 0, \text{ for all } k > k_2. \tag{3.44}$$

Consequently, the operators $\mathcal{A}_{k,S_{ik}}$ and $\mathcal{A}_{k,PS_{ik}}$ are L^2 coercive in the case when Γ is the unit circle in two dimensions.

Proof. We first study the properties of $\Re(p_m(k))$. We will use throughout the two representations of $\Re(p_m(k))$ expressed in equations (3.31) and (3.32). Just as before, let us fix k large enough and we consider several cases. In each case we will prove first that $\inf_{m\in\mathbb{N}} \Re(p_m(k)) > 0$, then we will establish the stronger result that $\inf_{m\in\mathbb{N}} \Re(p_m(k)) \geq C_{PS}$.

Case 1: $m > k > 0$. We use the results in Watson [48] (Section 15.3) which established that $J_m(k) > 0$ and $Y'_m(k) > 0$ to get that

$$\Re(p_m(k)) > 0 \text{ for } m > k > k_0.$$

Using the asymptotic expansions of the Bessel and Neumann functions for fixed k for large orders $\nu \to \infty$, namely [1] (Formula 9.3.1)

$$\begin{aligned} J_\nu(k) &\sim \frac{1}{\sqrt{2\pi\nu}}\left(\frac{ek}{2\nu}\right)^\nu \\ J'_\nu(k) &\sim \sqrt{\frac{\nu}{2\pi}}\left(\frac{e}{2\nu}\right)^\nu k^{\nu-1} \\ Y'_\nu(k) &\sim \sqrt{\frac{2\nu}{\pi}}\left(\frac{e}{2\nu}\right)^{-\nu} k^{-\nu-1} \end{aligned} \tag{3.45}$$

we get that

$$\Re(p_\nu(k)) \sim \frac{1}{2} + \frac{\eta}{8}\left(\frac{ek^2}{2\nu}\right)^\nu, \ \nu \to \infty \tag{3.46}$$

from which we conclude that for a fixed k large enough there exists a constant m_k such that

$$\Re(p_m(k)) \geq 0.4 \text{ for all } m_k \leq m.$$

Given that for all other indices $m > k$ we have that $\Re(p_m(k)) > 0$ it follows that for all large enough k we have the following estimate

$$\inf_{m>k} \Re(p_m(k)) > 0. \tag{3.47}$$

We derive in what follows a sharper lower bound on $\inf_{k\in(0,m)} \Re(p_m(k))$. To this end, we use the result established in Lemma 3.5 that there exist $\varepsilon_1 > 0$ and $m_1 > 1$ such that for all $m \geq m_1$ the function $f_m(k) = kJ_m(k)Y_m'(k)$ has a unique minimum point in the interval $(0, m)$, that actually lies in the interval $(m - \varepsilon_1 m^{\frac{1}{3}}, m]$. Given the transition asymptotic Formulas 9.3.23 and 9.3.28 in [1] we have that for a fixed $x \in [0, \varepsilon_1)$

$$(m - xm^{1/3})J_m(m - xm^{1/3})Y_m'(m - xm^{1/3}) \sim 2\mathrm{Ai}(2^{1/3}x)\ \mathrm{Bi}'(2^{1/3}x) + \mathcal{O}(m^{-\frac{2}{3}}),\ m \to \infty. \tag{3.48}$$

It can easily be checked that the function

$$G(x) = \mathrm{Ai}(x)\mathrm{Bi}'(x),\ x \geq 0$$

has a unique critical point $x_* \approx 0.72$ in $(0, \infty)$, and that x_* is a point of global minimum for G. Since $G(x_*) \approx 0.1238$ it follows that $G(x) \geq 0.12$ for all $x \geq 0$. Using this in equations (3.48) we obtain that there exists m_1 large enough such that

$$\Re(p_m(k)) \geq \frac{\pi}{2} f_m(k) \geq \frac{\pi}{2} \min_{k\in(m-\varepsilon_1 m^{1/3},m)} f_m(k) \geq 0.12\pi \approx 0.377, \text{ for all } m \geq m_1,\ k \in (0, m). \tag{3.49}$$

Thus, it follows from equation (3.49) that there exists a constant $C_{2,1}$ and a sufficiently large $k_{2,1}$ such that

$$\Re(p_m(k)) \geq C_{2,1}, \text{for all } k \geq k_{2,1}, \text{ and for all } m > k. \tag{3.50}$$

Case 2: $m \leq \frac{\sqrt{3}}{2}k$. In this case we use the representation of $\Re(p_m(k))$ given in formula (3.31). We have that

$$|J_m'(k)Y_m(k)| \leq \frac{1}{2}(J_m'(k))^2 + \frac{1}{2}(Y_m(k))^2 \leq \frac{1}{2}(J_m'(k))^2 + \frac{1}{2}(M_m(k))^2 \tag{3.51}$$

where the expression $M_m(k)$ is defined as $M_m(k) = \sqrt{J_m^2(k) + Y_m^2(k)}$. Using the estimate $M_m^2(k) \leq \frac{2}{\pi\sqrt{k^2-m^2}}$ [48] (Section 13.74), which is valid for all $k \geq m$ we obtain

$$\frac{\pi k}{2}|J_m'(k)Y_m(k)| \leq \frac{\pi k}{4}(J_m'(k))^2 + \frac{k}{2\sqrt{k^2 - m^2}}.$$

Since $m \leq \frac{\sqrt{3}}{2}k$ it follows that $\frac{k}{2\sqrt{k^2-m^2}} \leq 1$ and thus

$$\frac{\pi k}{2}|J_m'(k)Y_m(k)| \leq \frac{\pi\eta k^2}{4\sqrt{m^2 + k^2}}(J_m'(k))^2 + 1 \tag{3.52}$$

as long as $\frac{\sqrt{7}}{2} \leq \eta$. It follows then that if $\frac{\sqrt{7}}{2} \leq \eta$

$$-\frac{\pi k}{2} J'_m(k) Y_m(k) \leq \frac{\pi k}{2} |J'_m(k) Y_m(k)| \leq \frac{\pi \eta k^2}{4\sqrt{m^2+k^2}} (J'_m(k))^2 + 1$$

or equivalently

$$\Re(p_m(k)) \geq 0. \tag{3.53}$$

If $\frac{\sqrt{7}}{2} < \eta$, equality in the estimate (3.53) would be obtained if we had equality in equation (3.51), that is if $J'_m(k) = Y_m(k)$ and $J_m(k) = 0$. Furthermore, $\Re(p_m(k)) = 0$ and $J_m(k) = 0$ would imply via equation (3.67) that $J'_m(k) = 0$. Thus we get that $\Re(p_m(k)) = 0$ iff all the values $J_m(k)$, $Y_m(k)$, $J'_m(k)$ were equal to zero, which can never be the case since the real zeros of J_m and Y_m interlace [48](Section 15.3). In conclusion, if $\frac{\sqrt{7}}{2} < \eta$ we get that for any $k > k_0$

$$\inf_{m \leq \frac{\sqrt{3}}{2} k} \Re(p_m(k)) > 0. \tag{3.54}$$

In order to get a sharper estimate on the quantity $\inf_{m \leq \frac{\sqrt{3}}{2} k} \Re(p_m(k))$ we use again the representation (3.31) and we study the behavior of the quadratic function for values $m \leq \frac{\sqrt{3}}{2} k$:

$$g(x) = \frac{2}{\pi k} + Y_m(k)\ x + \frac{k\eta}{2(m^2+k^2)^{\frac{1}{2}}}\ x^2. \tag{3.55}$$

We have that

$$g(x) \geq \frac{-\Delta_g\ (m^2+k^2)^{\frac{1}{2}}}{2k\eta}$$

where Δ_g denotes the discriminant of the quadratic function g whose formula is given by

$$\Delta_g = Y_m^2(k) - \frac{4\eta}{\pi(m^2+k^2)^{\frac{1}{2}}}. \tag{3.56}$$

We first show that $\Delta_g < 0$. To this end we use again the estimate $M_m^2(k) \leq \frac{2}{\pi\sqrt{k^2-m^2}}$ and we get

$$\begin{aligned} -\Delta_g &= \frac{4\eta}{\pi(m^2+k^2)^{\frac{1}{2}}} - Y_m^2(k) \geq \frac{4\eta}{\pi(m^2+k^2)^{\frac{1}{2}}} - M_m^2(k) \\ &\geq \frac{4\eta}{\pi(m^2+k^2)^{\frac{1}{2}}} - \frac{2}{\pi\sqrt{k^2-m^2}} = \frac{4\eta}{\pi(m^2+k^2)^{\frac{1}{2}}} \left(1 - \frac{(m^2+k^2)^{\frac{1}{2}}}{2\eta(k^2-m^2)^{\frac{1}{2}}}\right) \\ &\geq \frac{4\eta}{\pi(m^2+k^2)^{\frac{1}{2}}} \left(1 - \frac{\sqrt{7}}{2\eta}\right) \geq 0. \end{aligned} \tag{3.57}$$

It follows then that for all $x \in \mathbb{R}$ we have

$$g(x) \geq \frac{-\Delta_g\ (m^2+k^2)^{\frac{1}{2}}}{2k\eta} \geq \frac{2}{\pi k}\left(1 - \frac{\sqrt{7}}{2\eta}\right)$$

which implies via representation (3.31) that there exists $k_{2,2}$ large enough such that

$$\Re(p_m(k)) \geq 1 - \frac{\sqrt{7}}{2\eta} > \frac{1}{2},\ \text{for all } k > k_{2,2},\ 0 \leq m \leq \frac{\sqrt{3}}{2} k. \tag{3.58}$$

We have used the fact that $\eta > \tilde{C}_0 k^{\frac{1}{3}}$ to establish estimate (3.58).

Case 3: $\frac{\sqrt{3}}{2}k < m \leq k$. We start with the case when $m \approx k$ and k is large. We use formulas (3.13) and (3.38) and we get that

$$\Re(p_\nu(\nu)) = \frac{\pi}{2}ab\sqrt{3} + \mathcal{O}(\nu^{-1/3}),\ \nu \to \infty, \tag{3.59}$$

which in particular implies that $\Re(p_m(m)) > 0$ for m large enough. We proceed with the analysis of the case when $k - m$ is small enough with respect to k. More precisely, we consider first the following subcase

Case 3a: $k - c_0 k^{\frac{1}{3}} \leq m \leq k$ and k large enough, where c_0 is a constant that will be specified later. In this case we consider the representation of $\Re(p_m(k))$ given by equations (3.32), and we can write $k = m + zm^{1/3}$ where $0 \leq z \leq z_0$ where we can take $z_0 = c_0(2/\sqrt{3})^{\frac{1}{3}}$. We consider a fixed z such that $0 \leq z \leq z_0$ and large values of the orders m, and we use Formulas 9.3.23 and 9.3.28 in [1] to obtain that

$$J_m(m + zm^{1/3})Y_m'(m + zm^{1/3}) = \frac{2}{m}\mathrm{Ai}(-2^{1/3}z)\mathrm{Bi}'(-2^{1/3}z) + \mathcal{O}(m^{-4/3}) \tag{3.60}$$

where Ai and Bi denote Airy functions. Denoting by

$$f(z) := \mathrm{Ai}(-2^{1/3}z)\mathrm{Bi}'(-2^{1/3}z),\ 0 \leq z \tag{3.61}$$

we get according to Formulas 10.4.4 and 10.4.5 in [1] that

$$f(0) = \frac{1}{2\pi} > 0,\ f'(0) = \frac{2^{1/3}}{3^{1/6}(\Gamma(1/3))^2} > 0.$$

Let $z_0 \approx 0.8712$ denote the first positive zero of the derivative f' of the function f defined in equation (3.61). It follows that f is increasing on the interval $[0, z_0]$ and thus $f(z) \geq f(0) = \frac{1}{2\pi}$ for all $0 \leq z \leq z_0$. It follows via the asymptotic formula (3.60) that there exists a m_0 large enough such that for all $0 \leq z \leq z_0$ and all $m > m_0$ we have

$$J_m(m + zm^{1/3})Y_m'(m + zm^{1/3}) \geq \frac{0.9}{\pi m}$$

from which it follows that

$$\Re(p_m(m + zm^{1/3})) \geq \frac{\pi(m + zm^{\frac{1}{3}})}{2}J_m(m + zm^{1/3})Y_m'(m + zm^{1/3}) \geq 0.45. \tag{3.62}$$

The last equation implies that there exists $c_0 = z_0(2/\sqrt{3})^{-\frac{1}{3}} \approx 0.8304$ and $k_{2,3}$ such that

$$\Re(p_m(k)) \geq 0.45,\ \text{for all } k_{2,3} \leq k,\ k - c_0 k^{\frac{1}{3}} \leq m \leq k \tag{3.63}$$

Case 3b: In the remaining case of orders m such that $\frac{\sqrt{3}}{2}k \leq m \leq k - c_0 k^{\frac{1}{3}}$, we investigate again the discriminant Δ_g of the quadratic function g defined in equation (3.55). We use one more time the estimate $M_m^2(k) \leq \frac{2}{\pi\sqrt{k^2-m^2}}$ and we get if we repeat the arguments that led to estimate (3.57)

$$\begin{aligned} -\Delta_g &\geq \frac{4\eta}{\pi(m^2+k^2)^{\frac{1}{2}}}\left(1-\frac{(m^2+k^2)^{\frac{1}{2}}}{2\eta(k^2-m^2)^{\frac{1}{2}}}\right) \\ &\geq \frac{4\eta}{\pi(m^2+k^2)^{\frac{1}{2}}}\left(1-\frac{\sqrt{2}k}{2\eta[c_0k^{\frac{1}{3}}(2k-c_0k^{\frac{1}{3}})]^{\frac{1}{2}}}\right) \\ &\geq \frac{4\eta}{\pi(m^2+k^2)^{\frac{1}{2}}}\left(1-\frac{k^{\frac{1}{3}}}{\eta\sqrt{2c_0}}\right). \end{aligned} \tag{3.64}$$

Clearly, taking $\eta > \tilde{C}_0 k^{\frac{1}{3}}$ with $\tilde{C}_0$ large enough such that $\tilde{C}_0 > \frac{1}{\sqrt{2c_0}}$, it follows from equation (3.64) that $\Delta_g < 0$ and thus $g(x) > 0$ for all $x \in \mathbb{R}$. Furthermore, we have that for all $x \in \mathbb{R}$

$$g(x) \geq \frac{-\Delta_g\ (m^2+k^2)^{\frac{1}{2}}}{2k\eta} \geq \frac{2}{\pi k}\left(1-\frac{1}{\sqrt{2c_0}\tilde{C}_0}\right)$$

which implies that

$$\Re(p_m(k)) \geq 1-\frac{1}{\sqrt{2c_0}\tilde{C}_0}, \text{ for all } k_{2,3} \leq k,\ \frac{\sqrt{3}}{2}k \leq m \leq k-c_0k^{\frac{1}{3}}. \tag{3.65}$$

