PHYSICS 111 HOMEWORK SOLUTION #10

April 10, 2013

0.1

Given $\vec{M} = 4\vec{i} + \vec{j} - 3\vec{k}$ and $\vec{N} = \vec{i} - 2\vec{j} - 5\vec{k}$, calculate the vector product $\vec{M} \times \vec{N}$.

By simply following the rules of the cross product:

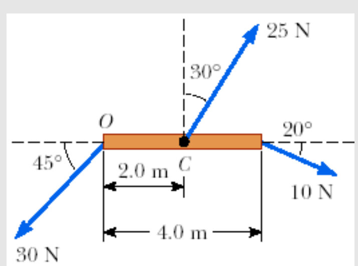
$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$$

$$\vec{i} \times \vec{j} = \vec{k} = -\vec{j} \times \vec{i}$$
$$\vec{j} \times \vec{k} = \vec{i} = -\vec{k} \times \vec{j}$$
$$\vec{k} \times \vec{i} = \vec{j} = -\vec{i} \times \vec{k}$$

$$\begin{aligned}\vec{M} \times \vec{N} &= (4\vec{i} + \vec{j} - 3\vec{k}) \times (\vec{i} - 2\vec{j} - 5\vec{k}) \\ &= -8\vec{k} + 20\vec{j} - \vec{k} - 5\vec{i} - 3\vec{j} - 6\vec{i} \\ &= -11\vec{i} + 17\vec{j} - 9\vec{k}\end{aligned}$$

0.2

Calculate the net torque (magnitude and direction) on the beam in the figure below about the following axes.



We will choose clockwise as our positive direction and apply the formula for a torque:

$$\vec{\tau}_{net} = \sum \vec{F}_i \times \vec{r}_i$$
$$\tau_{net} = \sum F_i r_i \sin\theta_i$$

a) About the O-axis:

$$\tau_{net} = -25 \times 2 \times \sin 60 + 10 \times 4 \times \sin 20 + 0$$
$$= -29.6 \text{ N.m}$$

This net torque is counterclockwise

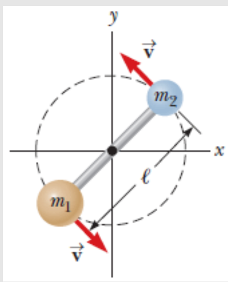
b) About the C-axis:

$$\tau_{net} = 0 + 10 \times 2 \times \sin 20 - 30 \times 2 \times \sin 45$$
$$= -35.6 \text{ N.m}$$

This net torque is again counterclockwise

0.3

A light, rigid rod of length l = 1.00 m joins two particles, with masses $m_1 = 4.00$ kg and $m_2 = 3.00$ kg, at its ends. The combination rotates in the xy plane about a pivot through the center of the rod (see figure below). Determine the angular momentum of the system about the origin when the speed of each particle is 2.00 m/s.



Angular momentum of the system :

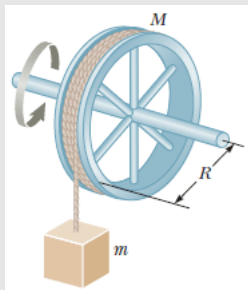
$$
\begin{aligned}
\vec{L} &= \vec{r} \times \vec{P} \\
&= \vec{r} \times m\vec{v} \\
&= \frac{l}{2}(m_1 + m_2)v\ \vec{z} \\
&= \frac{1}{2} \times (4+3) \times 2\ \vec{z} \\
&= 7\vec{z}\ \text{Kg.m}^2/\text{s}
\end{aligned}
$$

Angular momentum is on the $\vec{z}$ direction.

N.B. The right hand rule is of great help to visualize the torque (and any cross product) direction. In this case $\vec{r}$ and $\vec{v}$ are in the plane of the figure, the torque cross product must be oriented perpendicular to the plane.

0.4

A counterweight of mass m = 4.40 kg is attached to a light cord that is wound around a pulley as shown in the figure below. The pulley is a thin hoop of radius R = 9.00 cm and mass M = 2.50 kg. The spokes have negligible mass.



