Math 244-Final Exam
Name: ___________________________
Student #: _______________________

I pledge that I have not violated the NJIT code of honor.

1. How many 5-digit numbers can be formed from the integers 1, 2, ..., 9 if no digit can appear more than twice? (For instance, 41434 is not allowed.) (12 points)
(Self-test problems and exercises #10, page 20)

\[ 9^5 = \left[ 9 + \binom{5}{1}(9)(8) + \binom{5}{2}(9)(8)(7) \right] \]

Or
\[ 9^5 = \left[ 9 + \binom{5}{1}(9)(8)(7)(6) + \binom{5}{2}(9)(8)(7)(6)(5) \right] \]

2. A total of 30 percent of American smokes cigarettes, 6 percent smoke cigars and 3 percent smoke both cigars and cigarettes. What percentage of males smokes neither cigars nor cigarettes? What percentage smokes cigars but not cigarettes? (12 points) (Homework problem 11, page 51)

\[ P(C^c \cap I^c) = C: \text{cigarettes} \]
\[ I: \text{cigars} \]

\[ = 1 - P(C \cup I) = 1 - \left[ P(C) + P(I) - P(CI) \right] \]

\[ = 1 - \left[ .3 + .06 - .03 \right] = 1 - .33 = .67 \]

\[ P \left( I \cap C^c \right) = P(I) - P(I \cap C) = .06 - .03 \]
\[ = .03 \]
3. Let $X_1$ be Gamma with $(\lambda = 7, \alpha = 4)$ and $X_2$ be Gamma with $(\lambda = 7, \alpha = 5)$. If $X_1$ and $X_2$ are independent derive the distribution of $Y = X_1 + X_2$. (12 points)

(Done in class for general case, also on page 254-255)

$$M_{X_1}(t) = (\frac{7}{7-t})^4, \quad t < 7$$

$$M_{X_2}(t) = (\frac{7}{7-t})^5, \quad t < 7$$

$$M_{X_1+X_2}(t) = M_{X_1}(t)M_{X_2}(t) = (\frac{7}{7-t})^4(\frac{7}{7-t})^5 = (\frac{7}{7-t})^9, \quad t < 7$$

Hence $Y$ is Gamma $(\lambda = 7, \alpha = 9)$.

4. Let $f(x,y) = 24xy$ for $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq x+y \leq 1$ and let it equal zero otherwise.

(a) Compute the marginal density of $X$.

(b) Compute the probability $P(X < 2Y)$

(c) Is $X$ independent of $Y$? Why or why not? (18 points)

\[
\begin{align*}
\text{for } x \leq 1, y \leq 1, x+y \leq 1:
\end{align*}
\]

\[
\int_0^{1-x} \int_0^{1-y} 24xy \, dy \, dx, \quad 0 < x < 1
\]

\[
= 24x \int_0^{1-x} y^2 \, dy \\
= 12x(1-x)/3
\]

\[
= \begin{cases} 
12x(1-x)^2, & 0 < x < 1 \\
0, & \text{elsewhere}
\end{cases}
\]

\[
\text{Beta}(x=2, \beta=3).
\]

\[
\Gamma(\alpha + \beta) = \Gamma(2) = 1 \cdot 2! = 2
\]

\[
\Gamma(\alpha) \cdot \Gamma(\beta) = 1 \cdot 2! = 2
\]

\[
\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \cdot \Gamma(\beta)} = 1
\]

\[
\frac{\int_0^1 \int_0^{1-x} 24xy \, dy \, dx}{\int_0^{1/2} \int_0^{1-x/2} 24xy \, dy \, dx}
\]

\[
= \int_0^{1/2} \int_0^{1-x/2} 24xy \, dy \, dx \\
= \int_0^{1/2} 24x \left[ \frac{y^2}{2} \right]_{y=0}^{y=1-x/2} \, dx
\]

\[
= \int_0^{1/2} 24x \left( \frac{1-x^2}{4} \right) \, dx \\
= 6 \int_0^{1/2} x(1-x^2) \, dx
\]

\[
= \frac{6}{3} \int_0^{1/2} x - 3x^3 \, dx
\]

\[
= \frac{6}{3 \cdot 2} \left[ \frac{x^2}{2} - \frac{x^4}{4} \right]_0^{1/2}
\]

\[
= \frac{6}{3} \cdot \frac{1}{2} \cdot \frac{1}{4} - \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{4} - \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}
\]

\[
= 8/3 - \frac{64}{27} + \frac{16}{27} = 8/3 - \frac{64}{27} = \frac{24}{27} - \frac{64}{27} = -\frac{40}{27}
\]
5. If $X_1$ and $X_2$ are independent exponential random variables, each having parameter $\lambda$, find the joint density function of $Y_1 = X_1 + X_2$ and $Y_2 = c X_1$. (12 points) (page 290, #6.53)