Taking k_2 to be the maximum of $k_{2,1}, k_{2,2}$, and $k_{2,3}$ and C_{PS} to be the minimum of the constants that appear in the right hand sides of equations (3.50), (3.58), (3.63), and (3.65) (with the choice $\tilde{C}_0 = 1$ in the condition $\eta > \tilde{C}_0 k^{1/3}$, this would amount to an estimate $C_{PS} \approx 0.377$ which is fairly close to the value in the right panel of Figure 1), we obtain that for all $k > k_2$

$$\Re(p_m(k)) \geq C_{PS}, \text{ for all } m. \tag{3.66}$$

Finally, the real part of the eigenvalues $a_m(k)$ can be expressed as

$$\Re(a_m(k)) = \frac{\pi k}{2}\left(J_m(k)Y_m'(k)+\frac{k\pi\eta}{2}(J_m'(k))^2[iJ_m(ik)H_m^{(1)}(ik)]\right), \tag{3.67}$$

where we used the fact that $iJ_m(ik)H_m^{(1)}(ik) \in \mathbb{R}$. Using the result in Lemma 3.1 we obtain that for all $k \geq k_0$ and all m we have that

$$\frac{\pi k}{2}\left(J_m(k)Y_m'(k)+\frac{k\pi\eta C_1}{2(m^2+k^2)^{\frac{1}{2}}}(J_m'(k))^2\right) \leq \Re(a_m(k)).$$

Based on the result that we established above about uniform lower bounds on $\Re(p_m(k))$, it follows that if $\eta \geq \frac{C_1}{C_0}k^{\frac{1}{3}}$, there exists a constant k_2'' and a constant C_S such that for all $k > k_2''$

$$\Re(a_m(k)) \geq C_S, \text{ for all } m. \tag{3.68}$$

Clearly, combining equations (3.66) and (3.68), the result of the Theorem follows immediately. ∎

Remark 3.7 *A weaker version of the result in the Theorem 3.6 holds, that is if $\eta > \frac{\sqrt{7}}{2}$, then there exists $k_3 > 0$ such that*

$$\inf_{m\in\mathbb{N}} \Re(a_m(k)) > 0, \quad \inf_{m\in\mathbb{N}} \Re(p_m(k)) > 0, \text{ for all } k > k_2.$$

Furthermore, $\inf_{m\in\mathbb{N}} \Re(a_m(k)) \sim C|\eta|k^{-\frac{1}{3}}$, and a similar estimate holds for $\inf_{m\in\mathbb{N}} \Re(p_m(k))$.

Remark 3.8 *An immediate corollary of Theorem 3.6 is that if $\eta > C_0 k^{\frac{1}{3}}$, there exist $k_2 > 0$ and two constants $C_{1,p}$ and $C_{2,p}$ such that*

$$\|\mathcal{A}^{-1}_{k,S_{ik}}\|_{H^p(\mathbb{S}^1)\to H^p(\mathbb{S}^1)} \le C_{1,p}, \quad \|\mathcal{A}^{-1}_{k,PS_{ik}}\|_{H^p(\mathbb{S}^1)\to H^p(\mathbb{S}^1)} \le C_{2,p}, \quad 0 \le p.$$

In the three dimensional case, a direct calculation gives the following expression of the real part of the eigenvalues $b_m(k)$ of the operators $\mathcal{B}_{k,S_{ik}}$

$$\begin{aligned}\Re(b_m(k)) &= \Re(a_{m+\frac{1}{2}}(k)) - \frac{\pi}{4} J_{m+\frac{1}{2}}(k) Y_{m+\frac{1}{2}}(k) + \frac{\eta\pi^2}{16} J^2_{m+\frac{1}{2}}(k)[iJ_{m+\frac{1}{2}}(ik)H^{(1)}_{m+\frac{1}{2}}(ik)] \\ &- \frac{\eta\pi^2 k}{4} J_{m+\frac{1}{2}}(k) J'_{m+\frac{1}{2}}(k)[iJ_{m+\frac{1}{2}}(ik)H^{(1)}_{m+\frac{1}{2}}(ik)]. \end{aligned} \tag{3.69}$$

We use the following results established in [27]: there exists a constant $C > 0$ such that for all m and for all k large enough we have

$$|J_m(k)Y_m(k)| \le Ck^{-\frac{2}{3}}, \quad |J_m(k)| + |J'_m(k)| \le Ck^{-\frac{1}{3}}.$$

The second inequality above implies that $|J_m(k)J'_m(k)| \le \frac{1}{4}(|J_m(k)| + |J'_m(k)|)^2 \le Ck^{-\frac{2}{3}}$. Using these estimates together with the results in Lemma 3.1 we get that

$$\Re(b_m(k)) \ge \Re(a_{m+\frac{1}{2}}(k)) - Ck^{-\frac{2}{3}} - C|\eta|k^{-\frac{2}{3}}$$

for all $m \ge 0$ and all k large enough. The last estimate and the coercivity estimates in Theorem 3.6 imply the following result:

Theorem 3.9 *There exist positive constants C_0 and C independent of k and a wavenumber k_3 such that for all $k > k_3$ and coupling constants η with the property that $\eta \approx \tilde{C}_0\ k^{1/3}$ the following estimates hold:*

$$\inf_{m\in\mathbb{N}} \Re(b_m(k)) \ge C > 0, \text{ for all } k > k_2. \tag{3.70}$$

Consequently, the operators $\mathcal{B}_{k,S_{ik}}$ are coercive in $L^2(\mathbb{S}^2)$.

3.3 Wavenumber dependent bounds of the norms of the operators ICFIE-R and ICFIE-RPS in the case of general smooth boundaries Γ

The next set of results concerns establishing upper bounds on the 2-norm of the operators $\mathcal{A}_{k,S_{ik}}$ and $\mathcal{A}_{k,PS_{ik}}$ for general smooth curves Γ. For this we will need the following technical result:

Lemma 3.10 *For each $m = 0, 1, \ldots,$ there exists $C_m > 0$ independent of k such that $\int_0^{2\pi} K_m(2k|\sin\ t/2|)|\sin\ t/2|^m dt \le C_m k^{-1-m}$ for k sufficiently large, where K_m stands for the modified Bessel function of the third kind.*

Proof. We use

$$\int_0^{2\pi} K_m(2k|\sin\ t/2|)|\sin\ t/2|^m\ dt = 4\int_0^{\pi/2} K_m(2k\sin\ t)\sin^m t\ dt$$

and since $K_m(r) = \left(\frac{\pi}{2r}\right)^{1/2} e^{-r}(1 + \mathcal{O}(r^{-1}))$ as $r \to \infty$ [1] (Formula 9.7.2), we get

$$\int_{\pi/6}^{\pi/2} K_m(2k\sin\ t)\sin^m t\ dt \le C_m k^{-1/2} e^{-k}$$

with C_m independent of k. On the other hand, using the change of variables $x = 2k\sin t$ we obtain

$$\int_0^{\pi/6} K_m(2k\sin\ t)\sin^m t\ dt \le \frac{2}{\sqrt{3}(2k)^{m+1}}\int_0^k K_m(x)x^m dx.$$

The last integral can be split as

$$\int_0^k K_m(x)x^m dx = \int_0^L K_m(x)x^m dx + \int_L^k K_m(x)x^m dx \le C_L(1 + \int_L^k t^{-1/2}e^{-t}dt)$$

for $k \to \infty$ since $K_m(x)x^m$ is locally integrable in $[0, \infty)$ [1]. The result follows now readily. ■

We also need the following technical result:

Lemma 3.11 *There exists $C > 0$ such that for all $t > 0$ and all $\epsilon > 0$ the following estimate holds:*

$$\left|H_1^{(1)}(it) + \frac{2}{\pi t}\right| \le Ct^{-1+\epsilon}$$

.

Proof. We use the fact that $H_1^{(1)}(it) = -\frac{2}{\pi}K_1(t)$, where K_1 is the modified Bessel function of the third kind and order one. From the series representation of K_1 [1] (Formula 9.6.11) for $t > 0$

$$K_1(t) = \frac{1}{t} + t\ln t\sum_{j=0}^{\infty} b_j t^{2j} + t\sum_{j=0}^{\infty} a_j t^{2j}$$

we conclude that $\left|H_1^{(1)}(it) + \frac{2}{\pi t}\right| \le Ct^{-1+\epsilon}$ for small enough t. Using the asymptotic expansion [1] (Formula 9.7.2) $H_1^{(1)}(it) = \left(\frac{\pi}{2t}\right)^{1/2} e^{-t}(1+\mathcal{O}(t^{-1}))$, $t \to \infty$ we obtain that $\left|H_1^{(1)}(it) + \frac{2}{\pi t}\right| \le Ct^{-1+\epsilon}$ for all large enough t. Finally, since the function $H_1^{(1)}(it)+\frac{2}{\pi t}$ is continuous on any compact interval strictly included in $(0, \infty)$, the result of the Lemma follows. ■

Using the results of the previous Lemma 3.10 and Lemma 3.11 we will derive wavenumber dependent upper bounds on the norms of the operators $\mathcal{A}_{k,S_{ik}}$ and $\mathcal{A}_{k,PS_{ik}}$:

Theorem 3.12 *In the case when Γ is of class C^∞, there exist $k_0 > 0$ and $C > 0$ independent of k such that*

$$\|\mathcal{A}_{k,S_{ik}}\|_{L^2(\Gamma)\to L^2(\Gamma)} \le C(1+|\eta|)\ k^{1/2+\epsilon},\ \ \|\mathcal{A}_{k,PS_{ik}}\|_{L^2(\Gamma)\to L^2(\Gamma)} \le C(1+|\eta|)\ k^{1/2}$$

for any $\epsilon > 0$.

Proof. In the proof of the Theorem we will use generic constants C that do not depend on the wavenumber k. We start by noting that it was established in Theorem 4.14 in [27] that $\|K'\|_{L^2(\Gamma)\to L^2(\Gamma)} \le Ck^{1/2}$. We derive next a wavenumber k dependent bound on $\|S_{ik}\|_{L^2(\Gamma)\to L^2(\Gamma)}$. To this end, we write the integral operator S_{ik} in the form

$$(S_{ik}\psi)(s) = \int_0^{2\pi} S^k(s,t)\psi(t)dt$$

where

$$S^k(s,t) = \frac{i}{4}H_0^{(1)}(ik|x(s)-x(t)|)|x'(t)|.$$

With these notations we have that

$$\int_0^{2\pi}\left|\int_0^{2\pi} S^k(s,t)\psi(t)\ dt\right|^2 ds = \frac{1}{4}\int_0^{2\pi}\left|\int_0^{2\pi} W_{ik}(s,t)H_0^{(1)}(2ik|\sin((s-t)/2)|)\psi(t)\ dt\right|^2 ds$$

with

$$W_{ik}(s,t) = \frac{S^k(s,t)}{H_0^{(1)}(2ik|\sin((s-t)/2)|)}.$$

We note that the denominator of the kernel W_{ik} corresponds to the evaluation of its numerator on the unit circle $x(s) = (\cos s, \sin s)$. Since the singularity as $s \to t$ on the numerator and denominator of W_{ik} are the same, and for $s \neq t$, the asymptotic behavior of the top and the bottom of the expression $W_{ik}(s,t)$ as $k \to \infty$ are the same, it follows that $|W_{ik}(s,t)|$ can be bounded uniformly for all k large enough and $s,t \in [0,2\pi]^2$. Thus, we obtain using Young's inequality $\|f*g\|_2 \le \|f\|_1\|g\|_2$, and Lemma 3.10

$$\begin{aligned}\int_0^{2\pi}\left|\int_0^{2\pi} S^k(s,t)\psi(t)\ dt\right|^2 ds &\le C\left(\int_0^{2\pi} K_0(2k|\sin\ t/2|)|dt\right)^2\|\psi\|_2^2\\ &\le Ck^{-2}\|\psi\|_2^2,\end{aligned}\tag{3.71}$$

and consequently

$$\|S_{ik}\|_{L^2(\Gamma)\to L^2(\Gamma)} \le Ck^{-1}.\tag{3.72}$$

We derive next a bound on $\|N_k \circ S_{ik}\|_{L^2(\Gamma)\to L^2(\Gamma)}$. Based on equations (2.8), we write

$$\begin{aligned}([N_k\circ S_{ik}]\phi)(\mathbf{x}) &= k^2\int_\Gamma G_k(\mathbf{x}-\mathbf{y})(\mathbf{n}(\mathbf{x})\cdot\mathbf{n}(\mathbf{y}))(S_{ik}\phi)(\mathbf{y})ds(\mathbf{y})\\ &+ \int_\Gamma \partial_s[G_k(\mathbf{x}-\mathbf{y})-G_0(\mathbf{x}-\mathbf{y})]\partial_s(S_{ik}\phi)(\mathbf{y})ds(\mathbf{y})\\ &+ \mathrm{PV}\int_\Gamma \partial_s G_0(\mathbf{x}-\mathbf{y})\partial_s(S_{ik}\phi)(\mathbf{y})ds(\mathbf{y})\\ &= (\mathcal{A}_k^{(1)}\circ S_{ik}\phi)(\mathbf{x}) + [\mathcal{A}_k^{(2)}(\partial_s S_{ik}\phi)](\mathbf{x}) + [\mathcal{A}^{(3)}(\partial_s S_{ik}\phi)](\mathbf{x})\end{aligned}\tag{3.73}$$

where $G_0(\mathbf{x}-\mathbf{y}) = \frac{1}{2\pi}\log|\mathbf{x}-\mathbf{y}|$ is the free-space fundamental solution of Laplace's equation in 2D. We use the fact that $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq Ck^{-1/2}$, which was established in Theorem 4.14 in [27], from which it follows immediately that $\|\mathcal{A}_k^{(1)}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq Ck^{3/2}$, and thus

$$\|\mathcal{A}_k^{(1)} \circ S_{ik}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq Ck^{1/2}. \tag{3.74}$$

We derive next bounds on $\|\mathcal{A}_k^{(2)}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ based on the Riesz-Thorin interpolation techniques used in [16]. The main idea is to consider integral operators of the form

$$\mathcal{A}\phi(\mathbf{x}) = \int_\Gamma \kappa(\mathbf{x},\mathbf{y})\phi(\mathbf{y})ds(\mathbf{y}) \tag{3.75}$$

and to obtain estimates of the operator norms associated with such operators $\mathcal{A}$ as mappings from $L^2(\Gamma) \to L^2(\Gamma)$ via estimates on the operator norms of these operators as mappings from $L^1(\Gamma) \to L^1(\Gamma)$ and $L^\infty(\Gamma) \to L^\infty(\Gamma)$ respectively. Once the latter bounds are established, the bounds on the L^2 norms of operators $\mathcal{A}$ are obtained through the classical Riesz-Thorin interpolation theorem [25]

$$\|\mathcal{A}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq \|\mathcal{A}\|^{1/2}_{L^1(\Gamma)\to L^1(\Gamma)}\ \|\mathcal{A}\|^{1/2}_{L^\infty(\Gamma)\to L^\infty(\Gamma)}.$$

