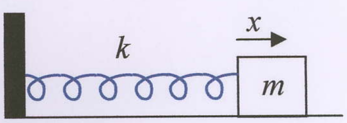
Physics 430, Classical Mechanics Name Solution
Exam 2, 2010 Nov 09

Instructions: No books, notes, or "cheat sheet" allowed. You may use a calculator, but no other electronic devices during the exam. Please turn your cell phone off.

Please note that the NJIT integrity code applies to this exam, as it does to all activities related to this course.

Each part of each question is worth the number of points noted, for a total of 90 points. You may use the back of the paper if you need more room to work. Clearly label the portion of problem on the back with the problem number. Show all work. Right answers with no work will be marked wrong.

1. (a) Consider a mass m on a massless spring (spring constant k), with one end fixed. The mass moves on a horizontal surface. Considering Newton's 2nd Law, what is the equation of motion for the mass position x relative to its equilibrium position, ignoring friction? (2 points)



$$\ddot{x} = -\frac{k}{m}x$$

(b) What is the frequency of oscillation, ω_0, of the undamped system, in terms of k and m? (2 points)

$$\omega_0 = \sqrt{\frac{k}{m}}$$

(c) When friction is included, one way to write the general solution is $x(t) = e^{-\beta t}\left[B_1 \cos(\omega_1 t) + B_2 \sin(\omega_1 t)\right]$. Solve for the unknown constants B_1 and B_2 when the mass is pushed a distance a to the *left*, and released from rest. Insert your results back into the above expression to give $x(t)$ for these initial conditions. (5 points)

$$x(0) = -a$$
$$v(0) = 0$$
$$x(0) = B_1 = -a$$
$$v(t) = -\beta e^{-\beta t}\left[B_1 \cos(\omega_1 t) + B_2 \sin(\omega_1 t)\right] + e^{-\beta t}\left[\omega_1 B_1 \sin \omega_1 t - \omega_1 B_2 \cos \omega_1 t\right]$$
$$v(0) = -\beta B_1 - \omega_1 B_2 = 0 \Rightarrow B_2 = -\frac{\beta a}{\omega_1}$$
$$x(t) = -e^{-\beta t}\left[\frac{\beta a}{\omega_1} \sin(\omega_1 t) + a\cos(\omega_1 t)\right]$$

2. Driven damped oscillators have amplitude A given by $A^2 = \dfrac{f_o^2}{(\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2}$,

where f_o is the driving amplitude, ω is the driving frequency, and ω_o and β are the natural frequency of oscillation and decay parameter of the undriven oscillator, respectively. The Q of the oscillator is defined as $Q = \omega_o/2\beta$.

(a) What is A^2 at resonance, in terms of f_o, ω_o and Q? (2 points)

At resonance, $\omega = \omega_o$

$\Rightarrow A^2 = \dfrac{f_o^2}{4\beta^2\omega_o^2}$ but $\beta = \dfrac{\omega_o}{2Q}$ $\Rightarrow A^2 = \dfrac{Q^2 f_o^2}{\omega_o^4}$

(b) The driving frequency is now shifted off resonance by β (i.e. $\omega = \omega_o + \beta$).

Expand $A^2 = \dfrac{f_o^2}{(\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2}$ for small β, i.e. $\beta/\omega_o \ll 1$, to show that A^2 is reduced by a factor of two. (5 points)

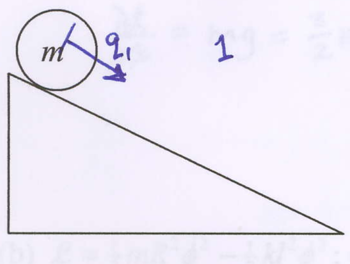
$$(\omega_o^2 - \omega^2)^2 = \left(\omega_o^2\left[1 - \underbrace{\left(1 + \frac{\beta}{\omega_o}\right)^2}\right]\right)^2$$

$$\omega_o^4\left[1 - 1 + \frac{2\beta}{\omega_o}\right]^2 = 4\beta^2\omega_o^2$$

So $A^2 = \dfrac{f_o^2}{8\beta^2\omega_o^2} = \dfrac{f_o^2 Q^2}{2\omega_o^4}$ which is half of what it was in part a

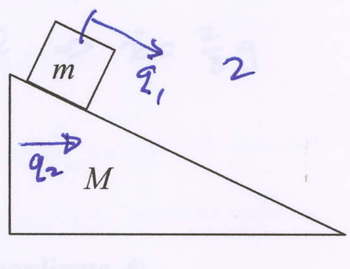
3. The first step in solving a problem using the Lagrange method is to identify generalized coordinates. For each system shown, identify how many coordinates are needed, and sketch them on the drawings. (4 points each)

(a)



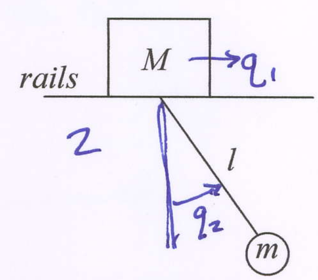
A disk rolling without slipping down a fixed incline

(b)



A block sliding without friction down an incline that is also free to slide without friction.

