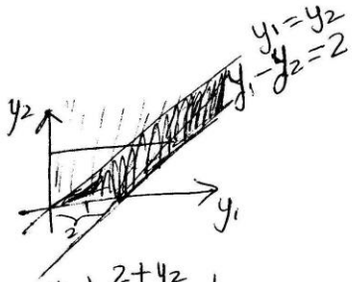
Math 341-001, Exam I, Spring 2014, Name_____

Student # _____

Monday, March 03. *Please show the complete solution (with all steps) to each problem to receive perfect score.*

1. The management at a fast-food outlet is interested in the joint behavior of the random variables Y_1 defined as the total time between a customer's arrival at the store and departure from the service window, and Y_2, the time a customer waits in line before reaching the service window. Since, Y_1 includes the time a customer waits in line, we must have $Y_1 \geq Y_2$. The joint distribution of observed values of Y_1 and Y_2 can be modelled by the probability density function with time measured in minutes. $f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \leq y_2 \leq y_1 < \infty, \\ 0, & \text{elsewhere.} \end{cases}$ Find the probability time spend at the service window Y_1 - Y_2 is at most 2 minutes, i.e. $P(Y_1 - Y_2 \leq 2)$.
(Reference problem 5.15, page 235) (15 points)

1.



$$\int_0^\infty \int_{y_2}^{2+y_2} e^{-y_1}\, dy_1\, dy_2 = \int_0^\infty \frac{e^{-y_1}}{-1}\Big|_{y_2}^{2+y_2} dy_2$$

$$= \int_0^\infty e^{-y_2} - e^{-(2+y_2)}\, dy_2$$

$$= (1 - e^{-2}) \int_0^\infty e^{-y_2}\, dy_2$$

$$= 1 - e^{-2} = 0.864665.$$

2. In problem 1. above, find the probability the total time Y_1 is at most 2 minutes given that the customer already waited in line exactly 1 minute, i.e., $Y_2 = 1$. (a) Thus, Find $P(Y_1 \leq 2 | Y_2 = 1)$.
(b) Are Y_1 and Y_2 independent? Why? Or why not?
(c) Find E(Y_2) and V(Y_2). (Reference problems 5.29 p. 244 & 5.77 p. 262) (25 points)

2. (a) $f_2(y_2) = \int_{y_2}^{\infty} e^{-y_1}\, dy_1 = e^{-y_2}, \quad 0 \leq y_2 < \infty$

$$P(Y_1 \leq 2 | Y_2 = 1) = \int_1^2 \frac{e^{-y_1}}{e^{-1}}\, dy_1 = \frac{e^{-1} - e^{-2}}{e^{-1}} = 1 - e^{-1} = .63212$$

(b) No because $Y_1 \geq Y_2$. (c) $Y_2 \sim Expo(\beta = 1)$ $EY_2 = 1$,
$= \Gamma(\alpha = 1, \beta = 1)$ $V(Y_2) = 1$.

3. Let W be a normal random variable with mean 0 and variance 1. Derive the distribution of $U = W^2$. (Reference example 6.11, page 319) (15 points)

EXAMPLE **6.11** Let Z be a normally distributed random variable with mean 0 and variance 1. Use the method of moment-generating functions to find the probability distribution of Z^2.

Solution The moment-generating function for Z^2 is

$$m_{Z^2}(t) = E(e^{tZ^2}) = \int_{-\infty}^{\infty} e^{tz^2} f(z)\,dz = \int_{-\infty}^{\infty} e^{tz^2} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z^2/2)(1-2t)}\,dz.$$

This integral can be evaluated either by consulting a table of integrals or by noting that, if $1 - 2t > 0$ (equivalently, $t < 1/2$), the integrand

$$\frac{\exp\left[-\left(\frac{z^2}{2}\right)(1-2t)\right]}{\sqrt{2\pi}} = \frac{\exp\left[-\left(\frac{z^2}{2}\right)\Big/(1-2t)^{-1}\right]}{\sqrt{2\pi}}$$

is proportional to the density function of a normally distributed random variable with mean 0 and variance $(1-2t)^{-1}$. To make the integrand a normal density function (so that the definite integral is equal to 1), multiply the numerator and denominator by the standard deviation, $(1-2t)^{-1/2}$. Then

$$m_{Z^2}(t) = \frac{1}{(1-2t)^{1/2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}(1-2t)^{-1/2}} \exp\left[-\left(\frac{z^2}{2}\right)\Big/(1-2t)^{-1}\right] dz.$$