The two norms that appear on the right-hand side of the previous estimate can be computed explicitly

$$\|\mathcal{A}\|_{L^1(\Gamma)\to L^1(\Gamma)} = \operatorname*{ess\,sup}_{\mathbf{y}\in\Gamma}\int_\Gamma |\kappa(\mathbf{x},\mathbf{y})|ds(\mathbf{x}),\ \ \|\mathcal{A}\|_{L^\infty(\Gamma)\to L^\infty(\Gamma)} = \operatorname*{ess\,sup}_{\mathbf{x}\in\Gamma}\int_\Gamma |\kappa(\mathbf{x},\mathbf{y})|ds(\mathbf{y}).$$

The integral operator $\mathcal{A}_k^{(2)}$ is an integral operator of the kind described in equations (3.75) with a kernel

$$\kappa_k(\mathbf{x},\mathbf{y}) = -\left(\frac{i}{4}kH_1^{(1)}(k|\mathbf{x}-\mathbf{y}|) - \frac{1}{2\pi|\mathbf{x}-\mathbf{y}|}\right)\ \frac{(\mathbf{x}-\mathbf{y})\cdot\tau(\mathbf{x})}{|\mathbf{x}-\mathbf{y}||\tau(\mathbf{x})|}$$

in terms of the tangent vectors $\tau(\mathbf{x})$ to the curve Γ at the points $\mathbf{x}$. We use the following estimate established in Equations 3.9 in [16]

$$\left|H_1^{(1)}(t) + \frac{2i}{\pi t}\right| \leq \frac{C}{t^{1/2}},\ \ t > 0$$

to derive that $|\kappa_k(\mathbf{x},\mathbf{y})| \leq C\tilde{\kappa}(\mathbf{x},\mathbf{y})$ where $\tilde{\kappa}(\mathbf{x},\mathbf{y}) = \frac{k^{1/2}}{|\mathbf{x}-\mathbf{y}|^{1/2}}$. We consider integral operators $\tilde{\mathcal{A}}$ defined by equations (3.75) with kernels $\tilde{\kappa}(\mathbf{x},\mathbf{y})$ defined above. On account of the symmetry of the kernels $\tilde{\kappa}$ we have that

$$\|\tilde{\mathcal{A}}\|_{L^1(\Gamma)\to L^1(\Gamma)} = \|\tilde{\mathcal{A}}\|_{L^\infty(\Gamma)\to L^\infty(\Gamma)} \leq Ck^{1/2}\operatorname*{ess\,sup}_{\mathbf{x}\in\Gamma}\int_\Gamma \frac{1}{|\mathbf{x}-\mathbf{y}|^{1/2}}ds(\mathbf{y}) \leq Ck^{1/2},$$

where we have used the fact that the kernel $\tilde{\kappa}$ is weakly singular. Since $|\kappa_k(\mathbf{x},\mathbf{y})| \leq C\tilde{\kappa}(\mathbf{x},\mathbf{y})$, it follows that

$$\|\mathcal{A}^{(2)}\|_{L^1(\Gamma)\to L^1(\Gamma)} \leq Ck^{1/2},\ \ \|\mathcal{A}^{(2)}\|_{L^\infty(\Gamma)\to L^\infty(\Gamma)} \leq Ck^{1/2},$$

from which we get based on the Riesz-Thorin interpolation theorem that

$$\|\mathcal{A}_k^{(2)}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq Ck^{1/2}. \tag{3.76}$$

The operator $\mathcal{A}^{(3)}$ can be expressed as

$$(\mathcal{A}^{(3)}\psi)(s) = \int_0^{2\pi} T(s,t)\psi(t)dt$$

in terms of the kernel $T_k(s,t)$ defined as follows

$$\begin{aligned} T(s,t) &= T^{(1)}(s,t) + T^{(2)}(s,t) \\ T^{(1)}(s,t) &= \left(2\frac{(x(s)-x(t))\cdot x'(s)}{|x(t)-x(s)|^2} - \cot\frac{s-t}{2}\right)\frac{|x'(t)|}{4\pi|x'(s)|} \\ T^{(2)}(s,t) &= \cot\frac{s-t}{2}\,\frac{|x'(t)|}{4\pi|x'(s)|}. \end{aligned} \tag{3.77}$$

Since the kernel $T^{(1)}(s,t)$ is continuous on $[0,2\pi]\times[0,2\pi]$ it follows that

$$\int_0^{2\pi}\left|\int_0^{2\pi} T^{(1)}(s,t)\psi(t)\,dt\right|^2 ds \leq C\|\psi\|_2^2. \tag{3.78}$$

The integral operators with kernels $T^{(2)}(s,t)$ are related to the classical operator H_0 with the Hilbert kernel

$$(T_0\psi)(s) = \frac{1}{2\pi i}\int_0^{2\pi}\left(\cot\frac{t-s}{2} + i\right)\psi(t)dt$$

which satisfies $H_0^2 = I$ on $L^2(\Gamma)$ [31]. From this, we immediately obtain

$$\int_0^{2\pi}\left|\int_0^{2\pi} T^{(2)}(s,t)\psi(t)\,dt\right|^2 ds \leq C\|\psi\|_2^2. \tag{3.79}$$

We conclude from equations (3.77)-(3.79) that

$$\|\mathcal{A}^{(3)}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C. \tag{3.80}$$

The expressions $\partial_s S_{ik}\psi$, in turn, can be written as

$$\partial_s S_{ik}\psi = \mathcal{A}_{ik}^{(2)}\psi + \mathcal{A}^{(3)}\psi \tag{3.81}$$

where the integral operator $\mathcal{A}_{ik}^{(2)}$ is an integral operator of the type described in equations (3.75) with kernel

$$\kappa_{ik}(\mathbf{x},\mathbf{y}) = \left(\frac{k}{4}H_1^{(1)}(ik|\mathbf{x}-\mathbf{y}|) + \frac{1}{2\pi|\mathbf{x}-\mathbf{y}|}\right)\frac{(\mathbf{x}-\mathbf{y})\cdot\tau(\mathbf{x})}{|\mathbf{x}-\mathbf{y}||\tau(\mathbf{x})|}.$$

The application of the same techniques that delivered the estimate (3.76) in conjunction with the result in Lemma 3.11 leads us via an estimate of the form $|\kappa_{ik}(\mathbf{x},\mathbf{y})| \leq C\frac{k^\epsilon}{|\mathbf{x}-\mathbf{y}|^{1-\epsilon}}$ to the estimate

$$\|\mathcal{A}_{ik}^{(2)}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq Ck^\epsilon \tag{3.82}$$

for all $\epsilon > 0$ and hence the following estimate also holds

$$\|\partial_s S_{ik}\psi\|_2 \leq Ck^\epsilon\|\psi\|_2,\ \epsilon > 0. \tag{3.83}$$

Consequently, combining the estimates (3.74), (3.76), (3.80), and (3.83) we derive the first estimate $\|\mathcal{A}_{k,S_{ik}}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C(1+|\eta|)k^{1/2+\epsilon}$.

For the second operator $\mathcal{A}_{PS_{ik}}$ we use formula (3.73) with S_{ik} replaced by PS_{ik}. Given the definition of the regularizing operator PS_{ik} (2.17) we have

$$\|PS_{ik}\phi\|_2^2 = \sum_{n\in\mathbb{Z}} \frac{1}{n^2+k^2} |(\widehat{\phi|x'|})(n)|^2 \leq Ck^{-2}\|\phi\|_2^2 \tag{3.84}$$

and thus $\|PS_{ik}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq Ck^{-1}$. Furthermore,

$$\|\partial_s(PS_{ik}\phi)\|_2^2 \leq C\sum_{n\in\mathbb{Z}} \frac{n^2}{n^2+k^2} |(\widehat{\phi|x'|})(n)|^2 \leq C\|\phi\|_2^2 \tag{3.85}$$

and thus the second estimate $\|\mathcal{A}_{k,PS_k}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C(1+|\eta|)k^{1/2}$ follows from (3.74), (3.76), (3.80), (3.84), and (3.85). ■

Remark 3.13 *The estimates in Theorem 3.12 can be extended to the three dimensional ICFIE-R operators $\mathcal{B}_{k,S_{ik}}$ introduced in Section 3.1. More precisely, there exists a constant C independent of the wavenumber k such that $\|\mathcal{B}_{k,S_{ik}}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C(1+|\eta|)k^{1+\epsilon}$ for all $\epsilon > 0$.*

Having discussed the various regularization techniques, we will outline next the high-order Nyström method based on trigonometric interpolation introduced in [30] to discretize the two-dimensional integral operators that appear in (2.16) and (2.18).

4 Numerical method

We present in this part Nyström discretizations of equations (2.16) and (2.18) that are based on extensions of the Nyström discretization introduced in [30] for the numerical solution of equations (2.6). We also derive error estimates for the solutions that are obtained through these discretizations.

4.1 Discretization of the ICFIE-R equations (2.16)

We first use Calderón's identities [20] to recast equations (2.14) with the choice of the regularizing operators described in equations (2.16) in a form that does not include hypersingular operators

$$\begin{aligned}(\mathcal{A}_{k,S_{ik}}\mu)(\mathbf{x}) = \frac{\partial u^{\text{inc}}(\mathbf{x})}{\partial \mathbf{n}(\mathbf{x})}\mathbf{x}\in\Gamma,\ \mathcal{A}_{k,S_{ik}}\mu \quad &= \quad \left(\frac{1}{2}+\frac{i\eta}{4}\right)\mu \\ &- \quad K_k'\mu - i\eta[(N_k-N_{ik})S_{ik}]\mu + i\eta(K_{ik}')^2\mu. \end{aligned} \tag{4.1}$$

We assume that the boundary curve Γ is analytic and given by the parametrization $\mathbf{x}(t) = (x_1(t), x_2(t)),\ t\in[0,2\pi]$ where the functions $x_j:\mathbb{R}\to\mathbb{R}$ are analytic and 2π periodic such that

$|x'(t)| > 0$ for all t. We write equations (4.1) in the following parametric form

$$\begin{aligned}\left(\frac{1}{2}+\frac{i\eta}{4}\right)\mu(t)|x'(t)| \quad &+ \quad \int_0^{2\pi} H_k(t,\tau)\mu(\tau)d\tau + \int_0^{2\pi} K_k(t,\tau)(S_{ik}\mu)(\tau)d\tau \\ &- \quad \int_0^{2\pi} K_{ik}(t,\tau)(S_{ik}\mu)(\tau)d\tau + i\eta\int_0^{2\pi} H_{ik}(t,\tau)(K'_{ik}\mu)(\tau)d\tau \\ &= \quad f(t),\ 0\le t\le 2\pi\end{aligned} \tag{4.2}$$

where $\mu(t) \equiv \mu(x(t))$ and $f(t) = |x'(t)|g(x(t))$ with $g = \frac{\partial u^{\rm inc}}{\partial \mathbf{n}}|_\Gamma$. According to the derivations in [30], the kernel $H_k(t,\tau)$ is given by

$$H_k(t,\tau) = \frac{ik}{4}\mathbf{n}(t)\cdot(x(t)-x(\tau))\frac{H_1^{(1)}(k|x(t)-x(\tau)|)}{|x(t)-x(\tau)|}|x'(\tau)||x'(t)|, \tag{4.3}$$

where $\mathbf{n}(t) = \frac{(x_2'(t),-x_1'(t))}{|x'(t)|}$, and the kernel $K_k(t,\tau)$ is given by

$$K_k(t,\tau) = -i\eta k^2 M_k(t,\tau)x'(t)\cdot x'(\tau) + i\eta N_k(t,\tau) \tag{4.4}$$

with

$$M_k(t,\tau) \quad = \quad \frac{i}{4}H_0^{(1)}(k|x(t)-x(\tau)|) \tag{4.5}$$

$$N_k(t,\tau) \quad = \quad \frac{\partial^2}{\partial t\,\partial\tau}\left\{\frac{i}{4}H_0^{(1)}(k|x(t)-x(\tau)|)+\frac{1}{4\pi}\ln\left(4\sin^2\frac{t-\tau}{2}\right)\right\}. \tag{4.6}$$

Similarly, the kernel $K_{ik}(t,\tau)$ is given by

$$K_{ik}(t,\tau) = i\eta k^2 M_{ik}(t,\tau)x'(t)\cdot x'(\tau) + i\eta N_{ik}(t,\tau) \tag{4.7}$$

with

$$M_{ik}(t,\tau) \quad = \quad \frac{i}{4}H_0^{(1)}(ik|x(t)-x(\tau)|) \tag{4.8}$$

$$N_{ik}(t,\tau) \quad = \quad \frac{\partial^2}{\partial t\,\partial\tau}\left\{\frac{i}{4}H_0^{(1)}(ik|x(t)-x(\tau)|)+\frac{1}{4\pi}\ln\left(4\sin^2\frac{t-\tau}{2}\right)\right\}. \tag{4.9}$$

The operators $S_{ik}\mu$ and $K'_{ik}\mu$, in turn, can be expressed themselves in parametric form as

$$(S_{ik}\mu)(t) = \int_0^{2\pi} M_{ik}(t,\tau)\mu(\tau)|x'(\tau)|d\tau \tag{4.10}$$

and

$$(K'_{ik}\mu)(t) = \int_0^{2\pi}\frac{H_{ik}(t,\tau)}{|x'(\tau)|}\mu(\tau)d\tau \tag{4.11}$$

where the kernels $M_{ik}(t,\tau)$ are defined in the same manner as the kernels $M_k(t,\tau)$ in equations (4.8) with k replaced by ik, and the kernels $H_{ik}(t,\tau)$ are defined like the kernels $H_k(t,\tau)$ in equations (4.3) with k replaced by ik.