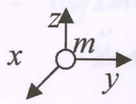
(c)



A pendulum attached to a cart that is free to slide on rails without friction.

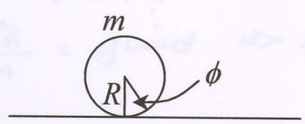
4. The second step is to determine the energies, T and U, and hence the Lagrangian $\mathcal{L}$, for the problem. Find T, U and $\mathcal{L}$ for each of the systems shown below, in terms of the generalized coordinates shown.

(a)



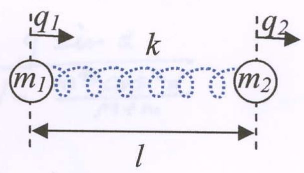
A free point particle able to move in any direction (unconstrained).

(b)



A disk rolling without slipping on a flat surface. Generalized coordinate is angle ϕ. Moment of inertia for a disk is ½ mR^2.

(c)



Two different masses attached to a spring of equilibrium length l.

(a) (5 points)

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \qquad U = mgz$

$\mathcal{L} = T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$

(b) (5 points)

$T = \frac{1}{2}mR^2\dot{\phi}^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\dot{\phi}^2 = \frac{3}{4}mR^2\dot{\phi}^2$

$U = 0$

$\mathcal{L} = T - U = \frac{3}{4}mR^2\dot{\phi}^2$

(c) (5 points)

$T = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2$

$U = \frac{1}{2}k(q_1 - q_2) \quad$ or $\quad U = \frac{1}{2}k(q_1 - q_2 + l)$

$\mathcal{L} = T - U = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 - \frac{1}{2}k(q_1 - q_2)$

5. For each of the Lagrangians given below, what is the equation of motion for each coordinate listed? (5 points each)

(a) $\mathcal{L} = \frac{3}{4}m\dot{x}^2 + mg(x - l)$; (coordinate x)

$\frac{\partial \mathcal{L}}{\partial x} = mg = \frac{3}{2}m\ddot{x} \Rightarrow \ddot{x} = \frac{2}{3}g$

(b) $\mathcal{L} = \frac{1}{2}mR^2\dot{\phi}^2 - \frac{1}{2}kl^2\phi^2$; (coordinate ϕ)

$\frac{\partial \mathcal{L}}{\partial \phi} = -kl^2\phi = mR^2\ddot{\phi} \Rightarrow \ddot{\phi} = -\frac{k}{m}\frac{l^2}{R^2}\phi$

(c) $\mathcal{L} = \frac{1}{2}(M+m)\dot{X}^2 + \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\dot{X}\cos\alpha) + mgx\sin\alpha$; (coordinates x and X)

$$\frac{\partial \mathcal{L}}{\partial \dot{X}} = 0 = (M+m)\ddot{X} + m\ddot{x}\cos\alpha \Rightarrow \ddot{X} = -\frac{m\ddot{x}\cos\alpha}{M+m} = -\frac{mg\sin\alpha\cos\alpha}{M+m(1-\cos^2\alpha)}$$

$$= -\frac{mg\sin\alpha\cos\alpha}{M+m\sin^2\alpha}$$

$$\frac{\partial \mathcal{L}}{\partial x} = mg\sin\alpha = m\ddot{x} + m\ddot{X}\cos\alpha$$

$$\ddot{x} - \frac{m\ddot{x}\cos^2\alpha}{M+m} = g\sin\alpha \Rightarrow \ddot{x} = \frac{g\sin\alpha}{1 - \frac{m\cos^2\alpha}{M+m}}$$

6. For each of the Lagrangians given below, which coordinate is ignorable? Give the equation of motion for the ignorable coordinate and show that it leads to a conservation relation. (4 points each)

(a) $\mathcal{L} = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}(m_1+m_2)\dot{x}^2 - \frac{1}{2}kx^2$; (coordinates X and x)

X is ignorable

$$\frac{\partial \mathcal{L}}{\partial \dot{X}} = M\dot{X} = \text{constant} \Rightarrow X \text{ momentum is conserved}$$

(b) $\mathcal{L} = \frac{1}{2}MR^2\dot{\phi}^2 + \frac{1}{2}(M+m)\dot{x}^2 - \frac{1}{2}mgl\cos\phi$; (coordinates x and ϕ)

x is ignorable

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = (M+m)\dot{x} = \text{constant} \Rightarrow x \text{ momentum is conserved}$$

(c) $\mathcal{L} = \frac{1}{2}(M+m)\dot{q}_1^2 + \frac{1}{2}m(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2\cos\alpha) + mgq_2\sin\alpha$; (coordinates q_1 and q_2)

[this is from prob. 3b]

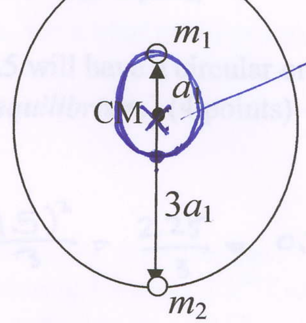
q_1 is ignorable

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_1} = (M+m)\dot{q}_1 + m\dot{q}_2\cos\alpha = \text{constant}$$

$\Rightarrow$ horizontal momentum is conserved.