- a) What is the net torque on the system about the axle of the pulley?
- b) When the counterweight has a speed v, the pulley has an angular speed = v/R. Determine the magnitude of the total angular momentum of the system about the axle of the pulley.
- Using your result from (b) and $\vec{\tau} = \frac{d\vec{L}}{dt}$, calculate the acceleration of the counterweight. (Enter the magnitude of the acceleration.)

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a)

The system about the axle of the pulley is under the torque applied by the cord. At rest, the tension in the cord is balanced by the counterweight $T = mg$. If we choose the rotation axle towards a certain $\vec{z}$, we should have:

$$\begin{aligned}
\vec{\tau}_{net} &= \vec{R} \times \vec{T} \\
&= Rmg\vec{z} \\
&= 0.09 \times 4.40 \times 9.8\vec{z} \\
&= 3.88\ \vec{z}
\end{aligned}$$

The net torque has a magnitude of $\tau = 3.88$N.m and its direction is along the rotation axis towards the right in the figure.

b)

Taking into account rotation of the pulley and translation of the counterweight , the total angular momentum of the system is:

$$\begin{aligned}
\vec{L} &= \vec{R} \times m\vec{v} + I\vec{\omega} \\
L &= mRv + MR\frac{v}{R} \\
&= (m+M)Rv \\
&= (4.40 + 2.50) \times 0.09 \\
&= 0.621 \text{ Kg.m}
\end{aligned}$$

c)

$$\begin{aligned}
\tau &= \frac{dL}{dt} \\
mgR &= (M+m)R\frac{dv}{dt} \\
&= (M+m)Ra \\
a &= \frac{mg}{m+M} \\
&= \frac{4.40 \times 9.8}{6.90} \\
&= 6.25 \text{ m/s}^2
\end{aligned}$$

0.5

A uniform solid disk of mass m = 2.94 kg and radius r = 0.200 m rotates about a fixed axis perpendicular to its face with angular frequency 6.02 rad/s.

- a)Calculate the magnitude of the angular momentum of the disk when the axis of rotation passes through its center of mass.
- b)What is the magnitude of the angular momentum when the axis of rotation passes through a point midway between the center and the rim?

a)

$$\begin{aligned}\vec{L} &= I\vec{\omega} \\ L &= \frac{1}{2}mr^2\omega \\ &= \frac{1}{2} \times 2.94 \times 0.2^2 \times 6.02 \\ &= 0.354 \text{ Kg.m}^2/\text{s}\end{aligned}$$

b)

If the rotation axis is shifted to a point midway the center and the rim, the moment of inertia will change from $\frac{1}{2}mr^2$ to $\frac{1}{2}mr^2 + m(\frac{r}{2})^2 = \frac{3}{4}mr^2$. The angular momentum will change to:

$$\begin{aligned}L &= \frac{3}{4}mr^2\omega \\ &= \frac{3}{4} \times 2.94 \times 0.2^2 \times 6.02 \\ &= 0.531 \text{ Kg.m}^2/\text{s}\end{aligned}$$

0.6

Model the Earth as a uniform sphere. Calculate the angular momentum of the Earth due to its spinning motion about its axis.

- a)Calculate the angular momentum of the Earth due to its spinning motion about its axis.
- b) Calculate the angular momentum of the Earth due to its orbital motion about the Sun.
- c) Explain why the answer in part (b) is larger than that in part (a) even though it takes significantly longer for the Earth to go once around the Sun than to rotate once about its axis.

a)

Earth as a solid sphere will have a moment of inertia of $\frac{2}{5}MR^2$, it's mean radius R is about 6378 Km, with a mass M of 5.9736 $\times 10^{24}$Kg, While spinning it makes a whole revolution in 24 hours. Angular momentum due to spinning is thus,

$$\begin{aligned} L &= I\omega \\ &= \frac{2}{5}MR^2\omega \\ &= \frac{2}{5} \times 5.9736 \times 10^{24} \times 6378000^2 \frac{2\Pi}{24 \times 3600} \\ &= 7.07 \times 10^{33} \text{ kg.m}^2/\text{s} \end{aligned}$$

Spinning is about the north celestial pole.

b)

For the motion about the sun, the moment of inertia will shift from $\frac{2}{5}MR^2$ to $\frac{2}{5}MR^2 + Md^2$, where d is the distance from earth to the sun averaging

