Math 341-001, Exam I, Spring 2014, Name__________________
Student #______________

Monday, March 03. Please show the complete solution (with all steps) to each problem to receive perfect score.

1. The management at a fast-food outlet is interested in the joint behavior of the random variables $Y_1$ defined as the total time between a customer’s arrival at the store and departure from the service window, and $Y_2$, the time a customer waits in line before reaching the service window. Since, $Y_1$ includes the time a customer waits in line, we must have $Y_1 \geq Y_2$. The joint distribution of observed values of $Y_1$ and $Y_2$ can be modelled by the probability density function with time measured in minutes. $f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \leq y_2 \leq y_1 < \infty, \\ 0, & \text{elsewhere.} \end{cases}$ Find the probability time spend at the service window $Y_1 - Y_2$ is at most 2 minutes, i.e. $P(Y_1 - Y_2 \leq 2)$. (Reference problem 5.15, page 235) (15 points)

\[
\int_0^\infty \int_0^{2+y_2} e^{-y_2} \, dy_1 \, dy_2 = \int_0^\infty e^{-y_1} \left( \int_0^{2+y_2} e^{-y_2} \, dy_2 \right) \, dy_1 = \int_0^\infty e^{-y_1} \left( \int_0^{2+y_2} e^{-y_2} \, dy_2 \right) \, dy_1 = \left( 1 - e^{-2} \right) \int_0^\infty e^{-y_2} \, dy_2 = \left( 1 - e^{-2} \right) e^2 = 0.864665. \\
\]

2. In problem 1. above, find the probability the total time $Y_1$ is at most 2 minutes given that the customer already waited in line exactly 1 minute, i.e., $Y_2 = 1$. (a) Thus, Find $P(Y_1 \leq 2 | Y_2 = 1)$. (b) Are $Y_1$ and $Y_2$ independent? Why? Or why not? (c) Find $E(Y_2)$ and $V(Y_2)$. (Reference problems 5.29 p. 244 & 5.77 p. 262) (25 points)

\[
P(Y_1 \leq 2 | Y_2 = 1) = \int_0^2 e^{-y_1} \, dy_1 = \frac{e^{-1}}{e^{-1}} = 1 - e^{-1} = 0.63212 \\
\]

(b) No because $Y_1 \geq Y_2$, (c) $Y_2 \sim \text{Expo} (\beta = 1)$, $EY_2 = 1$, $\beta (\alpha = 1, \beta = 1) = 1$, $V(Y_2) = 1$. 

3. Let $W$ be a normal random variable with mean 0 and variance 1. Derive the distribution of $U = W^2$. (Reference example 6.11, page 319) (15 points)

EXAMPLE 6.11 Let $Z$ be a normally distributed random variable with mean 0 and variance 1. Use the method of moment-generating functions to find the probability distribution of $Z^2$.

Solution The moment-generating function for $Z^2$ is

$$m_{Z^2}(t) = E(e^{tZ^2}) = \int_{-\infty}^{\infty} e^{t z^2} f(z) \, dz = \int_{-\infty}^{\infty} e^{t z^2} \frac{e^{-z^2/2}}{\sqrt{2\pi}} \, dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z^2/2)(1-2t)} \, dz.$$  

This integral can be evaluated either by consulting a table of integrals or by noting that, if $1 - 2t > 0$ (equivalently, $t < 1/2$), the integrand

$$\exp \left[ -\left( \frac{z^2}{2} \right) (1-2t) \right]$$

is proportional to the density function of a normally distributed random variable with mean 0 and variance $(1-2t)^{-1}$. To make the integrand a normal density function (so that the definite integral is equal to 1), multiply the numerator and denominator by the standard deviation, $(1-2t)^{-1/2}$. Then

$$m_{Z^2}(t) = \frac{1}{(1-2t)^{1/2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\left( \frac{z^2}{2} \right) \right] (1-2t)^{-1} \, dz.$$  