The kernels K_k and H_k are weakly singular and can be written in the form [30]

$$\begin{aligned} K_k(t,\tau) &= K_{k,1}(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right)+K_{k,2}(t,\tau) \\ H_k(t,\tau) &= H_{k,1}(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right)+H_{k,2}(t,\tau) \end{aligned} \tag{4.12}$$

for 2π-periodic analytic functions $K_{k,1}, H_{k,1}$ and $K_{k,2}, H_{k,2}$. The main idea in the derivation of equations (4.12) is to decompose the fundamental solution $H_0^{(1)}(z)$ in the form $H_0^{(1)}(z) = J_0(z) + iY_0(z)$ and to use the fact that $J_0(z)$ and $Y_0(z) - \frac{2}{\pi}\ln\frac{z}{2}J_0(z)$ are analytic functions of z; similar decompositions are available for $H_1^{(1)}(z)$. The same splitting strategy, unfortunately, does not work in the case when the kernels involve the Hankel function $H_0^{(1)}(iz)$. Indeed, the modified Bessel function $I_0(z) = J_0(iz)$ grows exponentially, while M_{ik_1} actually *decays* exponentially, as $|x(t)-x(\tau)|$ increases, and thus the splitting strategy generally gives rise to significant cancellation errors if used throughout the integration domain. In order to avoid subtraction of exponentially large quantities, we evaluate the operator S_{ik} by means of a slight modification of the approach used for the operator with kernels K_k and H_k: we use the truncated decomposition

$$\begin{aligned} M_{ik}(t,\tau) &= \chi(k|x(t)-x(\tau)|^4)\left\{\tilde{M}_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right)+\tilde{M}_2(t,\tau)\right\} \\ &+ (1-\chi(k|x(t)-x(\tau)|^4))M_{ik_1}(t,\tau) = M_{ik,1}(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right)+M_{ik,2}(t,\tau) \end{aligned} \tag{4.13}$$

where $\chi \in C_0^\infty(\mathbb{R})$ is a function such that $\chi(t) \equiv 1,\ |t| \leq 1/2$ and $\chi(t) \equiv 0,\ |t| \geq 1/2$. It can be checked easily that decomposition (4.13) does not suffer from cancellation errors and that the kernels $M_{ik,j}(t,\tau),\ j = 1,2$ in equation (4.13) are smooth (but not analytic) functions of t and τ. Applying the strategy outlined above for the case of the kernel M_{ik} to the kernels K_{ik} and H_{ik} leads to splittings of the form

$$K_{ik}(t,\tau) = K_{ik,1}(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right)+K_{ik,2}(t,\tau) \tag{4.14}$$

$$H_{ik}(t,\tau) = H_{ik,1}(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right)+H_{ik,2}(t,\tau) \tag{4.15}$$

for 2π-periodic smooth functions $K_{ik,1}, H_{ik,1}$ and $K_{ik,2}, H_{ik,2}$. For a given 2π periodic function ψ

we introduce the following operators $A_j^k, j=1,\ldots,4$, $A_j^{ik}, j=1,\ldots,6$, and A_7 by

$$
\begin{aligned}
(A_1^k\psi)(t) &= \int_0^{2\pi} H_{k,1}(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right)\psi(\tau)d\tau\\
(A_2^k\psi)(t) &= \int_0^{2\pi} H_{k,2}(t,\tau)\psi(\tau)d\tau\\
(A_1^{ik}\psi)(t) &= \int_0^{2\pi} H_{ik,1}(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right)\psi(\tau)d\tau\\
(A_2^{ik}\psi)(t) &= \int_0^{2\pi} H_{ik,2}(t,\tau)\psi(\tau)d\tau\\
(A_3^k\psi)(t) &= \int_0^{2\pi} K_{k,1}(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right)\psi(\tau)d\tau\\
(A_4^k\psi)(t) &= \int_0^{2\pi} K_{k,2}(t,\tau)\psi(\tau)d\tau\\
(A_3^{ik}\psi)(t) &= \int_0^{2\pi} K_{ik,1}(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right)\psi(\tau)d\tau\\
(A_4^{ik}\psi)(t) &= \int_0^{2\pi} K_{ik,2}(t,\tau)\psi(\tau)d\tau\\
(A_5^{ik}\psi)(t) &= \int_0^{2\pi} M_{ik,1}(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right)\psi(\tau)d\tau\\
(A_6^{ik}\psi)(t) &= \int_0^{2\pi} M_{ik,2}(t,\tau)\psi(\tau)d\tau\\
(A_7\psi)(t) &= \left(\frac{1}{2}+\frac{i\eta}{4}\right)|x'(t)|\psi(t)
\end{aligned}
\tag{4.16}
$$

so that the integral equation (4.2) can be written in the form

$$
\begin{aligned}
A_7\mu + (A_1^k + A_2^k)\mu &+ [(A_3^k + A_4^k)(A_5^{ik}+A_6^{ik})]\mu - [(A_3^{ik}+A_4^{ik})(A_5^{ik}+A_6^{ik})]\mu\\
&- i\eta[(A_1^{ik}+A_2^{ik})(A_1^{ik}+A_2^{ik})]\mu = f.
\end{aligned}
\tag{4.17}
$$

Owing to the weakly singular nature of their kernels, the operators $A_j^k, j=1,\ldots,4$ and the operators $A_j^{ik}, j=1,\ldots,6$ map the Sobolev spaces $H^p[0,2\pi]$ to the Sobolev spaces $H^{p+1}[0,2\pi]$ [31], where $H^p[0,2\pi]$ is the Sobolev space of periodic functions on the interval $[0,2\pi]$ defined in [31].

We describe next a Nyström method based on trigonometric interpolation that follows closely the quadrature method introduced by Kress in [30], which in turn relies on the logarithmic quadrature methods introduced by Kussmaul [34] and Martensen [36]. We choose $n\in\mathbb{N}$ and the equidistant mesh $t_j^{(n)} = \frac{j\pi}{n},\ j=0,1,\ldots,2n-1$. With respect to these nodal points the interpolation problem in the space T_n of trigonometric polynomials of the form

$$v(t) = \sum_{m=0}^{n} a_m\cos mt + \sum_{m=1}^{n-1} b_m \sin mt$$

is uniquely solvable [31]. We denote by $P_n : C[0,2\pi]\to T_n$ the corresponding interpolation operator and we will use in the error analysis the estimate [31]

$$\|P_n g - g\|_q \le C n^{q-p}\|g\|_p,\ 0\le q\le p,\ \frac{1}{2} < p \tag{4.18}$$

which is valid for all $g \in H^p[0, 2\pi]$ and a constant C depending on p and q, where we denoted by $\|g\|_p$ the norm of g in the Sobolev space $H^p[0, 2\pi]$. We use the quadrature rules [30]

$$\begin{aligned}\int_0^{2\pi} \ln\left(4\sin^2\frac{t-\tau}{2}\right) f(\tau)d\tau &\approx \int_0^{2\pi} \ln\left(4\sin^2\frac{t-\tau}{2}\right)(P_n f)(\tau)d\tau \\ &= \sum_{j=0}^{2n-1} R_j^{(n)}(t) f(t_j^{(n)}) \end{aligned} \tag{4.19}$$

where the expressions $R_j^{(n)}(t)$ are given by

$$R_j^{(n)}(t) = -\frac{2\pi}{n}\sum_{m=1}^{n-1}\frac{1}{m}\cos m(t - t_j^{(n)}) - \frac{\pi}{n^2}\cos n(t - t_j^{(n)}).$$

We also use the trapezoidal rule

$$\int_0^{2\pi} f(\tau)d\tau \approx \int_0^{2\pi}(P_n f)(\tau)d\tau = \frac{\pi}{n}\sum_{j=0}^{2n-1} f(t_j^{(n)}). \tag{4.20}$$

We apply the quadrature rules (4.19) and (4.20) to derive an approximation of the integral equations (4.17). To this end, we define the numerical quadrature operators

$$\begin{aligned}(A_{1,n}^k\psi)(t) &= \int_0^{2\pi} \ln\left(4\sin^2\frac{t-\tau}{2}\right)(P_n H_{k,1}(t,\cdot)\psi)(\tau)d\tau = \sum_{j=0}^{2n-1} R_j^{(n)}(t) H_{k,1}(t, t_j^{(n)})\psi(t_j^{(n)}) \\ (A_{2,n}^k\psi)(t) &= \int_0^{2\pi}(P_n H_{2,k}(t,\cdot)\psi)(\tau)d\tau = \frac{\pi}{n}\sum_{j=0}^{2n-1} H_{2,k}(t, t_j^{(n)})\psi(t_j^{(n)}) \\ (A_{3,n}^k\varphi)(t) &= \int_0^{2\pi} \ln\left(4\sin^2\frac{t-\tau}{2}\right)(P_n K_{k,1}(t,\cdot)\varphi)(\tau)d\tau = \sum_{j=0}^{2n-1} R_j^{(n)}(t) K_{1,k}(t, t_j^{(n)})\varphi(t_j^{(n)}) \\ (A_{4,n}^k\varphi)(t) &= \int_0^{2\pi}(P_n K_{2,k}(t,\cdot)\varphi)(\tau)d\tau = \frac{\pi}{n}\sum_{j=0}^{2n-1} K_{2,k}(t, t_j^{(n)})\varphi(t_j^{(n)})\end{aligned}$$

and similarly we define the numerical quadrature operators $A_{j,n}^{ik}$, $j = 1, \ldots, 6$. With these notations, we derive the approximating equation to equation (4.17)

$$\begin{aligned}A_7\mu_n + (A_{1,n}^k + A_{2,n}^k)\mu_n &+ [(A_{3,n}^k + A_{4,n}^k)P_n(A_{5,n}^{ik} + A_{6,n}^{ik})]\mu_n - [(A_{3,n}^{ik} + A_{4,n}^{ik})P_n(A_{5,n}^{ik} + A_{6,n}^{ik})]\mu_n \\ &- i\eta[(A_{1,n}^{ik} + A_{2,n}^{ik})P_n(A_{1,n}^{ik} + A_{2,n}^{ik})]\mu_n = f\end{aligned} \tag{4.21}$$

which we solve for $\mu_n \in T_n$; we have used the fact that $P_n(fg) = P_n(fP_n g)$ in the derivation of equation (4.21). Finally, we collocate equation (4.21) with the interpolation operator P_n in order to arrive at an approximating equation which is reduced to solving a finite dimensional linear system. We obtain the following linear system

$$\begin{aligned}A_7\mu_n &+ P_n(A_{1,n}^k + A_{2,n}^k)\mu_n + P_n[(A_{3,n}^k + A_{4,n}^k)P_n(A_{5,n}^{ik} + A_{6,n}^{ik})]\mu_n \\ &- P_n[(A_{3,n}^{ik} + A_{4,n}^{ik})P_n(A_{5,n}^{ik} + A_{6,n}^{ik})]\mu_n - i\eta P_n[(A_{1,n}^{ik} + A_{2,n}^{ik})P_n(A_{1,n}^{ik} + A_{2,n}^{ik})]\mu_n = P_n f\end{aligned} \tag{4.22}$$

which we solve for $\mu_n \in H^p[0,2\pi],\ p > \frac{1}{2}$. Further, we can write the linear system (4.22) in explicit form

$$
\begin{aligned}
&\sum_{j=0}^{2n-1} \mu_n(t_j^{(n)}) \left\{ R_{|\ell-j|}^{(n)} H_{k,1}(t_\ell^{(n)}, t_j^{(n)}) + \frac{\pi}{n} H_{k,2}(t_\ell^{(n)}, t_j^{(n)}) \right\} \\
&+ \sum_{j=0}^{2n-1} \left(\sum_{m=0}^{2n-1} \mu_n(t_m^{(n)}) \left\{ R_{|j-m|}^{(n)} M_{ik,1}(t_j^{(n)}, t_m^{(n)}) + \frac{\pi}{n} M_{ik,2}(t_j^{(n)}, t_m^{(n)}) \right\} \right) \times \\
&\qquad \left\{ R_{|\ell-j|}^{(n)} K_{k,1}(t_\ell^{(n)}, t_j^{(n)}) + \frac{\pi}{n} K_{k,2}(t_\ell^{(n)}, t_j^{(n)}) \right\} \\
&- \sum_{j=0}^{2n-1} \left(\sum_{m=0}^{2n-1} \mu_n(t_m^{(n)}) \left\{ R_{|j-m|}^{(n)} M_{ik,1}(t_j^{(n)}, t_m^{(n)}) + \frac{\pi}{n} M_{ik,2}(t_j^{(n)}, t_m^{(n)}) \right\} \right) \times \\
&\qquad \left\{ R_{|\ell-j|}^{(n)} K_{ik,1}(t_\ell^{(n)}, t_j^{(n)}) + \frac{\pi}{n} K_{ik,2}(t_\ell^{(n)}, t_j^{(n)}) \right\} \\
&+ i\eta \sum_{j=0}^{2n-1} \left(\sum_{m=0}^{2n-1} \mu_n(t_m^{(n)}) \left\{ R_{|j-m|}^{(n)} H_{ik,1}(t_j^{(n)}, t_m^{(n)}) + \frac{\pi}{n} H_{ik,2}(t_j^{(n)}, t_m^{(n)}) \right\} \right) \times \\
&\qquad \left\{ R_{|\ell-j|}^{(n)} H_{ik,1}(t_\ell^{(n)}, t_j^{(n)}) + \frac{\pi}{n} H_{ik,2}(t_\ell^{(n)}, t_j^{(n)}) \right\} \\
&+ \left(\frac{1}{2} + \frac{i\eta}{4} \right) |x'(t_\ell^{(n)})| \psi(t_\ell^{(n)}) = |x'(t_\ell^{(n)})| g(x(t_\ell^{(n)})),\ \ell = 0, 1, \dots, 2n-1, \qquad (4.23)
\end{aligned}
$$

with $R_k^{(n)}(t_j) = R_{|k-j|}^{(n)}$ where

$$
R_j^{(n)} = -\frac{1}{n} \left\{ \sum_{m=1}^{n-1} \frac{1}{m} \cos\frac{mj\pi}{n} + \frac{(-1)^j}{2n} \right\},\ j = 0, 1, \dots, 2n-1.
$$

We will prove the convergence $\|\mu_n - \mu\|_p \to 0,\ n \to \infty$ for all $\frac{1}{2} < p \le 1$ where $\|\psi\|_p$ denotes the norm of ψ in $H^p[0,2\pi]$. To this end, we use several results established in [31] concerning operators of the kind

$$
(A\varphi)(t) = \int_0^{2\pi} \ln\left(4\sin^2\frac{t-\tau}{2}\right) K(t,\tau)\varphi(\tau)d\tau
$$

$$
(B\varphi)(t) = \int_0^{2\pi} H(t,\tau)\varphi(\tau)d\tau
$$

and their collocated versions

$$
P_n A_n \varphi = P_n \int_0^{2\pi} \ln\left(4\sin^2\frac{t-\tau}{2}\right) P_n(K(t,\cdot)\varphi)(\tau))d\tau
$$

$$
P_n B_n \varphi = P_n \int_0^{2\pi} P_n(H(t,\cdot)\varphi)(\tau))d\tau.
$$

Under the assumption that K and H are infinitely differentiable and 2π-periodic with respect to both variables, the following results holds

