Physics 430, Classical Mechanics **Name _________________________**

Exam 1, 2010 Oct 05

Instructions: No books, notes, or "cheat sheet" allowed. You may use a calculator, but no other electronic devices during the exam. Please turn your cell phone off.

Please note that the NJIT honor code applies to this exam, as it does to all activities related to this course.

Each part of each question is worth 5 points, as noted, for a total of 80 points. You may use additional sheets of paper if you need more room to work. Clearly label the portion of each additional sheet with the problem number. Show all work. Right answers with no work will be marked wrong. **Be careful with notation** (e.g. vectors underlined, unit vectors with hats, etc.).

1. Given the two vectors **a** = (2, −1, −1) and **b** = (0, 0, 5)

   (a) (5 points) Find the dot product and use it to find the angle between the two vectors.

   Start with definition: $\mathbf{a}\cdot\mathbf{b} = ab\cos\theta$

   $$\mathbf{a}\cdot\mathbf{b} = a_x b_x + a_y b_y + a_z b_z = 2\times 0 - 1\times 0 - 1\times 5 = -5$$

   $$a = \sqrt{2^2 + (-1)^2 + (-1)^2} = \sqrt{6} \quad \text{and} \quad b = \sqrt{0^2 + 0^2 + 5^2} = 5$$

   $$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{ab} = -\frac{5}{5\sqrt{6}} = -0.4082 \quad \text{so} \quad \theta = 114.1^\circ$$

   (b) (5 points) Find the cross product and use its magnitude to find the (same) angle between the two vectors. If you get a different answer from (a), state why, and say which is the correct angle.

   Start with definition: $|\mathbf{a}\times\mathbf{b}| = ab\sin\theta$

   $$\mathbf{a}\times\mathbf{b} = (a_y b_z - a_z b_y)\hat{\mathbf{x}} + (a_z b_x - a_x b_z)\hat{\mathbf{y}} + (a_x b_y - a_y b_z)\hat{\mathbf{z}}$$

   $$= (-1\times 5 - 1\times 0)\hat{\mathbf{x}} + (-1\times 0 - 2\times 5)\hat{\mathbf{y}} + (2\times 0 - 1\times 0)\hat{\mathbf{z}} = -5\hat{\mathbf{x}} - 10\hat{\mathbf{y}}$$

   $$|\mathbf{a}\times\mathbf{b}| = \sqrt{(-5)^2 + (-10)^2} = \sqrt{125}$$

   $$\sin\theta = \frac{\mathbf{a}\times\mathbf{b}}{ab} = -\frac{5\sqrt{5}}{5\sqrt{6}} = 0.9129 \quad \text{so} \quad \theta = 65.9^\circ$$

   This is different from part (a), because of 180° ambiguity.

   The correct angle is 114.1° as in part (a).

2. A particle's potential energy is $U(\mathbf{r}) = x(y + z) - y^2$.

   (a) (5 points) What is the (vector) force on the particle?

   $$\mathbf{F} = -\nabla U = -\left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}\right)U(\mathbf{r}) = -\left[(y+z)\hat{\mathbf{x}} + (x-2y)\hat{\mathbf{y}} + x\hat{\mathbf{z}}\right]$$

   $$\text{or} \quad \mathbf{F} = -(y+z, x-2y, x)$$

(b) (5 points) What is the work $W = \int_1^2 \mathbf{F}\bullet\mathbf{dr}$ done against the force you derived in part (a) in going from the point (0,0,0) to the point (2,1,0) along a straight line path as shown in the Figure 1? Show your work. *[Hint: Write down the equation for the line, y in terms of x, then find dy in terms of dx. You can then convert the integral into one over a single variable x.]*

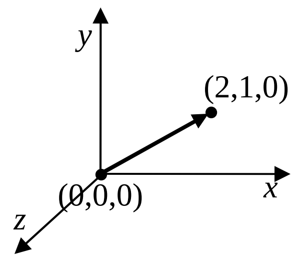


Figure 1.

Equation of the line is $y = \frac{1}{2}x$, so $dy = \frac{1}{2}dx$.

$$\int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_1^2 (F_x dx + F_y dy) = -\int_0^2 ydx + \left[x - 2(\tfrac{1}{2}x)\right]\tfrac{1}{2}dx$$

$$= -\int_0^2 \tfrac{1}{2}xdx = -\tfrac{1}{2}\left.\frac{x^2}{2}\right|_0^2 = -1$$

3. The equation of motion for a vertically falling body under linear air resistance (drag force $f(v) = -bv$) is $m\dot{v} = mg - bv$.
   (a) (5 points) Use this equation to derive the expressions for the terminal velocity. Explain your reasoning.

