Math 341, Exam I, Spring 2010, Name____________

Student # ______________

Wednesday, February 25. *Please show the complete solution (with all steps) to each problem to receive full credit.*

1.a. Let $Y_1, \ldots, Y_8$ be a random sample of size 8 from a standard normal distribution. What is the distribution of $\dfrac{\sqrt{3}(Y_7 - Y_8)}{\sqrt{\sum_{i=1}^{6}(Y_i)^2}}$? Why? (15 points)

b. In general define the random variable with F-distribution with 'n' as the numerator degrees of freedom (d.f.) and 'm' as the denominator d.f. (7 points)

b. Let $W_1 \sim \chi^2_{(n)}$

$W_2 \sim \chi^2_{(m)}$

W_1 is independent of W_2

Then $\dfrac{W_1/n}{W_2/m}$ is $F_{n,m}$.

(a.)

$\sum_{i=1}^{6} Y_i^2 \sim \chi^2_6$

$Y_7 - Y_8 \sim N(0, 2)$

$\dfrac{Y_7 - Y_8}{\sqrt{2}} \sim N(0,1)$

The expression in 1.a. $= \dfrac{(Y_7 - Y_8)/\sqrt{2}}{\sqrt{\dfrac{\sum_{i=1}^{6} Y_i^2}{6}}} = \dfrac{N(0,1)}{\sqrt{\dfrac{\chi^2(6)}{6}}} \sim t_6$

because $\sum_{i=1}^{6} Y_i^2$ is indep. of (Y_7, Y_8).

2. The times that a cashier spends processing individual customer's order are independent and identically distributed random variables with mean 2 minutes and standard deviations 1 minute. What is the approximate probability that it will take more than 3 hours to process the orders of 80 customers? Give justification for the method used to solve this problem. (15 points)

$$P\left(\sum_{i=1}^{80} Y_i > (3)(60)\right) = P\left(\sum_{i=1}^{80} Y_i > 180\right)$$

$$= P\left(\bar{Y} > \frac{180}{80} = \frac{9}{4} = 2.25\right) \doteq P\left(Z > \frac{2.25 - 2}{\frac{1}{\sqrt{80}}}\right)$$

$$= P(Z > 2.24) = .0125$$

Since $n = 80 > 30$ is large C.L.T gives $\dfrac{\bar{Y} - 2}{1/\sqrt{80}} \simeq Z$ (standard normal)

3. Let $Y_1, \ldots, Y_n$ be a random sample of size n from a population with mean 2. Suppose that $\hat{\theta}_2$ is the unbiased estimator of EY^2 and that $\hat{\theta}_3$ is the unbiased estimator of EY^3. Give an unbiased estimator of the third central moment, i.e., $E(Y-2)^3$ of the underlying distribution. (13 points)

$$(y-2)^3 = \binom{3}{3} y^3 (-2)^0 + \binom{3}{2} y^2 (-2)^1 + \binom{3}{1} y^1 (-2)^2 + \binom{3}{0} y^0 (-2)^3$$

$$= y^3 + 3(-2) y^2 + 12y + -8$$

$$= y^3 - 6y^2 + 12y - 8$$

$$E(Y-2)^3 = E(Y^3) - 6EY^2 + 12EY - 8$$

$$= E(Y^3) - 6EY^2 + 24 - 8$$

$$= E(Y^3) - 6EY^2 + 16$$

$\hat{\theta}_3 - 6\hat{\theta}_2 + 16$ is the unbiased estimator;

4. Suppose that Y_1, Y_2, Y_3 denote a random sample from the exponential density function given by $f(y) = \begin{cases} \frac{1}{\theta} e^{-y/\theta}, & y > 0 \\ 0, & elsewhere. \end{cases}$ $\sim Gamma(1, \theta)$ Consider the following two estimators of θ, $\hat{\theta}_1 = \bar{Y} = \frac{\sum_{i=1}^{3} Y_i}{3}$ and $\hat{\theta}_2 = \min(Y_1, Y_2, Y_3)$. a. Derive the bias of the estimators $\hat{\theta}_1$ and $\hat{\theta}_2$.

b. Find the estimator with the smallest mean square error. Show work.

