Chapter 4 - 2-D motion

How do you represent motion in 2-D?

\[ \begin{align*}
\text{Convert } x, y, a \text{ to vectors} \\
\Delta x &= \bar{x}_f - \bar{x}_i \\
\Delta \bar{r} &= \bar{r}_f - \bar{r}_i \\
\text{with } \bar{r} &= x\hat{i} + y\hat{j} + z\hat{k} \\
\text{Show PPT Slide Fig. 4.1}
\end{align*} \]

\[ \begin{align*}
V_{\text{av}} &= \frac{\Delta \bar{r}}{\Delta t} \\
\rightarrow \quad V_{\text{av}} &= \frac{\Delta \bar{r}}{\Delta t} \\
\rightarrow \quad \vec{V} &= \lim_{\Delta t \to 0} \frac{\Delta \bar{r}}{\Delta t} = \frac{d\bar{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \\
\text{Show PPT Slide Fig. 4.2, 4.3}
\end{align*} \]

Note: If plot out \((x, y)\) position of particle, \(\vec{V}\) determined by 

tangent to path.

\[ \text{Likewise} \quad \vec{a} = \frac{d\vec{V}}{dt} = \frac{dV_x}{dt}\hat{i} + \frac{dV_y}{dt}\hat{j} + \frac{dV_z}{dt}\hat{k} \]

What happens to all of our 1-D equations?

\[ V = V_0 + at \quad \Rightarrow \quad \vec{V} = \vec{V}_0 + \vec{a}t \]
Parallel and perpendicular components of acceleration.

Let's examine effect of acceleration on particle.

![Fig 3.10](image)

Acceleration parallel to path of particle changes particle's speed. Acceleration perpendicular to velocity only changes direction $\Rightarrow$ circular motion.

![Fig 3.11](image)

What happens to all of our 1-D equations?

\[ V = V_0 + at \Rightarrow V + V_0 + at \]
$$x = x_0 + v_0 t + \frac{1}{2} a t^2 \quad \Rightarrow \quad \vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2} \vec{a} t^2$$

what does this vector equation really mean?

$$\begin{align*}
\vec{r} &= x \hat{i} + y \hat{j} \\
\vec{a} &= a_x \hat{i} + a_y \hat{j} \\
\vec{V}_0 &= V_{0x} \hat{i} + V_{0y} \hat{j}
\end{align*}$$

$$x \hat{i} + y \hat{j} = x_0 \hat{i} + y_0 \hat{j} + (V_{0x} \hat{i} + V_{0y} \hat{j}) t + \frac{1}{2} (a_x \hat{i} + a_y \hat{j}) t^2$$

combine vector components

$$\begin{align*}
\vec{r}_0 &= (x_0 - x_0 - V_{0x} t - \frac{1}{2} a_x t^2) \hat{i} \\
\quad &\quad + (y_0 - V_{0y} t - \frac{1}{2} a_y t^2) \hat{j}
\end{align*}$$

$$\begin{align*}
x &= x_0 + v_{0x} t + \frac{1}{2} a_x t^2 \\
y &= y_0 + v_{0y} t + \frac{1}{2} a_y t^2
\end{align*}$$