Lemma 4.1 *Under the assumption above on the kernels K and H, for all $\frac{1}{2} < p \le 1$ and all $\varphi \in H^p[0,2\pi]$ we have the estimate*

$$
\|P_n A_n \varphi - A\varphi\|_p \le C n^{-p} \|\varphi\|_p,\ \|P_n B_n \varphi - B\varphi\|_p \le C n^{-p} \|\varphi\|_p.
$$

Proof. It follows from the result in Theorem 12.18 in [31] (page 215) that for all $p > \frac{1}{2}$ and all $\varphi \in H^p[0, 2\pi]$ we have the estimate

$$\|A_n\varphi - A\varphi\|_1 \leq C_1 n^{-p}\|\varphi\|_p. \tag{4.24}$$

We have for all p such that $\frac{1}{2} < p \leq 1$

$$\begin{aligned} \|P_nA_n\varphi - A\varphi\|_p &\leq \|P_nA_n\varphi - P_nA\varphi\|_p + \|P_nA\varphi - A\varphi\|_p \leq C_2\|A_n\varphi - A\varphi\|_p + \|P_nA\varphi - A\varphi\|_p \\ &\leq C_2\|A_n\varphi - A\varphi\|_1 + \|P_nA\varphi - A\varphi\|_p \end{aligned}$$

Using estimate (4.24) to estimate the first term in the right-hand side of the equation above and estimate (4.18) for $q = p$ and $p = p+1$ to estimate the second term in the right-hand side of the equation above, we obtain

$$\|P_nA_n\varphi - A\varphi\|_p \leq C_1C_2n^{-p}\|\varphi\|_p + C_3n^{-1}\|A\varphi\|_{p+1} \leq Cn^{-p}\|\varphi\|_p \tag{4.25}$$

where we used the fact that $A : H^p[0, 2\pi] \to H^{p+1}[0, 2\pi]$ is bounded [31]. The result for the operators B follows similarly. ■

Based on this result we establish the following

Lemma 4.2 *Let A_1 and A_2 be operators of either of the type A or B described above. Then for all $\frac{1}{2} < p \leq 1$ and all $\varphi \in H^p[0, 2\pi]$ we have the estimate*

$$\|P_nA_{1,n}P_nA_{2,n}\varphi - A_1A_2\varphi\|_p \leq Cn^{-p}\|\varphi\|_p.$$

Proof. We start with the estimate

$$\|P_nA_{1,n}P_nA_{2,n}\varphi - A_1A_2\varphi\|_p \leq \|P_nA_{1,n}P_nA_{2,n}\varphi - A_1P_nA_{2,n}\varphi\|_p + \|A_1P_nA_{2,n}\varphi - A_1A_2\varphi\|_p.$$

We estimate the second term in the right-hand side of the equation above using the result in Lemma 4.1.

$$\|A_1P_nA_{2,n}\varphi - A_1A_2\varphi\|_p \leq \|A_1\|_p\|P_nA_{2,n}\varphi - A_2\varphi\|_p \leq C_1n^{-p}\|\varphi\|_p.$$

A direct consequence of the result in Lemma 4.1 is that

$$\|P_nA_{2,n}\varphi\|_p \leq C\|\varphi\|_p.$$

From this estimate and one more application of Lemma 4.1 we obtain

$$\|P_nA_{1,n}P_nA_{2,n}\varphi - A_1P_nA_{2,n}\varphi\|_p \leq C_2n^{-p}\|P_nA_{2,n}\varphi\|_p \leq Cn^{-p}\|\varphi\|_p$$

from which the result of the Lemma follows. ■

We denote by $\mathcal{A}$ the continuous operator on the left-hand side of equation (4.1) and by $\mathcal{A}_n$ the discrete operator on the left-hand side of equation (4.22). We view both operators as operators from $H^p[0, 2\pi]$ to $H^p[0, 2\pi]$ for $\frac{1}{2} < p \leq 1$. We note that in the case of analytic curves Γ, the solution μ of equation (4.1) is itself analytic and thus it belongs to all of the Sobolev spaces $H^p[0, 2\pi]$ for all

$p > \frac{1}{2}$. It follows from the results in Lemma 4.1 and Lemma 4.2 that there exists a constant $C > 0$ such that

$$\|\mathcal{A}\psi - \mathcal{A}_n\psi\|_p \leq cn^{-p}\|\psi\|_p \tag{4.26}$$

for all $\psi \in H^p[0, 2\pi]$, $\frac{1}{2} < p \leq 1$. We formulate the convergence result of the solutions μ_n of equation (4.22) to the solution μ of equation (4.1).

Theorem 4.3 *For $\frac{1}{2} < p \leq 1$, if $f \in H^{2p}[0, 2\pi]$, then the approximating equation* (4.22) *has a unique solution μ_n for sufficiently large n and we have the following error estimate*

$$\|\mu_n - \mu\|_p \leq Cn^{-p}(\|f\|_{2p} + \|\mu\|_p)$$

for some constant $C = C(p)$. In the case when the boundary Γ is analytic and f is analytic, we have convergence of μ_n to μ in $H^p[0, 2\pi]$ of the order $\mathcal{O}(n^{-m})$ for all $m \in \mathbb{N}$.

Proof. We have that $\mathcal{A} : H^p[0, 2\pi] \to H^p[0, 2\pi]$ has a bounded inverse, and since by the estimates established in equation (4.26) we also have that the operators $\mathcal{A}_n$ converge in the p-norm to the operator $\mathcal{A}$. It follows from Neumann series considerations (see Theorem 10.1 in [31]) that the operators $\mathcal{A}_n$ are invertible for large enough n and the inverse operators $\mathcal{A}_n^{-1}$ are uniformly bounded. We write

$$\mu_n - \mu = \mathcal{A}_n^{-1}\{(P_n f - f) + (\mathcal{A} - \mathcal{A}_n)\mu\}$$

from which it follows that

$$\|\mu_n - \mu\|_p \leq C(\|P_n f - f\|_p + \|\mathcal{A}\mu - \mathcal{A}_n\mu\|_p).$$

Using estimates (4.18) and (4.26) together with the estimate above, we obtain the first result of the Theorem.

In the case when Γ is analytic and the incident field is analytic (and thus f is analytic), the solution μ of the integral equation (4.1) is itself analytic, and in particular infinitely differentiable. In the case of an infinitely differentiable function f, the estimate $\|P_n f - f\|_p$ can be improved to order $\mathcal{O}(n^{-m})$ for all $m \in \mathbb{N}$. This estimate leads in turn to the same improved order of convergence in the estimate in Theorem 12.18 in [31]. Based on this fact, and since the kernels $K(\cdot, \cdot)$ and $H(\cdot, \cdot)$ of the integral operators of the type A and B described above that enter the integral equation (4.17) are infinitely differentiable (however the integral operators which involve purely imaginary wavenumbers are not analytic because of the use of the cutoff function χ), we obtain the second result of the Theorem. ■

Remark 4.4 *The constants C that appear in Theorem 4.3 depend on $\|\mathcal{A}\|_{H^p(\Gamma)\to H^p(\Gamma)}\|\mathcal{A}^{-1}\|_{H^p(\Gamma)\to H^p(\Gamma)}$. According to the estimates established in Sections 3.1 and 3.2, in the case when $\Gamma = \mathbb{S}^1$, the latter quantities grow no faster than $\tilde{C}k^{\frac{1}{3}}$ if the coupling coupling constant η is such that $\eta \sim k^{\frac{1}{3}}$. Furthermore, our numerical evidence suggests that in the case of starlike boundaries Γ the condition numbers of the operators $\mathcal{A}$ can grow at most linearly with $k^{1/2}$ for a wide range of values of η.*

Remark 4.5 *We note that since $p > \frac{1}{2}$, the result established in Theorem 4.3 together with the observations above imply uniform convergence of μ_n to μ with superalgebraic order of convergence.*

4.2 Discretization of the ICFIER-RPS equations (2.18)

Assuming that the boundary curve Γ is analytic and given by the parametrization, equation (2.18) can be written in parametric form as

$$\begin{aligned}-\frac{i\eta}{4\pi}\int_0^{2\pi}\cot\frac{\tau-t}{2}(PS_{ik}\psi)'(\tau)d\tau \quad &+ \quad \int_0^{2\pi}K_k(t,\tau)(PS_{ik}\psi)(\tau)d\tau \\ &+ \quad \int_0^{2\pi}H_k(t,\tau)\psi(\tau)d\tau+\frac{\psi(t)\ |x'(t)|}{2}=f(t),\ 0\le t\le 2\pi\end{aligned} \tag{4.27}$$

where $\psi(t)\equiv\phi(x(t))$ and $f(t)=|x'(t)|g(x(t))$ with $g=\frac{\partial u^{\rm inc}}{\partial\mathbf{n}}|_\Gamma$, and the operators H_k and K_k were defined in equations (4.3) and (4.4) respectively. We introduce the following operators T_0 and A_5 by

$$\begin{aligned}(T_0\psi)(t) &= -\frac{1}{4\pi}\int_0^{2\pi}\cot\frac{\tau-t}{2}\psi'(\tau)d\tau\\ (A_5\psi)(t) &= \frac{|x'(t)|}{2}\psi(t)\end{aligned} \tag{4.28}$$

so that the integral equation (4.27) can be written in the form

$$i\eta A_0\psi+(A_{k,1}+A_{k,2})\psi+[(A_{k,3}+A_{k,4})PS_{ik}]\psi+A_5\psi=f, \tag{4.29}$$

where $A_0=T_0\circ PS_{ik}$. The operator PS_{ik} has the following mapping property: $PS_{ik}:H^p[0,2\pi]\to H^{p+1}[0,2\pi]$ which follows immediately from its spectral representation (2.17). The mapping properties of the operator A_0 can be also derived based on its spectral representation. Indeed, given that ψ is a periodic function on the interval $[0,2\pi]$ and hence the function $\psi(t)|x'(t)|$ can be expanded in Fourier series $\psi(t)|x'(t)|=\sum_{n\in\mathbb{Z}}\psi_n e^{int}$, the spectral representation of the operator A_0 is given by

$$(A_0\psi)(t)=\sum_{n\in\mathbb{Z}}\frac{|n|}{4(n^2+k^2)^{\frac12}}\psi_n e^{int} \tag{4.30}$$

where we have use the fact that $T_0u_n=\frac{|n|}{2}u_n,\ n\in\mathbb{Z}$ with $u_n(t)=e^{int}$ [31]. Equation (4.30) can be written as

$$(A_0\psi)(t)=\frac{\psi(t)|x'(t)|}{4}+(\tilde{A}_0\psi)(t)$$

where the operator $\tilde{A}_0$ is defined as

$$(\tilde{A}_0\psi)(t)=-\frac{k^2}{4}\sum_{n\in\mathbb{Z}}\frac{\psi_n}{n^2+k^2+|n|(n^2+k^2)^{\frac12}}e^{int}=-\frac{k^2}{4}\sum_{n\in\mathbb{Z}}\zeta_n(k)\psi_n e^{int} \tag{4.31}$$

and hence $\tilde{A}_0:H^p[0,2\pi]\to H^{p+2}[0,2\pi]$. With these notations, the integral equation (4.29) becomes

$$i\eta\tilde{A}_0\psi+(A_{k,1}+A_{k,2})\psi+[(A_{k,3}+A_{k,4})PS_{ik}]\psi+A_7\psi=f. \tag{4.32}$$

In order to discretize the operators PS_{ik} and $\tilde{A}_0$ we use the following interpolatory approximations

$$\begin{aligned}
(PS_{ik}\psi)(t) &\approx (PS_{ik}P_n\psi)(t) = ((PS)_n\psi)(t) = \sum_{m=0}^{n} \frac{\psi_m^{(1)}}{(m^2+k^2)^{\frac{1}{2}}}\cos mt + \sum_{m=1}^{n} \frac{\psi_m^{(2)}}{(m^2+k^2)^{\frac{1}{2}}}\sin mt \\
(\tilde{A}_0\psi)(t) &\approx (\tilde{A}_0P_n\psi)(t) = (\tilde{A}_{0,n}\psi)(t) = \sum_{m=0}^{n} \psi_m^{(1)}\zeta_m(k)\cos mt + \sum_{m=1}^{n} \psi_m^{(2)}\zeta_m(k)\sin mt \qquad (4.33)
\end{aligned}$$

where $(P_n\psi)(t) = \sum_{m=0}^{n} \psi_m^{(1)}\cos mt + \sum_{m=1}^{n-1} \psi_m^{(2)}\sin mt$.

We apply the quadrature rules (4.19) and (4.20) together with the approximations (4.33) to derive an approximation of the integral equations (4.32). We use the fact that $P_n(PS)_n = (PS)_n$ and we derive the approximating equation to equation (4.32)

$$i\eta\tilde{A}_{0,n}\psi_n + (A_{1,n}^k + A_{2,n}^k)\psi_n + (A_{3,n}^k + A_{4,n}^k)((PS)_n\psi_n) + A_7\psi_n = f \qquad (4.34)$$

which we solve for $\psi_n \in T_n$. Finally, we collocate equation (4.34) with the interpolation operator P_n in order to arrive at an approximating equation which is reduced to solving a finite dimensional linear system. We obtain

$$i\eta\tilde{A}_{0,n}\psi_n + P_n(A_{1,n}^k + A_{2,n}^k)\psi_n + P_n(A_{3,n}^k + A_{4,n}^k)((PS)_n\psi_n) + A_7\psi_n = P_nf \qquad (4.35)$$

for $\psi_n \in H^p[0,2\pi]$. We used the fact that $P_n\tilde{A}_{0,n} = \tilde{A}_{0,n}$ in order to derive equations (4.35). We write the linear system (4.35) in the form $(PSA)_n\psi_n = f_n$ and we solve this linear system using Krylov subspace iterative solvers (e.g. GMRES). Thus, it suffices to describe the matrix-vector product $(PSA)_n\psi_n$. The components of the latter matrix-vector product are given explicitly by

$$\begin{aligned}
((PSA)_n\psi_n)_\ell &= \sum_{j=0}^{2n-1} \psi_n(t_j^{(n)})\left\{R_{|\ell-j|}^{(n)}H_1(t_\ell^{(n)},t_j^{(n)}) + \frac{\pi}{n}H_2(t_\ell^{(n)},t_j^{(n)})\right\} \\
&+ \sum_{j=0}^{2n-1} \left\{R_{|\ell-j|}^{(n)}K_1(t_\ell^{(n)},t_j^{(n)}) + \frac{\pi}{n}K_2(t_\ell^{(n)},t_j^{(n)})\right\} \times \\
&\quad \mathcal{F}_{2n}^{-1}\{\sigma(k,2n).*\mathcal{F}_{2n}\{\psi_n\}\} + i\eta\mathcal{F}_{2n}^{-1}\{\zeta(k,2n).*\mathcal{F}_{2n}\{\psi_n\}\} \\
&+ \left(\frac{1}{2} + \frac{i\eta}{4}\right)|x'(t_\ell^{(n)})|)\psi_n(t_\ell^{(n)}),\ \ell = 0,1,\ldots,2n-1 \qquad (4.36)
\end{aligned}$$

in terms of the discrete direct and indirect discrete Fourier transforms using $2n$ nodes, where we have used the MATLAB notation .* for multiplication in the Fourier space. We effect in practice the discrete Fourier transforms $\mathcal{F}_{2n}$ and $\mathcal{F}_{2n}^{-1}$ via the Fast Fourier Transform (FFT).