7. The orbits of two stars about their common center of mass are ellipses with apastron (greatest) distances a_1 and $a_2 = 3a_1$, as shown in the figure below.
(a) The positions of both stars are shown. Carefully sketch the orbit of the other star. (2 points)

draw X

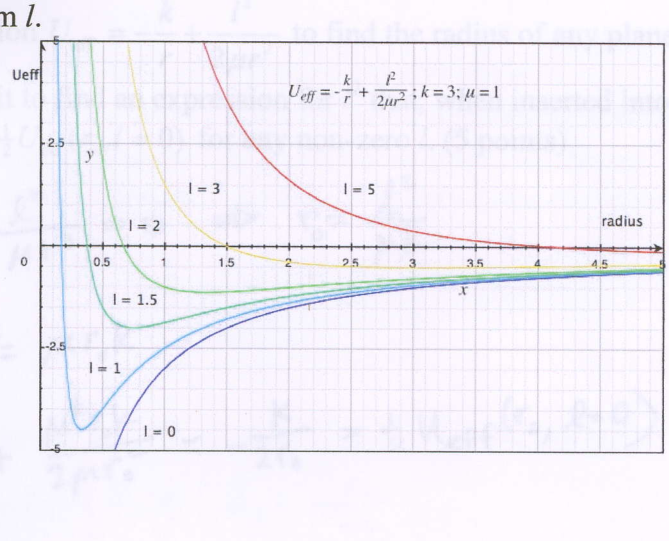


(b) If the mass of the more massive star is m_1, what is the mass of the less massive star in terms of m_1? What is the reduced mass μ and apastron distance, in terms of m_1 and a_1, for the equivalent *one-body problem* where the star of mass m_1 is considered fixed? (4 points)

$\frac{m_1}{m_2} = \frac{r_2}{r_1} = \frac{3a_1}{a_1} = 3$ $\quad \mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{\frac{1}{3}m_1^2}{\frac{4}{3}m_1} = \frac{1}{4}m_1$

$\Rightarrow m_2 = \frac{1}{3}m_1$ $\quad$ apastron distance is $a_1 + 3a_1 = 4a_1$

8. The curves in the figure below show the effective potential function for a planet orbiting a star, $U_{\text{eff}} = -\frac{k}{r} + \frac{l^2}{2\mu r^2}$, where $k = GMm = 3$, and $\mu = 1$, for various values of angular momentum l.



(a) What is U_{eff} for a planet with $l = 0$ [call it $U_{\text{eff}}(l=0)$]? What sort of "orbit" would such a planet have? (2 points)

$$U_{\text{eff}}(l=0) = -\frac{k}{r} = -\frac{3}{r}$$

With $l=0$, the planet has no perpendicular motion and would simply fall straight into the star.

(b) Calculate the radius r_o at which a planet with $l = 1.5$ will have a circular orbit. [*Hint: This is the point where $U_{\text{eff}}(l = 1.5)$ has an equilibrium.*] (4 points)

$$U_{\text{eff}}(l=1.5) = -\frac{3}{r} + \frac{(1.5)^2}{2r^2}$$

$$\frac{dU_{\text{eff}}}{dr} = \frac{3}{r^2} - \frac{(1.5)^2}{r^3} = 0 \Rightarrow r_o = \frac{(1.5)^2}{3} = \frac{2.25}{3} = 0.75$$

(c) What is the value of $U_{\text{eff}}(r_o, l = 1.5)$ at this radius? Compare with the value of $U_{\text{eff}}(r_o, l = 0)$ at the same radius, to show $U_{\text{eff}}(r_o, l = 1.5) = \frac{1}{2}U_{\text{eff}}(r_o, l = 0)$. (4 points)

$$U_{\text{eff}}(r_o, l=1.5) = -\frac{3}{0.75} + \frac{(1.5)^2}{2(0.75)^2} = -4 + \frac{(2.25)8}{9} = -2$$

$$U_{\text{eff}}(r_o, l=0) = -\frac{3}{0.75} = -4$$

Hence $U_{\text{eff}}(r_o, l=1.5) = \frac{1}{2}U_{\text{eff}}(r_o, l=0)$

(d) Use the general expression $U_{\text{eff}} = -\frac{k}{r} + \frac{l^2}{2\mu r^2}$ to find the radius of any planet in a circular orbit. Use it to find an expression for l^2 that, when inserted into U_{eff}, shows $U_{\text{eff}}(r_o, l) = \frac{1}{2}U_{\text{eff}}(r_o, l = 0)$ for any non-zero l. (5 points)

$$\frac{dU_{\text{eff}}}{dr} = \frac{k}{r^2} - \frac{l^2}{\mu r^3} = 0 \Rightarrow r_o = \frac{l^2}{\mu k}$$

$$l^2 = \mu r_o k$$

$$U_{\text{eff}}(r_o, l) = -\frac{k}{r_o} + \frac{\mu r_o k}{2\mu r_o^2} = -\frac{k}{2r_o} = \frac{1}{2}U_{\text{eff}}(r_o, l=0)$$