motion in $\hat{i}$ and $\hat{j}$ direction can be considered separately but coupled through time $t$. 

```nolines```
What about \( V^2 = V^2_0 + 2a(x-x_0) \) ?

\[
\Rightarrow \quad V_x = V^2_{x0} + 2a_x(x-x_0) \\
\quad \text{and} \\
V_y = V^2_{y0} + 2a_y(y-y_0)
\]

How do we choose "best" coordinate system?

I-Chiicker Question

It is not wrong to choose a), b) or c).

If you do, there is both \( a_x \) and \( a_y \).

\( \Rightarrow \) acceleration in 2D

* However if choose one axis in direction of acceleration, acceleration is orthogonal direction.

\( \Rightarrow \) zero

\[
x = x_0 + V^2_{x0} + a_x t^2 \\
y = y_0 + V^2_{y0} + a_y t^2
\]
Example: Projectile motion.

Note: By choice of coordinate system, \( \alpha_x = 0 \)

so:

\[
\begin{align*}
\dot{\ddot{x}} &= x_0 + v_{0x} t + \frac{1}{2} a_x t^2 \\
v_x &= v_{0x} + a_x t \\
v_x &= v_{0x} \quad \text{horizontal velocity, constant, independent of } y \\
\end{align*}
\]

\( \ddot{y} = v_{0y} - gt \quad \text{independent of } x \\
\]

Motion of \( x, y \) direction decoupled except through time variable.

Show Demo

\[
\begin{align*}
\text{doors} &\quad \text{down} \\
\text{which reaches ground first?}
\end{align*}
\]
Let's try some "standard" projectile motion problems.

For given initial magnitude of velocity and launch angle, what is $h$? and range $R$?

**I-Clicker Question**

At peak height $V_y = 0$

Divide motion into $x$, $y$ directions

- $y = y_0 + V_{0y}t - \frac{1}{2}gt^2$
- $h = y - y_0 = V_{0y}t - \frac{1}{2}gt^2$

need value for $t$
\[ V_y = V_{0y} - gt \]

\[ \Rightarrow \text{at Peak, } V_y = 0 \]

\[ t = \frac{V_{0y}}{g} \]

\[ h = V_{0y}^2 \left( \frac{V_{0y}}{g} \right) - \frac{1}{2} g \left( \frac{V_{0y}^2}{g^2} \right) \]

**Rearrange**

\[ h = \frac{1}{2} \frac{V_{0y}^2}{g} \]

**Note:** get same answer using

\[ v_y^2 = v_{0y}^2 - 2g(y - y_0) \]

What is \( v_{0y}^2 \)? \( v_{0y} = V_0 \sin \Theta \)

Now calculate range \( R = X - X_0 \)

\[ \Rightarrow \text{when projectile hits ground, } y - y_0 = 0 \]

Since \( y_0 = 0 \), \( y = 0 \)
\[ Y - Y_0 = 0 = V_{oy}t - \frac{1}{2} gt^2 \]

\[ 0 = (V_{oy} - \frac{1}{2} gt)t \leq \text{two roots} \]

Which do we discard?

\[ t = 0 \text{ is discarded... why?} \]

So \( V_{oy} = \frac{1}{2} gt \implies t = \frac{2V_{oy}}{g} \)

How far does the projectile move horizontally in this time?

\[ R = x - x_0 = V_{ox}t \]

\[ R = V_{ox} 2V_{oy} = \frac{2V_0 \cos \theta \cdot V_0 \sin \theta}{g} \]

\[ R = \frac{V_0^2 \sin \theta \cos \theta}{g} = \frac{V_0^2 \sin(2\theta)}{g} \]

What must launch angle be to maximize range?

maximize \( \sin(2\theta) = 1 \)

\[ \theta = \frac{\pi}{4} \implies 45^\circ \]
Calculus aside: find maximum of function

\[ \frac{\partial R}{\partial \theta} = 0 = \frac{V_0^2 \cos(2\theta)}{g} \]

\[ 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4} \]

\[ 2\theta \cos(2\theta) = 0 \]

Note: these equations for range only apply if launch height and final height are the same.

For a given range, is launch angle and velocity unique? No!

Another classic example \iff shooting a falling object \iff PPT Slide

how is this different than 

how do you aim at the falling Warthog?
Lesson 4.4 Uniform circular motion

$|v| = \text{constant}$.

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Since $|\vec{v}_i| = |\vec{v}_f|$, and $|\vec{v}_i| = |\vec{v}_f| = V$

then $\Delta \theta$ same for both $\vec{v}$ vectors and $\vec{r}$ vectors.

Geometry of arc length:

$s = R \phi$

so

$\Delta \phi = R \Delta \theta$

and

$\Delta \vec{v} = V \Delta \theta$

Define

$\Delta v_{\text{avg}} = \frac{|\Delta \vec{v}|}{\Delta t} = \frac{V \Delta \theta}{\Delta t} = \frac{V \Delta \phi}{\Delta t} = \frac{V}{R} \frac{\Delta \phi}{\Delta t}$

so

$\Delta v_{\text{avg}} = \frac{V^2}{R}$

Time to go around circle once

$T = \frac{2\pi R}{V}$

total distance

speed.
For many cases, we will consider uniform circular motion (since it's easy to solve since $|\vec{v}|$ is constant.) However, an object can still move in circular motion, but the velocity may not be constant and the total acceleration may not be towards center.

$$a_{rad} = a_\perp = \frac{v^2}{r} \quad a_{tang} = \frac{d|\vec{v}|}{dt} = a_\parallel$$

$$\vec{a} = -a_{rad} \hat{r} + a_{tang} \hat{t}$$

Relative Velocity $\Rightarrow$ how motion is viewed in one coordinate system relative to another.

$$\vec{r}_{P/A} = \vec{r}_{P/B} + \vec{r}_{B/A}$$

$$\frac{d}{dt} \vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A}$$

In frame of train (which is moving), passenger is walking at 1 m/s.
In frame of cyclist on platform, passenger moves at 3.2 m/s.

$\Rightarrow$ related by $\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A}$

*Velocity of train.*
Another projectile example

\[ Y - Y_0 = V_{0y} t - \frac{1}{2} gt^2 \]

\[ V_{0y} = V_0 \sin \theta \]

\[ H = V_{0y} t - \frac{1}{2} gt^2 \]

\[ 0 = \frac{1}{2} gt^2 - V_{0y} t + H \Rightarrow \text{quadratic equation} \]

\[ t = \frac{V_{0y} \pm \sqrt{V_{0y}^2 - 4 \frac{g}{2} H}}{g} = \frac{V_{0y} \pm \sqrt{V_{0y}^2 - 2gh}}{g} \]

Note: one gets two roots - which do you choose? Largest time since hit wall after maximum height.
What is the distance to easy wall D?

\[ D = x - x_0 = v_{ox} t \]

time that projectile hits wall.

\[ D = v_{ox} \left( \frac{v_{oy} + \sqrt{v_{oy}^2 - 2gH}}{g} \right) \]

with \( v_{ox} = v_0 \cos \theta \)