Step 1: Solve for $x_1, x_2$

\[ x_1 = \ln y_2 \]
\[ x_2 = y_2 - \ln y_2 \]

Step 2: \[
\begin{vmatrix}
\lambda e^{-\lambda x_1} y_1 & y_2 \\\(y_2 - \ln y_2) & y_2
\end{vmatrix}
\]

\[ J = 1 - \frac{1}{y_2} \]

Step 3: Substitute $x_1 = \ln y_2$

\[ f(x_1, x_2) = \lambda e^{-\lambda (x_1 + x_2)}, x_1 > 0, x_2 \geq 0 \]

6. Let $X$ and $Y$ have joint density given by

\[ f(x, y) = \begin{cases} 2e^{-2x}, & 0 \leq x < \infty, 0 \leq y \leq x, \\ 0, & \text{elsewhere}. \end{cases} \]

Compute $E(XY)$, $E(X)$ and $E(Y)$ (10 points) (7.4, page 373 and 7.38, page 375)

\[
E(X) = \int_0^\infty x 2e^{-2x} \, dx = \int_0^\infty \frac{x^2}{2} e^{-2x} \, dx = \frac{E(X^2)}{2}
\]

\[
E(Y) = \int_0^\infty y 2e^{-2x} \, dx = \int_0^\infty \frac{y}{x} e^{-2x} \, dx = \frac{EW}{2}
\]

\[
E(X^2) = \int_0^\infty x^2 2e^{-2x} \, dx = \int_0^\infty \frac{x^2}{2} e^{-2x} \, dx = \frac{E(W^2)}{2}
\]

\[
EW = \frac{1}{\lambda} \sqrt{\frac{2}{\pi}}
\]

\[
E(Y) = \frac{1}{2\lambda} E(X)
\]
7. Suppose that a die is rolled twice. Find the probability mass function associated with the random variable $X$: minimum value to appear in the two rolls. Compute the variance of $X$. (14 points) (4.7 and 4.8(b) page 173)

$$
\begin{array}{c|cccccc}
\text{min}(X \cup Y) &=& Y_1 = \text{Dice 2 values} & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
X_1 &=& \text{Dice 1 values} & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 2 & 2 & 2 & 2 & 2 \\
3 & 1 & 2 & 3 & 3 & 3 & 3 \\
4 & 1 & 2 & 3 & 4 & 4 & 4 \\
5 & 1 & 2 & 3 & 4 & 5 & 5 \\
6 & 1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
$$

$$
X = M = X_{\text{max}} + Y_1
$$

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(m)$</th>
<th>$mP(m)$</th>
<th>$m^2P(m)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$11/36$</td>
<td>$11/36$</td>
<td>$11/36$</td>
</tr>
<tr>
<td>2</td>
<td>$9/36$</td>
<td>$18/36$</td>
<td>$26/36$</td>
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<td>3</td>
<td>$7/36$</td>
<td>$21/36$</td>
<td>$63/36$</td>
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<td>$5/36$</td>
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<td>$80/36$</td>
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<tr>
<td>5</td>
<td>$3/36$</td>
<td>$15/36$</td>
<td>$75/36$</td>
</tr>
<tr>
<td>6</td>
<td>$1/36$</td>
<td>$6/36$</td>
<td>$36/36$</td>
</tr>
</tbody>
</table>

$$
EX = EM = \frac{91}{36} = 2.527
$$

$$
\text{Var}(X) = \frac{361}{36} - \left(\frac{91}{36}\right)^2 = 1.9 + 1.45617
$$

$$
= 8.361 - 6.389660494
$$
8. A parallel system functions whenever at least one of its components works. Consider a parallel system of \( n \) components, and suppose that each component works independently with probability 1/3. Find the conditional probability that component 1 works given that the system is functioning. (10 points)

\[
P(\text{Component 1 works} | \text{System is functioning})
\]

\[
= \frac{P(\text{Component 1 works} \land \text{System is functioning})}{P(\text{System is functioning})}
\]

\[
= \frac{P(\text{Component 1 works})}{P(\text{System is functioning})}
\]

\[
= \frac{\frac{1}{3}}{1 - P(\text{System is not functioning})}
\]

\[
= \frac{\frac{1}{3}}{1 - P(\text{None of the components work})}
\]

\[
= \frac{\frac{1}{3}}{1 - (\frac{2}{3})^n}
\]

END