Because the integral equals 1, if $t < 1/2$,

$$m_{Z^2}(t) = \frac{1}{(1-2t)^{1/2}} = (1-2t)^{-1/2}.$$

A comparison of $m_{Z^2}(t)$ with the moment-generating functions in Appendix 2 shows that $m_{Z^2}(t)$ is identical to the moment-generating function for the gamma-distributed random variable with $\alpha = 1/2$ and $\beta = 2$. Thus, using Definition 4.10, Z^2 has a χ^2 distribution with $\nu = 1$ degree of freedom. It follows that the density function for $U = Z^2$ is given by

$$f_U(u) = \begin{cases} \dfrac{u^{-1/2}e^{-u/2}}{\Gamma(1/2)2^{1/2}}, & u \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$ ■

OR

#3 $f(w) = \frac{1}{\sqrt{2\pi}} e^{-w^2/2}$

Step1 $P(U \leq u) = P(W^2 \leq u)$, $u > 0$

$= P(-\sqrt{u} \leq W \leq \sqrt{u})$, $u > 0$

$= \int_{-\sqrt{u}}^{\sqrt{u}} \frac{e^{-w^2/2}}{\sqrt{2\pi}}\,dw$

$= 2\int_{0}^{\sqrt{u}} \frac{e^{-w^2/2}}{\sqrt{2\pi}}\,dw$, $u > 0$

Step2 $F_U'(u) = \cancel{2}\,\frac{e^{-u/2}}{\sqrt{2\pi}} \cdot \frac{1}{\cancel{2}} u^{-1/2}$, $u > 0$

Chain rule

Answer

4. Let $Y_1, \ldots, Y_{10}$ be independent Exponential distributed with mean = 5.

a) Find the distribution of $Y_{(1)} = min(Y_1, \ldots Y_{10})$.

b) Identify the distribution obtained in 4. a).

c) Find $P(Y_{(1)} \leq 3)$.

(Reference problem # 6.81, page 339) (25 points)

4. (a) $P(Y_{(1)} > y) = P(Y_1 > y)^{10}, \quad 0 < y < \infty$

$= \left(e^{-y/5}\right)^{10}$

$= e^{-2y}$

$F_{Y_{(1)}}(y) = \begin{cases} 1 - e^{-2y}, & 0 < y < \infty \\ 0, & y \leq 0 \end{cases}$ $\quad f_{Y_{(1)}}(y) = \begin{cases} 2e^{-2y}, & y > 0 \\ 0 & \text{elsewhere} \end{cases}$

(b) $Y_{(1)}$ is Exponential with $\beta = 1/2$.

(c) $\int_0^3 2e^{-2y}\,dy = \left.\frac{e^{-2y}}{-1}\right|_0^3 = 1 - e^{-6} = .9975.$

5. The joint distribution for the length of life of two different types of components operating in a system is given by $f(y_1, y_2) = \begin{cases} \frac{1}{2} e^{-y_1} e^{-y_2/2}, & y_2 > 0, y_1 > 0, \\ 0, & \text{elsewhere.} \end{cases}$

Find the probability density function for the relative efficiency of the two types of components measured by $U = Y_2 / Y_1$. (20 points)

5. Fix $Y_1 = y_1$

$u y_1 = Y_2 \qquad \frac{dy_2}{du} = y_1$

$$f(y_1, u y_1) = \begin{cases} \frac{1}{2} e^{-y_1} e^{-u y_1/2}, & u y_1 > 0,\ y_1 > 0 \\ 0, & \text{elsewhere} \end{cases}$$

$$f_{Y_1, U}(y_1, u) = \begin{cases} \frac{1}{2} y_1 e^{-y_1(1+\frac{u}{2})}, & u > 0,\ y_1 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$f_U(u) = \int_0^\infty \frac{1}{2} y_1 e^{-y_1(1+\frac{u}{2})} dy_1 = \frac{1}{(1+\frac{u}{2})^2} \cdot \frac{1}{2} \int_0^\infty w e^{-w} dw$$

$[y_1(1+\frac{u}{2}) = w]$

$$\frac{(1+\frac{u}{2})^{-1}}{-2 \cdot \frac{1}{2}} \Big|_0^\infty \overset{\checkmark}{=} 1$$

$$= \begin{cases} \frac{1}{2(1+\frac{u}{2})^2}, & u > 0 \\ 0, & \text{elsewhere} \end{cases}$$

$$= \begin{cases} \frac{2}{(2+u)^2}, & u > 0 \\ 0, & \text{elsewhere.} \end{cases}$$