149.60×10^9 m, completing its revolution within 365 days.

$$
\begin{aligned}
L &= I\omega \\
&= (\frac{2}{5}R^2 + d^2)M\omega \\
&= (\frac{2}{5} \times 6378^2 \times 10^6 + 149.60^2 \times 10^{18}) \times 5.9736 \times 10^{24} \frac{2\Pi}{365 \times 24 \times 3600} \\
&= 2.66 \times 10^{40} \text{ kg.m}^2/\text{s}
\end{aligned}
$$

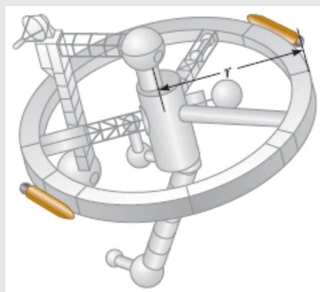
The direction of this orbital motion is towards the north ecliptic pole.

c)

The periods differ only by a factor of 365 (365 days for orbital motion to 1 day for rotation). Because of the huge distance from the Earth to the Sun, however, the moment of inertia of the Earth around the Sun is six orders of magnitude larger than that of the Earth about its axis.

0.7

A space station is constructed in the shape of a hollow ring of mass 5.35×10^4 kg. Members of the crew walk on a deck formed by the inner surface of the outer cylindrical wall of the ring, with radius 130 m. At rest when constructed, the ring is set rotating about its axis so that the people inside experience an effective free-fall acceleration equal to g. (See figure below.) The rotation is achieved by firing two small rockets attached tangentially to opposite points on the rim of the ring.



- a)What angular momentum does the space station acquire?
- b) For what time interval must the rockets be fired if each exerts a thrust of 110 N?

a)

We need the passengers to experience a free-fall acceleration equal to g i-e. $a = \frac{v^2}{r} = r\omega^2 = g$ is the acceleration. The angular momentum that the space station acquires is then:

$$L = I\omega$$

$$
\begin{aligned}
&= mr^2\sqrt{\frac{g}{r}} \\
&= m\sqrt{gr^2} \\
&= 5.35 \times 10^4\sqrt{\frac{9.81}{130}} \\
&= 2.48 \times 10^8 \text{ kg.m}^2/\text{s}
\end{aligned}
$$

b)

If the two rockets supply a thrust of F=110 N each, we can write Newton 2^{nd} law as follows:

$$
\begin{aligned}
2Fr &= I\alpha \\
&= mr^2\alpha \\
\alpha &= \frac{2F}{mr}
\end{aligned}
$$

The average time interval is then calculated from α and $\omega = \sqrt{\frac{g}{r}}$:

$$
\begin{aligned}
t &= \frac{\omega}{\alpha} \\
&= \frac{mr}{2F}\sqrt{\frac{g}{r}} \\
&= \frac{m\sqrt{rg}}{2F} \\
&= 8680 \text{ s}
\end{aligned}
$$

0.8

A playground merry-go-round of radius R = 1.60 m has a moment of inertia I = 255 kg $.m^2$ and is rotating at 9.0 rev/min about a frictionless vertical axle. Facing the axle, a 22.0-kg child hops onto the merry-go-round and manages to sit down on the edge. What is the new angular speed of the merry-go-round?

Without the child the merry-go-round has a moment of inertia I which will change to $I' = I + mr^2$ when the child hops onto the edge. However, the moment of inertia should be conserved.

$$\begin{aligned} L' &= L \\ I'\omega' &= I\omega \\ \omega' &= \frac{I}{I'}\omega \\ &= \frac{255}{255 + 22 \times 1.60^2} \times 9 \\ &= 7.37 \text{ rev/s} \end{aligned}$$

0.9

A uniform cylindrical turntable of radius 1.80 m and mass 26.1 kg rotates counterclockwise in a horizontal plane with an initial angular speed of 4π rad/s. The fixed turntable bearing is frictionless. A lump of clay of mass 2.39 kg and negligible size is dropped onto the turntable from a small distance above it and immediately sticks to the turntable at a point 1.70 m to the east of the axis.