We denote by $\mathcal{B}$ the continuous operator on the left-hand side of equation (4.27) and by $\mathcal{B}_n$ the discrete operator on the left-hand side of equation (4.35). We view both operators as operators from $H^p[0,2\pi]$ to $H^p[0,2\pi]$ for $\frac{1}{2} < p \leq 1$. We note that in the case of analytic curves Γ, the solution ψ of equation (4.27) is itself analytic and thus it belongs to all of the Sobolev spaces $H^p[0,2\pi]$ for all $p \geq 0$. It follows immediately from their definition that

$$\|(PS)_n\psi - PS_{ik}\psi\|_p \leq Cn^{-1}\|\psi\|_p,\ \|\tilde{A}_{0,n}\psi - \tilde{A}_0\psi\|_p \leq Cn^{-2}\|\psi\|_p,$$

and thus the results in Lemma 4.2 can be extended to derive similar bounds on the differences between the compositions of discrete and continuous operators that enter equation (4.35). Using this fact and the results in Lemma 4.1, we obtain that there exists a constant $C > 0$ such that

$$\|\mathcal{B}\psi - \mathcal{B}_n\psi\|_p \leq cn^{-p}\|\psi\|_p \tag{4.37}$$

for all $\psi \in H^p[0, 2\pi]$, $\frac{1}{2} < p \leq 1$. We formulate the convergence result of the solutions ψ_n of equation (4.35) to the solution ψ of equation (4.27). The proof of this result follows the lines of the proof of Theorem 4.3. In the case when the boundary Γ is analytic, the estimate in Theorem 12.18 in [31], which is the cornerstone of the error analysis presented above, can be improved to order $\mathcal{O}(e^{-ns})$ for some positive constant s. Since the kernels $K(\cdot, \cdot)$ and $H(\cdot, \cdot)$ of the integral operators of the type A and B described above that enter the integral equation (4.27) are analytic, we get the following

Theorem 4.6 *For $\frac{1}{2} < p \leq 1$, if $f \in H^{2p}[0, 2\pi]$, then the approximating equation* (4.35) *has a unique solution ψ_n for sufficiently large n and we have the following error estimate*

$$\|\psi_n - \psi\|_p \leq Cn^{-p}(\|f\|_{2p} + \|\psi\|_p)$$

for some constant $C = C(p)$, where ψ is the solution of equation (4.27)*. In the case when the boundary Γ is analytic and the function f is analytic, we can deduce convergence of the order $\mathcal{O}(e^{-ns})$ for some positive constant s in the estimate above.*

Remark 4.7 *We note that since $p > \frac{1}{2}$, the result established in Theorem 4.6 together with the observations above imply uniform convergence of ψ_n to ψ with exponential order of convergence.*

5 Numerical Results

We present in this section a variety of numerical results that demonstrate the properties of the regularized combined field integral equations (2.16) and (2.18) constructed in the previous sections. Solutions of the linear systems arising from the Nyström discretizations of both types of Regularized Combined Field Integral Equations according to the description in Section 4 are obtained by means of the fully complex version of the iterative solver GMRES [41]. We used the coupling constant $\eta = 2$ in both equations (2.16) and (2.18); our extensive numerical experiments suggest that this value of the coupling constant η leads to nearly optimal numbers of GMRES iterations to reach desired (small) GMRES relative residuals. The results presented in the Tables 1–3 in this section were obtained by prescribing a GMRES relative residual tolerances equal to 10^{-16} in the case of circular geometries (for which exact solutions are available) or equal to either 10^{-8} or 10^{-12} otherwise.

We present scattering experiments concerning the following smooth geometries: (a) a unit circle, (b) an object with five petals whose parametrization is given in polar coordinates as $x_1(t) = r(t)\cos t$, $x_2(t) = r(t)\sin t$ with $r(t) = 1 + 0.3\cos 5t$, and (c) a cavity whose parametrization is given by $x(t) = (x_1(t), x_2(t))$, $x_1(t) = (\cos t + 2\cos 2t)/2.5$, $x_2(t) = Y(t)/2 - Y_s(t)/48$, where $Y(t) = \sin t + \sin 2t + 1/2\sin 3t$, $Y_s(t) = -4\sin t + 7\sin 2t - 6\sin 3t + 2\sin 4t$, see Figure 2. We note that each of these geometries has a diameter equals to 2.

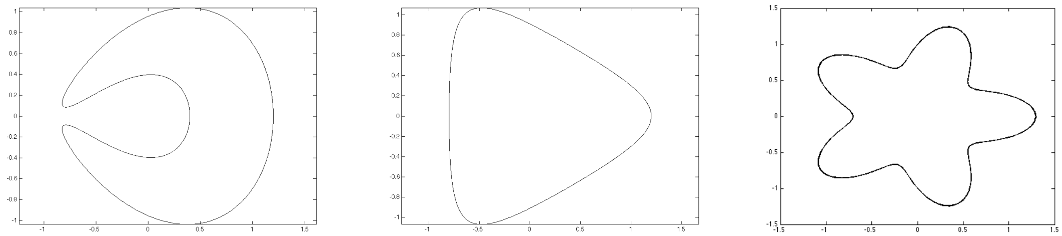

Figure 2: Cavity geometry (left), rounded triangle (center), and five petal (right).

For every scattering experiment we consider plane-wave incidence u^{inc} and we present maximum far-field errors, that is we choose sufficiently many directions $\frac{\mathbf{x}}{|\mathbf{x}|}$ and for each direction we compute the far-field amplitude $u^s_\infty(\hat{\mathbf{x}})$ defined as

$$u^s(\mathbf{x}) = \frac{e^{ik|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}}\left(u^s_\infty(\hat{\mathbf{x}}) + \mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right)\right), \ |\mathbf{x}| \to \infty. \tag{5.1}$$

The maximum far-field errors were evaluated through comparisons of the numerical solutions $u^{s,\mathrm{calc}}_\infty$ corresponding to either formulation (2.16) and (2.18) with reference solutions $u^{s,\mathrm{ref}}_\infty$ by means of the relation

$$\varepsilon_\infty = \max|u^{s,\mathrm{calc}}_\infty(\hat{\mathbf{x}}) - u^{s,\mathrm{ref}}_\infty(\hat{\mathbf{x}})| \tag{5.2}$$

The latter solutions $u^{s,\mathrm{ref}}_\infty$ were produced using either exact Mie-series solutions in the case of circular geometries or solutions corresponding with refined discretizations (more precisely discretizations that use 12 points per wavelength) based on the formulation ICFIE-R (2.16) with GMRES residuals of 10^{-12} for all other geometries. Besides far field errors, Tables 1–3 display the numbers of iterations required by the GMRES solver to reach a relative residual of 10^{-16} in the case of circular geometries and of 10^{-8} for the other two geometries. We used discretizations ranging from 4 to 10 discretization points per wavelength, for frequencies in the medium to the high-frequency range, i.e. $k = 2^i, i = 3, \ldots, 9$ corresponding to acoustic scattering problems of sizes ranging from 2.5λ to 163.2λ. The columns "Unknowns" in all Tables 1–3 display the numbers of unknowns used in each case, which equal to the value $2n$ defined in Section 4. In all of the scattering experiments we considered plane-wave incident fields; in the case of the cavity the direction of the incident field was taken to be equal to $d = (1, 0)$ in order to generate multiple reflections; in the other two cases we considered plane-wave incident fields of direction $d = (0, -1)$. As it can be seen in Tables 1–3, our solvers converge with high-order, as predicted by the error analysis in Section 4.

We conclude from the results displayed in Tables 1–3 that the formulations ICFIE-R (2.16) and ICFIE-RPS (2.18) lead to small numbers of GMRES residuals to reach desired tolerances, the iteration count being slightly more favorable in the case of the ICFIE-R formulation (2.16). However, for the same discretization, our implementation of a matrix-vector product corresponding to formulation ICFIE-RPS (2.18) is about 1.28 less expensive than a matrix-vector product corresponding to formulation ICFIE-R (2.16). For instance, using 512 unknowns in the case of the five-petal geometry, our MATLAB implementation of a matrix-vector product corresponding to the formulation ICFIE-R (2.18) took 43.68 seconds whereas a matrix-vector product corresponding to the formulation ICFIE-RPS (2.16) took 56.19 seconds on a MacPro machine with $2 \times$ 3GHz Quad-core Intel Xeon. We also remark that the number of GMRES iterations varies modestly with

Table 1: Scattering experiments for the unit circle, plane wave incidence, maximum errors in the far-field, and numbers of iterations needed to reach GMRES relative residuals equal to 10^{-16}

Size	Unknowns	ICFIE − RPS		ICFIE − R	
		Iter.	ϵ_∞	Iter.	ϵ_∞
2.5λ	32	18	3.5×10^{-5}	18	1.4×10^{-3}
2.5λ	48	18	3.4×10^{-9}	18	3.4×10^{-9}
2.5λ	64	18	5.0×10^{-15}	18	3.5×10^{-15}
5.1λ	64	26	5.1×10^{-4}	29	5.0×10^{-4}
5.1λ	96	26	2.7×10^{-14}	27	2.6×10^{-14}
10.2λ	128	39	2.0×10^{-4}	34	1.8×10^{-4}
10.2λ	160	39	1.2×10^{-14}	33	8.4×10^{-15}
20.4λ	384	51	6.9×10^{-5}	39	6.8×10^{-5}
20.4λ	640	51	1.4×10^{-14}	38	1.0×10^{-14}
40.8λ	768	71	2.5×10^{-5}	47	2.4×10^{-5}
40.8λ	1280	71	5.1×10^{-14}	46	3.5×10^{-14}
81.6λ	768	91	6.9×10^{-6}	61	8.8×10^{-6}
81.6λ	1280	91	7.2×10^{-14}	61	6.4×10^{-14}
163.2λ	1536	105	3.3×10^{-6}	69	3.1×10^{-6}
163.2λ	2560	105	2.8×10^{-13}	67	1.5×10^{-13}

Table 2: Scattering experiments for the five petal object, plane wave incidence, maximum errors in the far-field, and numbers of iterations needed to reach GMRES relative residuals equal to 10^{-8}

Size	Unknowns	ICFIE − RPS		ICFIE − R	
		Iter.	ϵ_∞	Iter.	ϵ_∞
2.5λ	48	19	3.5×10^{-5}	19	3.6×10^{-5}
2.5λ	64	19	1.2×10^{-8}	19	1.3×10^{-8}
5.1λ	96	21	3.8×10^{-5}	21	3.8×10^{-5}
5.1λ	128	21	1.5×10^{-7}	21	1.4×10^{-7}
10.2λ	192	24	4.0×10^{-6}	22	4.0×10^{-6}
10.2λ	256	24	1.5×10^{-8}	22	1.1×10^{-8}
20.4λ	384	27	1.1×10^{-3}	23	1.1×10^{-3}
20.4λ	512	27	1.4×10^{-8}	23	1.0×10^{-8}
40.8λ	768	30	1.3×10^{-3}	25	1.5×10^{-3}
40.8λ	1024	30	1.4×10^{-8}	25	2.2×10^{-8}
81.6λ	1536	37	1.1×10^{-3}	30	2.6×10^{-3}
81.6λ	2048	37	1.6×10^{-8}	30	1.7×10^{-8}
163.2λ	3072	45	2.9×10^{-3}	36	3.7×10^{-3}
163.2λ	4096	45	1.8×10^{-8}	35	2.5×10^{-8}

Table 3: Scattering experiments for the cavity geometry, plane wave incidence, maximum errors in the far-field, and numbers of iterations needed to reach GMRES relative residuals equal to 10^{-8}

Size	Unknowns	ICFIE − RPS		ICFIE − R	
		Iter.	ϵ_∞	Iter.	ϵ_∞
2.5λ	64	21	8.7×10^{-4}	22	9.2×10^{-4}
2.5λ	96	21	1.6×10^{-7}	22	5.6×10^{-8}
5.1λ	128	25	3.9×10^{-4}	26	3.4×10^{-4}
5.1λ	160	25	3.6×10^{-7}	26	3.1×10^{-7}
10.2λ	256	35	2.1×10^{-4}	35	1.6×10^{-4}
10.2λ	320	35	1.8×10^{-8}	35	1.6×10^{-8}
20.4λ	512	52	2.0×10^{-4}	50	1.4×10^{-4}
20.4λ	640	52	1.9×10^{-8}	50	1.2×10^{-8}
40.8λ	1024	85	2.0×10^{-4}	82	1.7×10^{-4}
40.8λ	1280	85	1.8×10^{-8}	82	1.7×10^{-8}
$81.6\ \lambda$	2034	126	1.4×10^{-4}	119	1.2×10^{-4}
$81.6\ \lambda$	2560	126	2.2×10^{-8}	119	2.3×10^{-8}
$163.2\ \lambda$	4096	233	2.2×10^{-4}	207	1.9×10^{-4}
$163.2\ \lambda$	5120	233	3.8×10^{-8}	207	3.7×10^{-8}

the frequency in the case of starlike geometries (Tables 1 and 2) but it tends to increase almost linearly with the frequency in the case of the cavity geometry in Table 3. We mention that the use of other coupling parameters such as $\eta = 4$ or $\eta = k^{\frac{1}{3}}$ leads to larger iteration counts; in the former case the numbers of GMRES iterations are only slightly larger, while in the latter case the numbers of GMRES iterations can be about two times larger than those presented in Tables 1–3 to reach the same GMRES residuals.

We use next the high-order discretizations of the ICFIE-R operators defined in equations (2.16) to investigate numerically their coercivity in the case of starlike smooth domains.