The body accelerates until the drag force equals the gravity force.
When that occurs, the body stops accelerating and $\dot{v} = 0$.
The velocity at that time is $v = v_{ter}$. Then:

$mg - bv_{ter} = 0$ so $v_{ter} = \dfrac{mg}{b}$ (linear case)

(b) (5 points) What is the corresponding equation for a horizontally moving body under linear air resistance? Write it in separated form (terms involving *dv* on one side, and *dt* on the other). Write the quantity *m*/*b* as τ (time constant).

$$m\dot{v} = -bv \Rightarrow \frac{dv}{dt} = -\frac{b}{m}v.$$

In separated form, this is:

$$\frac{dv}{v} = -\frac{b}{m}dt = -\frac{1}{\tau}dt.$$

(c) (5 points) Solve your equation derived in part (b), for $v(t)$, then integrate to get the equation for $x(t)$, for the initial conditions $x(0) = 0$, $v(0) = v_o$. What is the maximum distance the body will reach after infinite time?

Integrating both sides:

$$\int \frac{dv}{v} = -\frac{1}{\tau}t$$

so

$$\ln\frac{v}{v_o} = -\frac{1}{\tau}t \Rightarrow v(t) = v_o e^{-t/\tau}.$$

Integrating again:

$$\int v dt = x(t) = -v_o \tau (e^{-t'/\tau}\Big|_0^t) = v_o \tau (1 - e^{-t/\tau}).$$

At $t \to \infty$, $x = v_o \tau$.

4. An ion rocket engine has an exhaust velocity of 29,000 m/s (almost 65,000 mph), but very low thrust $-\dot{m}v_{ex}$ = 950 mN (milli-Newtons). If a small satellite has a payload (non-fuel part of the rocket) of 100 kg, and carries 50 kg of fuel in the form of Xenon gas to supply the ions, what is its maximum speed when its fuel is exhausted? What is the rate of fuel consumption $\dot{m}$ in kg/s? How long will the rocket take to reach maximum speed, if the rate of fuel consumption is constant? Recall that the rocket equation is $v - v_o = v_{ex} \ln(m_o/m)$. (5 points)

The initial mass is payload + fuel, so m_o = 150 kg, while the final mass is m = 100 kg.

$$v = v_{ex} \ln(m_o / m)$$

so

$$v = 29000 \ln(150/100) = 11,758 \text{ m/s}.$$

The rate of fuel consumption is

$$\dot{m} = \frac{-950 \text{ mN}}{v_{ex}} = \frac{-0.950 \text{ N}}{29000 \text{ m/s}} = 3.3\times10^{-5} \text{ kg/s}.$$

Since 50 kg are lost at the above rate, it takes

$$\Delta t = \frac{50 \text{ kg}}{3.3\times10^{-5} \text{ kg/s}} = 1.5\times10^{-6} \text{ s} \approx 17.6 \text{ days}.$$

5. (a) (5 points) The element of volume in Cartesian coordinates is $dV = dxdydz$. Give the corresponding expression in cylindrical coordinates ρ, ϕ, z.

The element of volume in cylindrical coordinates is $dV = \rho\, d\rho d\phi dz$.

(b) (5 points) By direct integration, $V = \int dV$, over the appropriate limits, find the volume of the one-quarter cylinder of radius R and height h, shown in Fig. 1. Show all steps, including the limits on each of the three integrals needed. Show that your volume is 1/4th of a complete cylinder.

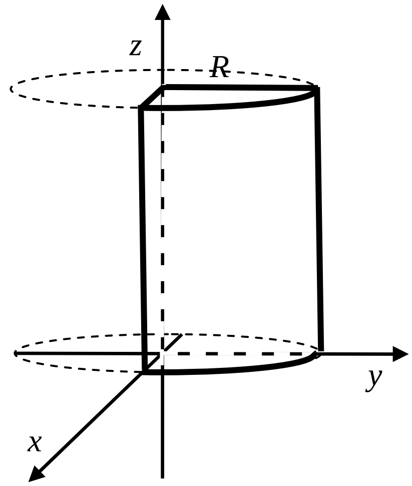


Fig. 1. Slice of a cylinder of radius R, height h.

$$V = \int dV = \int_0^R \rho\, d\rho \int_0^{\pi/2} d\phi \int_0^h dz = \frac{R^2}{2}\frac{\pi}{2}h = \frac{\pi}{4}R^2 h$$

$$\frac{V}{V_{cyl}} = \frac{\frac{\pi}{4}R^2 h}{\pi R^2 h} = \frac{1}{4}$$

(c) (5 points) Using the definition of center of mass

$$\mathbf{R} = \frac{1}{M}\int \sigma \mathbf{r}\, dV$$

write down the expression for the x-component of the CM vector (call it X), in terms of integrals over $d\rho$, $d\phi$, and dz, assuming the density σ (we are using σ for density to avoid confusion with the coordinate ρ) is constant and the total mass is M. Indicate the appropriate limits for each coordinate. Do not do the integration yet, you'll do it in part (d).