- a) Find the final angular speed of the clay and turntable.
- b) Is mechanical energy of the turntable-clay system constant in this process? What, if any, is the change in internal energy?
- c) Is momentum of the system constant in this process? What, if any, is the amount of impulse imparted by the bearing?

a)

The lump of clay of negligible size will only change the moment of inertia by a shift of mr^2, the angular momentum will be conserved in this inelastic process:

$$\begin{aligned} L &= L' \\ I\omega &= I'\omega' \\ \frac{MR^2}{2}\omega &= (\frac{MR^2}{2} + mr^2)\omega' \\ \omega' &= \frac{MR^2/2}{MR^2/2 + mr^2} \\ &= \frac{1}{1 + \frac{2m}{M}\frac{r^2}{R^2}} \\ &= \frac{1}{1 + \frac{2\times 2.39}{26.1}\frac{1.70^2}{1.80^2}} \\ &= 10.80 \text{ rad/s} \end{aligned}$$

Its direction is counterclockwise

b)

The inelastic colliding and sticking of the lump of clay will not allow mechanical energy conservation. And there will be a loss in internal energy. This loss can be calculated by taking the difference in kinetic energy:
Let's calculate the moments of inertia first:

$$I = \frac{MR^2}{2} = 42.28 \text{ kg.M}^2$$

$$I' = I + mr^2 = 42.28 + 2.39 \times 1.70^2 = 49.18 \text{ kg.m}^2$$

$$\begin{aligned} \Delta Ki &= \frac{1}{2}I'\omega'^2 - \frac{1}{2}I\omega^2 \\ &= \frac{1}{2}(49.18 \times 10.80^2 - 42.28 \times (16\pi^2)) \\ &= -469.8 \text{ J} \end{aligned}$$

c)

The linear momentum of the system is not conserved and the impulse imparted by the bearing is just the change in linear momentum which corresponds to the momentum gained by the lump of clay as it sticks to the rotating turntable:

$$\begin{aligned} Impulse &= mv' \\ &= mr\omega' \\ &= 2.39 \times 1.70 \times 10.80 \\ &= 43.9 \text{ kg.m/s} \end{aligned}$$

Once it sticks on the east side of the turntable, the rotation is being counter-clockwise, the bearing will take direction towards north.

0.10

A student sits on a freely rotating stool holding two dumbbells, each of mass 3.02 kg (see figure below). When his arms are extended horizontally (figure a), the dumbbells are 0.99 m from the axis of rotation and the student rotates with an angular speed of 0.749 rad/s. The moment of inertia of the student plus stool is 2.61 kg m^2 and is assumed to be constant. The student pulls the dumbbells inward horizontally to a position 0.296 m from the rotation axis (figure b).

- a)Find the new angular speed of the student.
- b) Find the kinetic energy of the rotating system before and after he pulls the dumbbells inward.

a)

Let's first calculate the moments of inertia.

- First position:

$$\begin{aligned} I_i &= I + 2mr_i^2 && (1)\\ &= 2.61 + 2 \times 3.02 \times 0.99^2 && (2)\\ &= 8.53 \text{ kg.m}^2 && (3) \end{aligned}$$

- Second position

$$\begin{aligned} I_f &= I + 2mr_f^2 && (4)\\ &= 2.61 + 2 \times 3.02 \times 0.296^2 && (5)\\ &= 3.14 \text{ kg.m}^2 && (6) \end{aligned}$$

Angular momentum is conserved:

$$\begin{aligned} L_f &= L_i \\ I_i\omega_i &= I_f\omega_f \\ \omega_f &= \frac{I_i}{I_f}\omega \\ &= \frac{8.53}{3.14} \times 0.749 \\ &= 2.04 \text{ rad/s} \end{aligned}$$

- Before:

$$\begin{aligned} K &= \frac{1}{2}I_i\omega_i^2 \\ &= \frac{1}{2} \times 8.53 \times 0.749^2 \\ &= 2.39 \text{ J} \end{aligned}$$

- After:

$$\begin{aligned} K &= \frac{1}{2}I_f\omega_f^2 \\ &= \frac{1}{2} \times 3.14 \times 2.04^2 \\ &= 6.53 \text{ J} \end{aligned}$$