5.1 Numerical evidence of coercivity of the ICFIE-R operators for starlike scatterers

We provide in this section numerical evidence of the coercivity of the ICFIE-R operators $\mathcal{A}_{k,S_{ik}}$ defined in equation (2.16) for starlike geometries. For definiteness, we say that Γ is starlike with respect to the origin O if $\text{ess inf}_{x\in\Gamma}\ x \cdot n(x) > 0$, where $n(x)$ denotes the normal to the curve Γ at the point x [16]. Since the boundary integral operators $\mathcal{A}_{k,S_{ik}}$ are not normal (a proof of this fact is missing in the literature) except for the case of circular geometries, their coercivity cannot be inferred from information on their spectra alone. Instead, and just as in [11], we investigate numerically the coercivity of the operators $\mathcal{A}_{k,S_{ik}}$ through computation of their numerical ranges. For an operator $T : H \to H$, where H is a Hilbert space, the numerical range of T is defined as the following (convex) set

$$W(T) = \left\{ \frac{\langle Tv, v\rangle}{\langle v, v\rangle},\ v \in H,\ v \neq 0 \right\}. \tag{5.3}$$

If $0 \notin W(T)$, it follows that the operator T is coercive with the coercivity constant γ equal to the distance from the origin to the set $W(T)$. We investigate the coercivity of the operators $\mathcal{A}_{k,S_{ik}}$ for various geometries and wavenumbers via their numerical ranges. Specifically, for each test geometry, we considered four wavenumbers $k = 16, 32, 64, 128$ and we computed approximations of the numerical ranges of the operators $\mathcal{A}_{k,S_{ik}}$ by using the high-order Nyström discretization described in Section 4 using $N = [6k]$ discretization points for each wavenumber k. Specifically, we computed numerical ranges of the discrete operators by building explicitly the matrices corresponding to the discrete operators. Given the very high-order convergence of the Nyström discretization presented in Section 4, the numerical ranges of the discrete operators approximate very accurately the numerical ranges of the continuous operators. For the computation of the numerical ranges of the discrete operators we relied on an a well-known algorithm [24], which for the sake of completeness we describe next (we note that the same algorithm has been used in [11]). Given the matrix $A_N \in \mathbb{C}^{N\times N}$ corresponding to the Nyström discretization with N points of the operators $\mathcal{A}_{k,S_{ik}}$, and for each direction $\theta_j = j\pi/P,\ j = 0, 1, \ldots, P-1$ we computed the Hermitian part of the matrix $e^{i\theta_j} A_N$, that is $A^j_{N,H} = \frac{1}{2}(e^{i\theta_j} A_N + e^{-i\theta_j} A^*_N)$ and the eigenvectors $v^j_{\min}$ and $v^j_{\max}$ corresponding to the smallest and respectively largest eigenvalues of the matrix $A^j_{N,H}$. For each of the vectors $v^j_{\min}$ and $v^j_{\max}$ we computed the complex quantities

$$w^j_{\min} = \frac{\langle A_N v^j_{\min}, v^j_{\min}\rangle}{\langle v^j_{\min}, v^j_{\min}\rangle}, \qquad w^j_{\max} = \frac{\langle A_N v^j_{\max}, v^j_{\max}\rangle}{\langle v^j_{\max}, v^j_{\max}\rangle}$$

and then we computed the numerical range of A_N as the convex hull of the pairs $(w^j_{\min}, w^j_{\max}),\ j = 0, 1, \ldots, P-1$. For each geometry and wavenumber k considered, we took $P = 100$ in the algorithm described above to compute the numerical ranges.

First, we present numerical ranges in the case of two strictly convex geometries: (a) a rounded triangle whose parametrization is given by $x(t) = (\cos t + 0.2 \cos 2t, \sin t - 0.2 \sin 2t)$ —see Figure 2, and (b) an ellipse with semi-axes $a = 1$ and $b = 0.25$. We present in Figure 3 and Figure 4 computed numerical ranges for these two geometries and four wavenumbers together with the spectra of the operators $\mathcal{A}_{k,S_{ik}}$. For these geometries, and for certain choices of the coupling parameter η the operators $\mathcal{A}_{k,S_{ik}}$ are coercive, just as it was proved to be the case for circular geometries. Specifically, in the case $\eta = 4$ (which is used in Figure 3 and Figure 4) and $\eta = k^{1/3}$ the operators $\mathcal{A}_{k,S_{ik}}$ are coercive since the origin $(0, 0)$ is outside their numerical ranges. Furthermore, we computed approximations of the coercivity constants in each of the cases above. We computed numerical ranges for two discretizations: one corresponding to six discretization points per wavelength (used in Figure 3 and Figure 4, i.e. $N = [6k]$) and one corresponding to twelve discretization points per wavelength (i.e. $N = [12k]$) and the corresponding coercivity constants were computed as the distances from the origin to the numerical ranges. We note that the coercivity constants computed using the coarser discretization have three exact digits when compared against those corresponding to the finer discretizations. Specifically, in the case of the triangular geometry and coupling constant $\eta = 4$ we obtained the following values for the coercivity constants $\gamma \approx 0.450,\ 0.416,\ 0.387,\ 0.351$ for $k = 16, 32, 64$ and 128 respectively; in the case of the coupling constant $\eta = k^{1/3}$ we obtained the following values for the coercivity constants $\gamma \approx 0.493,\ 0.486,\ 0.479,\ 0.475$ for $k = 16, 32, 64$ and 128 respectively. In the case of the elliptical geometry and coupling constant $\eta = 4$ we obtained the following values for the coercivity constants $\gamma \approx 0.439,\ 0.393,\ 0.387,\ 0.367$ for $k = 16, 32, 64$ and 128 respectively; in the case of coupling constant $\eta = k^{1/3}$ we obtained the following values for the coercivity constants $\gamma \approx 0.483,\ 0.478,\ 0.483,\ 0.487$ for $k = 16, 32, 64$ and 128 respectively.

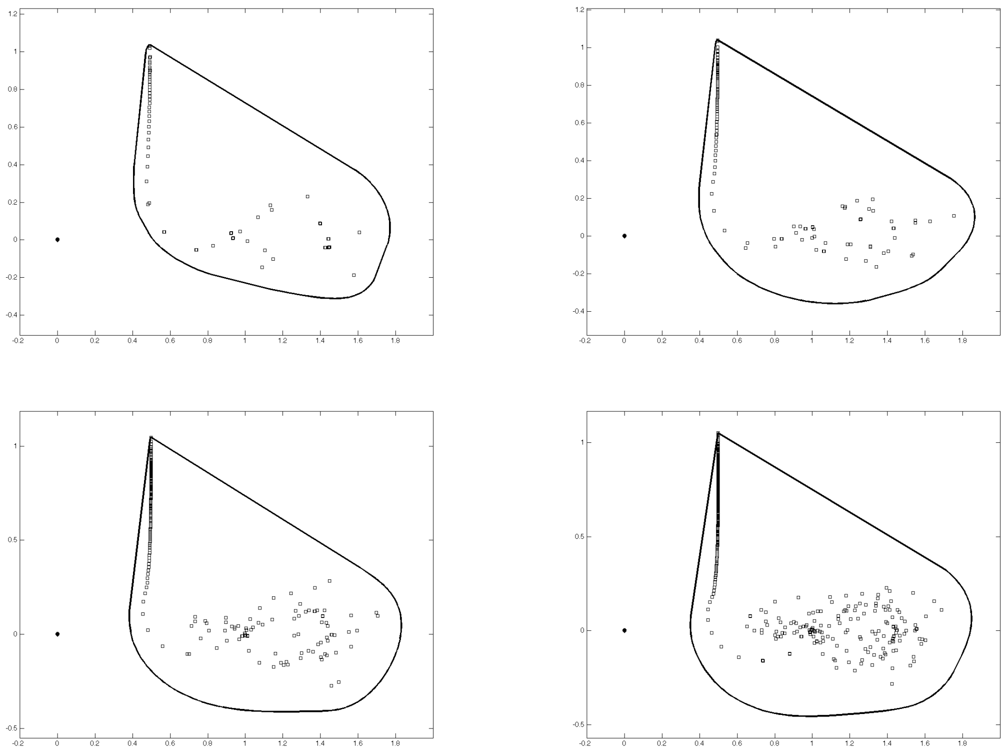

Figure 3: Computed numerical ranges and spectra of the operators $\mathcal{A}_{k,S_{ik}}$ with $\eta = 4$ for the triangle geometry for four wavenumbers $k = 16, 32, 64, 128$ pictured top left, top right, bottom left and bottom right respectively. The numerical ranges are the bounded domains enclosed by the closed curves in the figures; for reference, we also displayed the origin which is not contained in any of these domains. The coercivity constants are $\gamma \approx 0.4507,\ 0.4163,\ 0.3877,\ 0.3510$ for $k = 16, 32, 64$ and 128 respectively.

We observe from these numerical experiments that for the choice of a coupling constant $\eta = k^{\frac{1}{3}}$ the coercivity constants associated with the operators $\mathcal{A}_{k,S_{ik}}$ appear to be independent of the wavenumber k for strictly convex geometries.

For certain choices of the coupling parameter η, the numerical evidence presented in Figure 5 suggests that the operators $\mathcal{A}_{k,S_{ik}}$ are also coercive in the case of non-convex but starlike geometries such as the five petal geometry. For instance, in the case of the five-petal geometry, taking $\eta = 18$ for all four wavenumbers $k = 8, 16, 32, 128$ we see in Figure 5 that the origin is outside the computed numerical ranges. Furthermore, we get the following approximations for the coercivity constants $\gamma \approx 0.3775,\ 0.3114,\ 0.204,\ 0.0829$ for $k = 16, 32, 64$ and 128 respectively (again here, these approximations of the coercivity constant are accurate with three digits).

We conclude this section with two remarks. First, our numerical evidence suggests that the ICFIE-R operators $\mathcal{A}_{k,S_{ik}}$ are not coercive in the case of the cavity geometry, regardless of the choice of the coupling constant η. Second, by constructing explicitly the matrices corresponding to the ICFIE-RPS operators $\mathcal{A}_{k,PS_{ik}}$ defined in equation (2.18) (note that the discretization of these operators presented in Section 4 delivers only the matrix-vector product associated with these

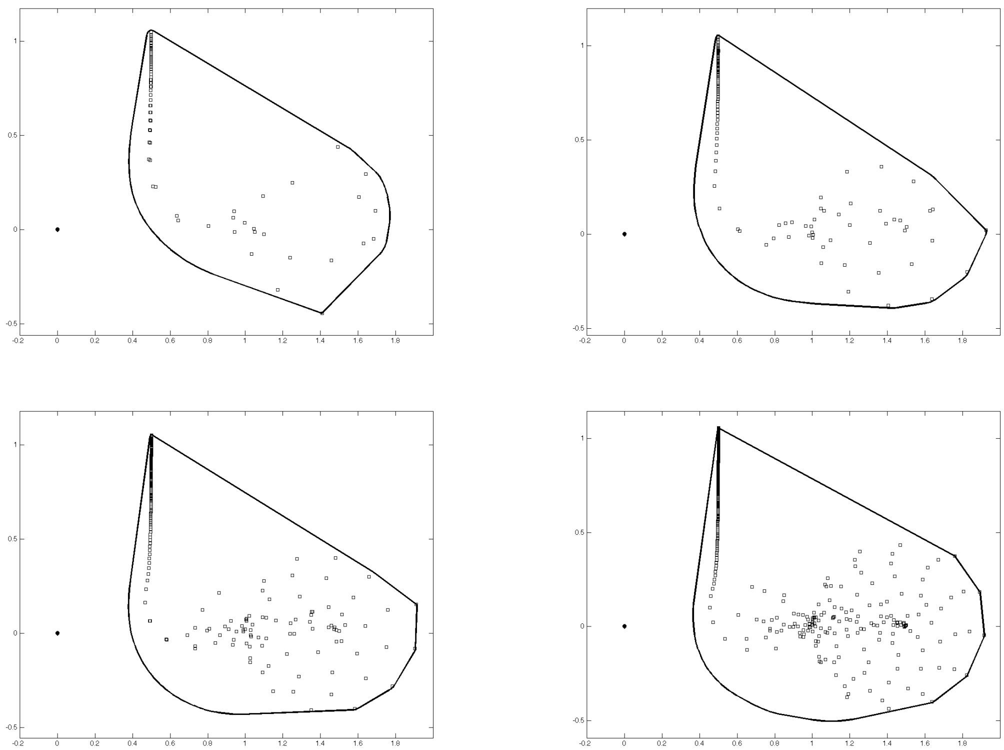

Figure 4: Computed numerical ranges and spectra of the operators $\mathcal{A}_{k,S_{ik}}$ with $\eta = 4$ for an elliptical geometry of semi-axes $a = 1$ and $b = 0.25$ for four wavenumbers $k = 16, 32, 64, 128$ pictured top left, top right, bottom left and bottom right respectively. The numerical ranges are the bounded domains enclosed by the closed curves in the figures; for reference, we also displayed the origin which is not contained in any of these domains. The coercivity constants are $\gamma \approx 0.439,\ 0.393,\ 0.387,\ 0.367$ for $k = 16, 32, 64$ and 128 respectively.

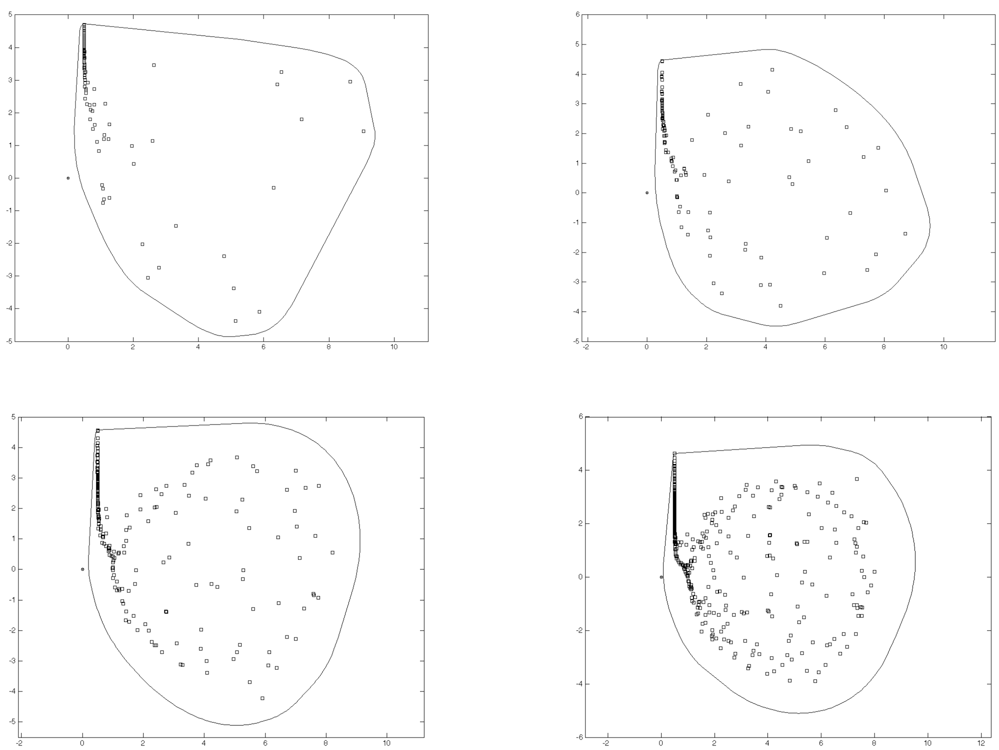

Figure 5: Computed numerical ranges and spectra of the operators $\mathcal{A}_{k,S_{ik}}$ with $\eta = 18$ for the five petal geometry and for four wavenumbers $k = 16, 32, 64, 128$ pictured top left, top right, bottom left and bottom right respectively. The numerical ranges are the bounded domains enclosed by the closed curves in the figures; for reference, we also displayed the origin which is not contained in any of these domains. The coercivity constants are $\gamma \approx 0.3775,\ 0.3114,\ 0.204,\ 0.0829$ for $k = 16, 32, 64$ and 128 respectively.

linear operators), we obtained similar numerical ranges to the ones pertaining to the ICFIE-R operators $\mathcal{A}_{k,S_{ik}}$ defined in equation (2.16).

