WK 14 - Oscillatory Motion

After an object is in a static equilibrium, what happens if we give it a little "push" to start vibrating?

Restoring force makes oscillatory motion possible

$\vec{F} = -k\vec{x}$

- force opposes displacement.

$F_s = -kx = ma_x$

$\alpha_x = -\frac{kx}{m}$

$\alpha = \frac{d\vec{v}_x}{dt} = \frac{d^2\vec{x}}{dt^2} = -\frac{kx}{m}$

$k$ has units of $\frac{1}{\text{m}}$

let $\omega \equiv \sqrt{\frac{k}{m}}$

$\frac{d^2\vec{x}}{dt^2} = -\omega^2 \vec{x}$
What is the solution to this second order differential equation?

\[ x(t) = A \cos(\omega t + \phi) \]

\[ \uparrow \text{constants} \]

**Proof:**

\[ \frac{dx}{dt} = -A \sin(\omega t + \phi) \omega \]

\[ \frac{d^2x}{dt^2} = -A \omega^2 \cos(\omega t + \phi) \]

\[ -A \omega^2 \cos(\omega t + \phi) = -\omega^2 \left[A \cos(\omega t + \phi)\right] \]

so \( x(t) = A \cos(\omega t + \phi) \) where \( \omega = \sqrt{\frac{k}{m}} \)

The motion is repetitive & in a time duration of a period.

\[ T = \frac{2\pi}{\omega} \]

the frequency of oscillation in Hertz

\[ f = \frac{1}{T} = \frac{\omega}{2\pi} \]
What about \( V(t) \) and \( a(t) \)?

\[
V = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi) \\
a = \frac{d^2x}{dt^2} = -\omega^2 A \cos(\omega t + \phi)
\]

\( V_{\text{max}} \)
\( a_{\text{max}} \)

How do you determine \( A \) and \( \phi \)? \( \Rightarrow \) Initial condition.

Example: A mass of 0.2 kg oscillates on a spring with \( K = 1000 \frac{N}{m} \). When \( t = 0 \), the mass is 0.01 m from its equilibrium position but the velocity is 0.1 m/s. What is \( x(t) \)?

\[
W = \sqrt{\frac{K}{m}} = \sqrt{\frac{1000}{0.2}} = \sqrt{5000}
\]

\[
x(t) = A \cos(\omega t + \phi) \\
x(0) = A \cos \phi = 0.01
\]

\[
x'(t) = V(t) = -A\omega \sin(\omega t + \phi)
\]

\[
V(0) = -A\omega \sin \phi = -1
\]

\[
-\frac{A\omega \sin \phi = -1}{A \cos \phi = 0.01} \Rightarrow -\frac{1}{0.01} \tan \phi = 10
\]

\[
\tan \phi = -10 \sqrt{5000}
\]

\[
A^2 \omega^2 \cos^2 \phi + A^2 \omega^2 \sin^2 \phi = (0.01)^2 (5000)^2 + (1)^2
\]

\[
A^2 \omega = 2500 + 0.01
\]

\[
A = \frac{50}{\sqrt{5000}}
\]
For frictionless surface, total energy of oscillator conserved.

\[ E = KE + PE = \frac{1}{2} m v^2 + \frac{1}{2} k x^2 \]

\[ x(t) = A \cos(\omega t + \phi) \quad v(t) = -A\omega \sin(\omega t + \phi) \]

\[ E = \frac{1}{2} m A^2 \omega^2 \sin^2(\omega t + \phi) + \frac{1}{2} k A^2 \cos(\omega t + \phi) \]

\[ \omega^2 = \frac{k}{m} \]

\[ E_{kin} = \frac{1}{2} k A^2 \]

\[ \frac{1}{2} k A^2 = \frac{1}{2} m v^2 + \frac{1}{2} k x^2 \]

Solve for \( v \) \( \Rightarrow \)

\[ v = \pm \sqrt{\frac{k}{m}} (A^2 - x^2) \]

\[ v = \pm \omega \sqrt{A^2 - x^2} \]

Corresponds to velocities and positions.

\[ \text{PPT - Action} \]

\[ x \text{ times 3} \]

Simple harmonic motion is related to uniform circular motion.