Because the integral equals 1, if $t < 1/2$,

$$m_{Z^2}(t) = \frac{1}{(1-2t)^{1/2}} = (1-2t)^{-1/2}.$$  

A comparison of $m_{Z^2}(t)$ with the moment-generating functions in Appendix 2 shows that $m_{Z^2}(t)$ is identical to the moment-generating function for the gamma-distributed random variable with $\alpha = 1/2$ and $\beta = 2$. Thus, using Definition 4.10, $Z^2$ has a $\chi^2$ distribution with $\nu = 1$ degree of freedom. It follows that the density function for $U = Z^2$ is given by

$$f_U(u) = \begin{cases} \frac{u^{-1/2} e^{-u/2}}{\Gamma(1/2) 2^{1/2}}, & u \geq 0, \\ 0, & \text{elsewhere}. \end{cases}$$

OR

\#3 $f(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$

\textbf{Step 1} \quad P(U \leq u) = P(W^2 \leq u), \quad u > 0

= P(-\sqrt{u} \leq W \leq \sqrt{u}), \quad u > 0

= \int_{-\sqrt{u}}^{\sqrt{u}} e^{-w^2/2} \, dw

= 2 \int_{0}^{\sqrt{u}} e^{-w^2/2} \, dw, \quad u > 0

\textbf{Step 2} \quad f_U(u) = \frac{u^{1/2} e^{-u/2}}{\sqrt{2\pi}} \quad u > 0$
4. Let $Y_1, \ldots, Y_{10}$ be independent Exponential distributed with mean $= 5$.

a) Find the distribution of $Y_{(1)} = \min(Y_1, \ldots, Y_{10})$.

b) Identify the distribution obtained in 4. a).

c) Find $P(Y_{(1)} \leq 3)$.

(Reference problem # 6.81, page 339) (25 points)

\[ \begin{align*}
4. & \quad P(Y_{(1)}>y) = P(Y_1>y)^{10}, \quad 0<y<\infty \\
& \quad = (e^{-y/5})^{10} \\
& \quad = e^{-2y} \\
F_{Y_{(1)}}(y) &= \begin{cases} 
1-e^{-2y}, & 0<y<\infty \\
0, & y \leq 0 
\end{cases} \\
\text{where,} \\
(b) & \quad Y_{(1)} \text{ is Exponential with } \beta = \frac{1}{2}. \\
(c) & \quad \int_{0}^{\infty} 2e^{-2y} dy = \left. \frac{e^{-2y}}{-2} \right|_{0}^{\infty} = 1 - e^{-6} \approx 0.9975.
\end{align*} \]
5. The joint distribution for the length of life of two different types of components operating in a system is given by

\[ f(y_1, y_2) = \begin{cases} \frac{1}{2} e^{-y_1} e^{-y_2/2}, & y_2 > 0, y_1 > 0, \\ 0, & \text{elsewhere}. \end{cases} \]

Find the probability density function for the relative efficiency of the two types of components measured by \( U = \frac{Y_2}{Y_1} \).

\[ F_U(U) = \begin{cases} y_1 = y_1, \\ Y_U = Y_2, \quad \frac{dy_2}{du} = y_1, \\ f(y_1, u) = \begin{cases} \frac{1}{2} e^{-y_1} e^{-uy_1/2}, & uy_1 > 0, y_1 > 0 \\ 0, & \text{elsewhere} \end{cases} \\ f_U(u) = \begin{cases} \int_0^{\infty} \frac{1}{2} y_1 e^{-y_1 (1+u^2)/2} dy_1 = \frac{1}{2} e^{-u^2/2} \\ \int_0^{\infty} e^{-w} dw \\ \frac{(1+u^2)^{1/2}}{2} \\ = \begin{cases} \frac{2}{(1+u^2)^2}, \quad u > 0 \\ 0, \quad \text{elsewhere} \end{cases} \end{cases} \]