6 Conclusions

We analyzed the wavenumber dependence of the norms of the operators that enter a class of Regularized Combined Field Integral Equations (CFIER) for the solution of two and three dimensional scattering problems with Neumann boundary conditions. We established that these norms grow modestly with the wavenumber in the high frequency regime for smooth boundaries. Furthermore, we proved that for certain choices of the coupling parameters, the CFIER operators are coercive in the case of circular and spherical geometries with coercivity constants that are independent of the wavenumber. We presented discretizations of the two-dimensional CFIER integral operators that relies on collocation methods based on global trigonometric interpolation. The ensuing discrete operators converge with high-order to the continuous ones. Based on these very accurate discretizations, we compute the discrete numerical ranges of the two dimensional CFIER operators and we demonstrate that the two dimensional CFIER operators are coercive for (a) smooth and strictly convex domains with coercivity constants that are independent of the wavenumber, and (b) smooth, starlike non-convex domains with coercivity constants that appear to depend on the wavenumber.

Acknowledgments

Yassine Boubendir gratefully acknowledge support from NSF through contract DMS-1016405. Catalin Turc gratefully acknowledge support from NSF through contract DMS-1008076.

7 Appendix

We will prove that the principal symbol of the pseudodifferential operator S_{ik} is equal to $\frac{1}{2\sqrt{k^2+\xi^2}}$ and that the next term in the asymptotic expansion of the symbol of S_{ik} is of order $|\xi|^{-3}$. Our presentation uses techniques originally developed in [44]. Using a parametrization of the closed curve Γ, i.e. $\Gamma = \{x(t) : t \in I = [0, 2\pi]\}$, we get that $|x(t) - x(t_0)| = (|x'(t)| + L(t, t_0))|t - t_0|$ for all t and t_0 in $I \times I$, where $L(t, t_0)$ is smooth on $I \times I$ and $L(t_0, t_0) = 0$. Using the fact [31] that

$$\frac{i}{4}H_0^{(1)}(kz) = -\frac{1}{2\pi}\log z + \frac{i}{4} - \frac{1}{2\pi}\left(\log\frac{k}{2} + C\right) + \mathcal{O}(z^2\log z)$$

where C is Euler's constant, we get that the kernel of S_{iK} can be expressed as

$$\begin{aligned} \frac{i}{4}H_0^{(1)}(ik|x(t) - x(t_0)|)|x'(t)| &= \frac{i}{4}H_0^{(1)}(ik|t - t_0|)|x'(t)| \\ &+ \frac{i}{4}H_0^{(1)}(ik|t - t_0|)r(t_0)(t - t_0)^2 + \Phi(t, t_0) \end{aligned} \tag{7.1}$$

for two smooth functions $r(t)$ and $\Phi(t,t_0)$. Since any operator A acting on functions $u \in C_0^\infty(I)$ can be lifted to an operator A_Γ acting on $C_0^\infty(\Gamma)$ via $A = A_\Gamma(u\circ x^{-1})\circ x$, we get that the operator $S_{ik} : C_0^\infty(\Gamma) \to C^\infty(\Gamma)$ can be represented using (7.1) as the S_Γ of the following operator:

$$\begin{aligned}(Su)(t_0) &= \int_{\mathbb{R}} \frac{i}{4} H_0^{(1)}(ik|t-t_0|)|x'(t)|u(t)dt \\ &+ \int_{\mathbb{R}} \frac{i}{4} H_0^{(1)}(ik|t-t_0|)r(t,t_0)(t-t_0)^2 u(t)dt + \int_{\mathbb{R}} \eta(|t-t_0|)\Phi(t,t_0)u(t)dt \qquad (7.2)\end{aligned}$$

where $u \in C_0^\infty(I)$ and η is a smooth cut-off function such that $\eta(|t-t_0|) = 1$ for all $t,\ t_0 \in I$. Since the integral operators in (7.2) are of convolution type, we get that

$$\begin{aligned}(Su)(t_0) &= \frac{1}{2\pi}\int_{\mathbb{R}} \sigma(t_0,\xi)e^{it_0\xi}\hat{u}(\xi)d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{2\sqrt{k^2+\xi^2}} e^{it_0\xi}\hat{u}(\xi)d\xi + \frac{1}{2\pi}\int_{\mathbb{R}} \sigma_1(t_0,\xi)e^{it_0\xi}\hat{u}(\xi)d\xi, \qquad (7.3)\end{aligned}$$

where we denoted by $\hat{u}$ the Fourier transform of the function $|x'(t)|u(t)$ and we have used the fact that the Fourier transform of $\frac{i}{4}H_0^{(1)}(i|z|) = \frac{1}{2\pi}K_0(|z|)$ is equal to $\frac{1}{2\langle\xi\rangle}$ with $\langle\xi\rangle = \sqrt{1+\xi^2}$ [26]. Furthermore, it can be shown that the Fourier transform of $\frac{i}{4}H_0^{(1)}(ik|z|)z^2$ is of order $\langle\xi\rangle^{-3}$ (it can be shown to equal to $k^2/4\langle\xi\rangle^{-3}$ [35]) whereas the Fourier transform of $\eta(|z|)\Phi(z)$ decays exponentially on account of the smoothness of Φ. Thus, the principal symbol of the pseudodifferential operator S_{ik} of order -1 is equal to $\frac{1}{2\sqrt{k^2+\xi^2}}$ and the remainder is a pseudodifferential operator of order -3 as claimed in the text.

References

[1] Abramowitz, M., and I.Stegun, "Handbook of mathematical functions with formulas, graphs, and mathematical tables", vol. 55 of National Bureau of Standards Applied Mathematics Series. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington D.C., 1964.

[2] Adams, R. , *Combined Field Integral Equation Formulations for Electromagnetic Scattering From Convex Geometries*, IEEE Trans. Antennas Propag., 52 (2004), 1294-1303

[3] Amini, S, and P. Harris, *A comparison between various boundary integral formulations of the exterior acoustic problem*, Computer Meth. Appl. Mech. Eng. 84, 1990, 59-75

[4] Antoine X. and Darbas M., *Generalized combined field integral equations for the iterative solution of the three-dimensional Helmholtz equation*, Mathematical modeling and numerical analysis, 41, 2007, pp. 147–167.

[5] Antoine, X., Darbas, M., *Alternative integral equations for the iterative solution of acoustic scattering problems*, Q. Jl. Mech. Appl. Math. 58 (2005), no. 1, 107-128

[6] Banjai, L., and S. Sauter, *A refined Galerkin error and stability analysis for highly indefinite variational problems*, SIAM J. Numer. Anal., Vol. 45, No 1, pp 37–53

[7] Brackhage, H., and P. Werner, *Uber das Dirichletsche Aussenraumproblem fur die Helmholtsche Schwingungsgleichung*, Arch.Math, 16, 325-329, 1965.

[8] Bruno, O., and L. Kunyansky, *Surface scattering in three dimensions: an accelerated high-order solver*, R. Soc. Lon. Proc. Ser. A Math. Phys. Eng. Sci., 2016, 2921-2934, 2001.

[9] Bruno, O., Elling, T., Turc, C., *Regularized integral equations and fast high-order solvers for sound-hard acoustic scattering problems*, submitted to Int. J. Eng. Math., 2011.

[10] Anand, A., Ovall, J., Turc, C., *Well-conditioned boundary integral equations for two-dimensional sound-hard scattering problems in domains with corners*, to appear in Journal Integral Equations and Applications, 2011.

[11] Betcke, T., E. Spence, *Numerical estimation of coercivity constants for boundary integral operators in acoustic scattering*, SIAM J Num Analysis, 2011.

[12] Burton, A., Miller J., *The application of integral equation methods to the numerical solution of some exterior boundary-value problems*, Proc. Royal Soc. London 323 (1971), 201-210

[13] Burton, A., Numerical solution of acoustic radiation problems, *NPL Contract Rept. OC5/S35 National Physical Laboratory, Teddington, Middlesex*, (1976).

[14] Chen, G., and J. Zhou, "Boundary element methods", Academic Press, Harcourt Press Jovanovitch Publishers, 1992.

[15] Calderón, A., *The multipole expansion of radiation fields*, Arch. Rat. Mech. Anal. 3 (1954), 523-537

[16] Chandler-Wilde, S., I. G. Graham, S. Langdon, and M. Lindner, *Condition number estimates for the combined potential boundary integral operators in acoustic scattering*, J. Int. Eqns. Appl., 21 (2009) 229-279

[17] Chandler-Wilde, S., P. Monk, *Wave-number explicit bounds in time-harmonic scattering*, SIAM J. Math. Anal, 39:1428-1455, 2008.

[18] Chew, W. C., Jin, J., Michielssen, E., and J. Song, Fast and efficient algorithms in computational electromagnetics, *Artech House, Norwood*, 2001

[19] Christiansen, S., Nédélec, J. C., Des préconditioneurs pour la résolution numérique des équations intégrales de frontière de l'acoustique, *C.R. Acad. Sci. Paris, Sér. I 330*, 2000, 617-622

[20] Colton, D., Kress, R., Integral equation methods in scattering theory, *John Wiley & Sons, Inc., New York*, (1983)

[21] Colton, D., and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer-Verlag,New York, Sec. Ed. 1992.

[22] Contopanagos H., Dembart B., Epton M., Ottusch J., Rohklin V., Visher J., Wandzura, S., *Well-conditioned boundary integral equations for three-dimensional electromagnetic scattering*, IEEE Trans. Antennas Propag., 50 (2002), no. 12, 1824-1830

[23] Costabel, M., Stephan, E., *A direct boundary integral equation method for transmission problems*, J. Math. Anal. Appl. 106 (1985), 367-415

[24] Cowen, C., E. Harel, *An effective algorithm for computing the numerical range*, available online at http://www.math.iupui.edu/ ccowen/Downloads/33NumRange.html

[25] Dunford, N., and J. T. Schwartz, Linear Operators, Wiley 1988.

[26] Erdélyi, A., "Tables of integral transforms", McGraw-Hill, New York, 1954.

[27] Dominguez, V., Graham, I., and V. Smyshlyaev, *A hybrid numerical-asymptotic boundary integral method for high-frequency acoustic scattering*, Numerische Mathematik, 106, (2007), 471–510

[28] Hiptmair, R., Buffa, A., *Regularized combined field integral equations*, Numerische Mathematik, (2005)

[29] Hiptmair, R., Buffa, A., *A coercive combined field integral equation for electromagnetic scattering*, SIAM J. Numer. Anal. 42 (2004), no. 2, 621-640

[30] Kress, R., *On the numerical solution of a hypersingular integral equation in scattering theory*, J. Comput. Appl. Math 61 (1995), 345-360

[31] Kress, R., *Linear Integral Equations*, Springer-Verlag, Sec. Ed., New York, 1999.

[32] Kress, R., Minimizing the condition number of boundary integral operators in acoustic and electromagnetic scattering ,*Q. Jl. Mech. Appl. Math.*, 38, 323-341, 1985.

[33] Kriegsman, G., Taflove, A., Umashankar, K., A new formulation of electromagnetic wave scattering using the on-surface radiation condition method, *IEEE Trans. Antennas Propag. 35*, 153-161, 1987

[34] Kussmaul, R., *Ein numerisches Verfahren zur Losung des Neumannschen Aussenraumproblems fur die Helmholtzsche Schwingungsgleichung*, Computing 4, 246-273 (1969)

[35] Levadoux, D., Etude d'une équation intégrale adaptaeé à la resolution haute-fréquance de l'equation D'Helmholtz, *Thèse de doctorat de l'Université de Paris VI France*, 2001

[36] Martensen, E., *Uber eine Methode zum raumlichen Neumannschen Problem mit einer ANwendung fur torusartige Berandungen*, Acta Math. 109, 75–135, (1963)

[37] Maue, A. W., *Toward formulation of a general diffraction problem via an integral equation*, Zeitschrift für Physik, 126, 601-618

[38] Melenk, J. M., *Mapping properties of combined field Helmholtz boundary integral operators*, Tech. Rep. ASC Report 1/2010, Institute for Analysis and Scientific Computing - Vienna University of Technology, 2010

[39] Nédélec, J. C., "Acoustic and electromagnetic equations", Springer Verlag, New York, 2001.

[40] Panich, I., *On the question of the solvability of the exterior boundary problem for the wave equation and Maxwell's equation*, Uspekhi Mat. Nauk 20, (1965), 221-226

[41] Saad, Y., and M. H. Schultz, *GMRES: A generalized minimal residual algorithm for solving non-symmetric linear systems*, SIAM J. Sci. Stat. Comput., 3, vol. 7, 1986, 856–869.

[42] Saranen, J., and G. Vainikko, "Periodic integral and pseudodiffferential equations with numerical approximation", Springer Monographs in Mathematics, Berlin, New York, 2002

[43] Seeley, R., *Topics in pseudo-differential operators,* Pseudo-Differential Operators, L. Nirenberg ed., Roma, C.I.M.E., Cremonese, 168–305, 1969.

[44] Smirnov, Yu. G., *The solvability of vector integro-differential equations for the problem of the diffraction of an electromagnetic field by screens of arbitrary shape*, Comp. Maths. Math. Phys., 10, vol. 34, 1995, 1265-1276.

[45] Spence, E., S. Chandler-Wilde, I. Graham, and V. Smyshlyaev, *A new frequency-uniform coercive boundary integral equation for acousting scattering*, preprint 2011.

[46] Stein, E., "Harmonic Analysis", *Princeton Univ. Press*, (1993)

[47] Taylor, M. E., "Pseododifferential Operators", Princeton University Press, Princeton, New Jersey, 1981.

[48] Watson, G.N., "A treatise on the theory of Bessel functions", Second Edition, Cambridge University Press, 1944.