Easiest way to visualise connection. \( \Rightarrow \)

\[ \text{PPT Action} \]

\[ x(t) = A \cos(\omega t + \phi) \]

\[ y(t) = A \sin(\omega t + \phi) \]

Both \( x(t) \), \( y(t) \) are in simple harmonic motion.

\[ x^2 + y^2 = (t) = A^2 \cos^2(\omega t + \phi) + A^2 \sin^2(\omega t + \phi) = A^2 \]

\[ \Rightarrow r \text{ fixed in time}, \]
Taylor Series \( f(x) \) near \( x = a + \Delta x \):

\[
f(x) = f(a) + \frac{f'(a)}{1!} \Delta x + \frac{f''(a)}{2!} \Delta x^2 + \frac{f'''(a)}{3!} \Delta x^3 + \ldots
\]

For example, \( \sin(x) \) near \( x = 0 \):

\( \sin(x) \approx x - \frac{x^3}{3} + \ldots \)

The pendulum:

\[ T - mg \cos \theta = \frac{mv^2}{r} \]

\[ -mg \sin \theta = ma_\theta = m \frac{d^2s}{dt^2} \]

\[ S = L \Theta \]

The force acts opposite to increase \( \theta \) as \( s \).

\[ \frac{d\dot{\theta}}{dt} = -\frac{g}{L} \sin \theta \approx -\frac{g}{L} \theta \] for small angles

\[ \frac{d\theta}{dt^2} = -\frac{g}{L} \theta \]

\[ \omega = \frac{g}{L} \]

\[ \Theta(t) = \Theta_0 \cos(\omega t + \phi) \]
What about an extended object? (Physical Pendulum)

\[
\vec{r} = \vec{r} \times \vec{F}
\]

\[
z = \frac{L}{2} \, mg \sin \theta = \frac{L}{2} \, I \frac{d^2 \theta}{dt^2}
\]

\[
\frac{d^2 \theta}{dt^2} = \frac{-L \, mg \, \sin \theta}{2I} = -\frac{Lmg}{2I} \theta.
\]

\[
\omega^2 = \frac{Lmg}{2I} = \frac{Lmg}{2 \frac{1}{3}ML^2} = \frac{3g}{2L}
\]

Let's try a more complicated problem (PPT)

In equilibrium

\[
\Sigma F = 0
\]

\[
mg = F_p + k \Delta x
\]

\[
\Sigma \tau = 0
\]

\[
\frac{L}{2} \, mg - Lk \Delta x = 0
\]

Find frequency of small oscillations.
\[ \Delta x = \frac{mg}{2k} \]

Now consider small oscillations.

\[ \xi = -I \xi \]

\[ LF_s - \frac{L}{2} mg = -I \frac{d^2 \xi}{dt^2} \]

\[ F_s = L k (\Delta x + \Delta \xi) - \frac{L}{2} mg = -I \frac{d^2 \theta}{dt^2} \]

\[ L \left( \frac{K \Delta x - \frac{1}{2} mg}{2} \right) + L k \Delta \xi = -I \frac{d^2 \theta}{dt^2} \]

\[ = 0 \]

Static equilibrium.

\[ L k \Delta s = -I \frac{d^2 \theta}{dt^2} \]

\[ \Delta s = \theta L \]

\[ K \theta L^2 = -I \frac{d^2 \theta}{dt^2} \]

\[ \frac{d^2 \theta}{dt^2} = - \frac{KL^2}{I} \theta \]

\[ \omega^2 = \frac{KL^2}{I} = \frac{KL^2}{\frac{1}{3} I M L^2} = \frac{3k}{m} \]

\[ \omega = \sqrt{\frac{3k}{m}} \]