Using $x = \rho\cos\phi$, then $x\,dV = \rho^2 d\rho \cos\phi\, d\phi\, dz$, so

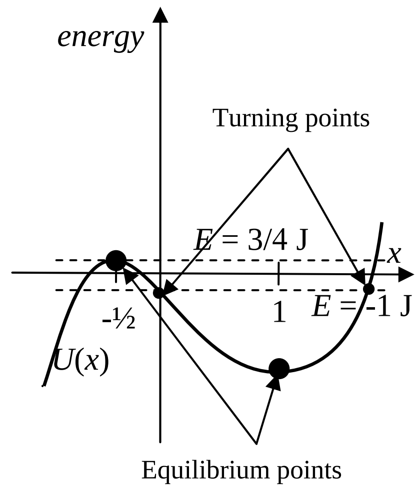
$$X = \frac{\sigma}{M}\int_0^R \rho^2 d\rho \int_0^{\pi/2} \cos\phi\, d\phi \int_0^h dz = \frac{4}{\pi R^2 h}\int_0^R \rho^2 d\rho \int_0^{\pi/2} \cos\phi\, d\phi \int_0^h dz$$

(d) (5 points) Do the integration to show $X = \frac{4}{3\pi}R$.

$$X = \frac{4}{\pi R^2 h}\frac{R^3}{3}h\int_0^{\pi/2}\cos\phi\, d\phi = \frac{4}{\pi R^2 h}\frac{R^3}{3}h = \frac{4}{3\pi}R.$$

6. The plot below schematically shows the graph of the one-dimensional potential energy, $U(x) = 4x^3 - 3x^2 - 6x - 1$ J, acting without friction on an object of mass 10 kg, where x is in m.

(a) (5 points) Find all equilibrium points x_n of the object under potential energy $U(x)$, mathematically, and determine which are stable and which are unstable. Plot these equilibrium points on the graph and label their locations on the x axis.



The equilibria occur where the slope of the potential energy curve is zero, i.e.

$$\frac{dU}{dx} = 12x^2 - 6x - 6 = 0. \quad \text{This has roots } x = +1 \text{ m}, -\tfrac{1}{2} \text{ m}.$$

To find if they are stable or unstable, check the curvature (2nd derivative) at each point:

$$\left.\frac{d^2U}{dx^2}\right|_{x=1} = 24(1) - 6 = 18 > 0 \Rightarrow \text{stable}$$

$$\left.\frac{d^2U}{dx^2}\right|_{x=-3} = 24(-\tfrac{1}{2}) - 6 = -18 < 0 \Rightarrow \text{unstable}$$

(b) (5 points) Let the total energy of the object moving in the vicinity of the stable equilibrium point be $E = -1$ J. Show that $x = 0$ is a turning point of the object. Sketch the total energy and both turning points on the graph [do not calculate the second turning point]. What is the force on the object at $x = 0$, in N?

The turning points occur where the potential energy equals the total energy, $E = U(x)$.

$-1 \text{ J} = 4x^3 - 3x^2 - 6x - 1$ J. Thus $4x^3 - 3x^2 - 6x = 0$, which clearly has a root at $x = 0$.

The force at this location is found from:

$$\mathbf{F} = -\nabla U = -\frac{dU}{dx}\hat{\mathbf{x}} = \left(-12x^2 + 6x + 6\right)\hat{\mathbf{x}} = 6\hat{\mathbf{x}} \text{ N at } x = 0.$$

(c) (5 points) For the case of $E = -1$ J, what is the speed (m/s) of the object when it is at the $x = 0$ turning point? What is its speed when it is at the equilibrium point?

At a turning point, the kinetic energy is zero, so the object's speed is zero there.

At the equilibrium point, $x = 1$, we have $E = T + U(x=1)$:

$T = E - U(x=1) = -1 - 4x^3 + 3x^2 + 6x + 1 \text{ J} = -1 - 4 + 3 + 6 + 1 \text{ J} = 5 \text{ J}.$

The speed is determined from $T = \frac{1}{2}mv^2$, where $m = 10$ kg, so

$v = \sqrt{\frac{2T}{m}} = 1$ m/s.

(d) (5 points) What is the maximum total energy the object could have and still be bound, i.e. remain always within a finite distance of the stable equilibrium point? Sketch and label this total energy on the graph.

Looking at the graph, the left turning point ceases to exist when the total energy exceeds the $U(x = -\frac{1}{2})$ for the unstable turning point, i.e.

$E = U(x = -\frac{1}{2}) = 4(-\frac{1}{2})^3 - 3(-\frac{1}{2})^2 - 6(-\frac{1}{2}) - 1 = -\frac{1}{2} - \frac{3}{4} + 3 - 1 = \frac{3}{4}$ J.