(20 points)

a. $E\bar{Y} = EY = \theta$ hence $\boxed{Bias(\hat{\theta}_1) = 0}$

$$P(\hat{\theta}_2 > y) = \left(\int_y^\infty \frac{1}{\theta} e^{-x/\theta} dx \right)^3$$

$$= P(Y_1 > y, Y_2 > y, Y_3 > y) = \left(\left. \frac{e^{-x/\theta}}{-1} \right|_y^\infty \right)^3$$

$$= P(Y_1 > y)^3 = e^{-3y/\theta}, \quad y > 0$$

$$F_{\hat{\theta}_2}(y) = 1 - e^{-3y/\theta}, \quad y > 0$$

$$f_{\hat{\theta}_2}(y) = \begin{cases} \frac{3}{\theta} e^{-3y/\theta}, & y > 0 \\ 0 & elsewhere \end{cases}$$

Gamma$(\alpha = 1, \beta = \frac{\theta}{3})$

$$E[\hat{\theta}_2] = \frac{\theta}{3} \qquad Var(\hat{\theta}_2) = \frac{\theta^2}{9}$$

$$\boxed{Bias(\hat{\theta}_2) = \theta/3 - \theta = \frac{-2\theta}{3}}$$

(b) $MSE(\hat{\theta}_1) = Var(\bar{Y}) = \frac{\theta^2}{3}$

$$MSE(\hat{\theta}_2) = Var(\hat{\theta}_2) + (Bias\,\hat{\theta}_2)^2 = \frac{\theta^2}{9} + \frac{4\theta^2}{9}$$

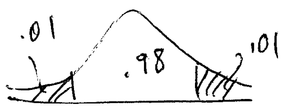
$MSE(\hat{\theta}_2) = \frac{5\theta^2}{9}$, which is larger than that of $\hat{\theta}_1$. Answer is $\hat{\theta}_1$.

5. The manager of a supermarket wants to obtain information about the proportion of customers who dislike a new policy on cashing checks. How many customers should he or she sample if the manager wants the sample fraction to be within 0.15 of the rue fraction with probability 0.98? Use p = 0.5 because it maximizes the standard error of the sample fraction. (15 points)

$$P(|\hat{p}-p| < .15) = .98 \qquad Z = \frac{\hat{p}-p}{\sqrt{\frac{(.5)(.5)}{n}}}$$

$$P\left(\frac{-.15}{\sqrt{\frac{.25}{n}}} < Z < \frac{.15}{\sqrt{\frac{.25}{n}}}\right) = .98$$

$$= P(-.3\sqrt{n} < Z < .3\sqrt{n}) = .98$$



$$.3\sqrt{n} = 2.33$$

$$\sqrt{n} = \frac{2.33}{.3} \qquad n = \left(\frac{2.33}{.3}\right)^2 = \left(\frac{23.3}{3}\right)^2 = 60.32\dot{1}$$

$$\approx 61$$

6. Briggs and King developed the technique of nuclear transplantation in which the nucleus of a cell from one of the later stages of an embryo's development is transplanted into a zygote (a single-cell, fertilized egg) to see if the nucleus can support normal development. If the probability that a single transplant from the early gastrula stage will be successful is 0.70, what is the approximate probability that more than 50 transplants out of 75 will be successful? Justify the validity of any approximation used. (15 points)

Y: # of successful transplants out of n=75)

$$P(Y>50) = P(Y \geq 51)$$

$$EY = np = (75)(.7) = 52.5$$

$$= P(Y \geq 50.5)$$

$$Var(Y) = (52.5)(.3) = 15.75$$

$$= P\left(Z \geq \frac{50.5-52.5}{\sqrt{15.75}}\right)$$

$$= P\left(Z \geq \frac{-2}{\sqrt{15.75}} = -.5\right)$$

$$0 \leq 52.5 \pm 3\sqrt{15.75} \leq 75$$

$0 \leq 52.5 \pm 11.91 \leq 75$, hence good approxim.

by symmetry of Z density $= P(Z \leq .5) = 1 - .3085 = 0.6915$.