Math 341-001, Exam I, Spring 2014, Name ____________________
Student # ____________________

Monday, March 03. Please show the complete solution (with all steps) to each problem to receive perfect score.

1. The management at a fast-food outlet is interested in the joint behavior of the random variables \( Y_1 \) defined as the total time between a customer’s arrival at the store and departure from the service window, and \( Y_2 \), the time a customer waits in line before reaching the service window. Since, \( Y_1 \) includes the time a customer waits in line, we must have \( Y_1 \geq Y_2 \). The joint distribution of observed values of \( Y_1 \) and \( Y_2 \) can be modelled by the probability density function

\[
f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \leq y_2 \leq y_1 < \infty, \\ 0, & \text{elsewhere.} \end{cases}
\]

Find the probability time spend at the service window \( Y_1 - Y_2 \) is at most 2 minutes, i.e. \( P(Y_1 - Y_2 \leq 2) \).

(Reference problem 5.15, page 235) (15 points)

2. In problem 1. above, find the probability the total time \( Y_1 \) is at most 2 minutes given that the customer already waited in line exactly 1 minute, i.e., \( Y_2 = 1 \).

(a) Thus, Find \( P(Y_1 \leq 2|Y_2 = 1) \).

(b) Are \( Y_1 \) and \( Y_2 \) independent? Why? Or why not?

(c) Find \( E(Y_2) \) and \( V(Y_2) \). (Reference problems 5.29 p. 244 & 5.77 p. 262) (25 points)
3. Let $W$ be a normal random variable with mean 0 and variance 1. Derive the distribution of $U = W^2$. (Reference example 6.11, page 319) (15 points)
4. Let $Y_1, \ldots, Y_{10}$ be independent Exponential distributed with mean = 5.
   a) Find the distribution of $Y_{(1)} = \min(Y_1, \ldots, Y_{10})$.
   b) Identify the distribution obtained in 4. a).
   c) Find $P(Y_{(1)} \leq 3)$.

(Reference problem # 6.81, page 339) (25 points)
5. The joint distribution for the length of life of two different types of components operating in a system is given by

\[
f(y_1, y_2) = \begin{cases} 
\frac{1}{2}e^{-y_1}e^{-y_2/2}, & y_2 > 0, y_1 > 0, \\
0, & \text{elsewhere}.
\end{cases}
\]

Find the probability density function for the relative efficiency of the two types of components measured by \( U = \frac{y_2}{y_1} \). (20 points)