

Vector Definitions

- Experiments tell us which physical quantities are scalars and vectors
- E&M uses vectors for fields, vector products for magnetic field and force

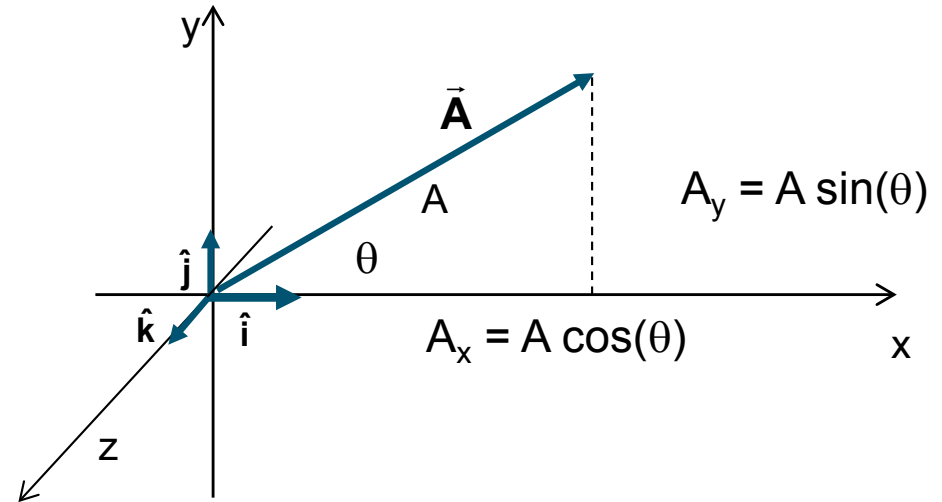
Representations in 2 Dimensions:

- Cartesian (x,y) coordinates

$$\vec{\mathbf{A}} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}}$$

- Magnitude & direction

$$A = \sqrt{A_x^2 + A_y^2} \quad \theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$



- Addition and subtraction of vectors:

$$\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}} \text{ means } C_x = A_x + B_x \text{ and } C_y = A_y + B_y$$

$$\vec{\mathbf{C}} = -\vec{\mathbf{A}} \text{ means } C_x = -A_x \text{ and } C_y = -A_y$$

- Notation for vectors $\mathbf{F} = m\mathbf{a}$

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}}$$

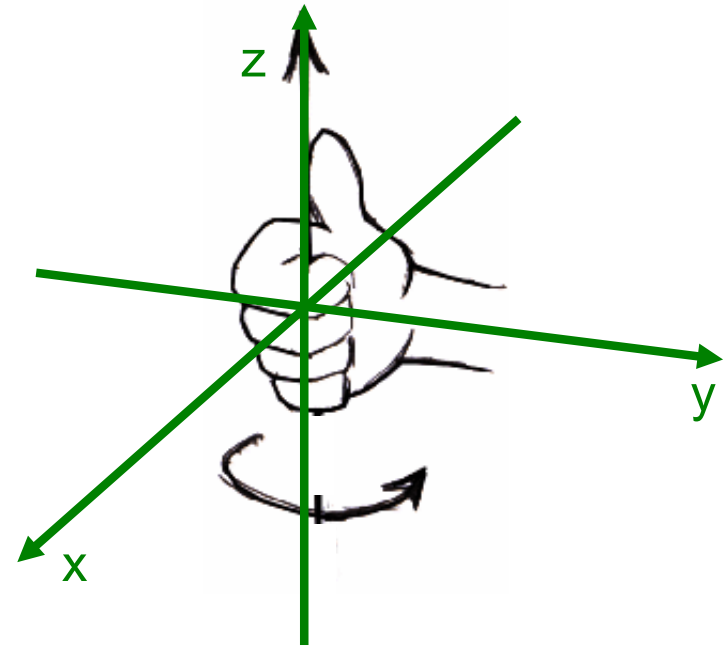
$$\underline{\mathbf{F}} = m\underline{\mathbf{a}}$$

Definition: Right-Handed Coordinate Systems

We **always** use right-handed coordinate systems.

In three-dimensions the right-hand rule determines which way the positive axes point.

Curl the fingers of your **RIGHT HAND** so they go from x to y . Your thumb will point in the positive z direction.



This course uses several right hand rules related to this one!

Vectors in 3 dimensions

Unit vector (Cartesian) notation:

$$\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

Spherical polar coordinate representation:

1 magnitude and 2 directions

$$\vec{a} \equiv (a, \theta, \phi)$$

Conversion into x, y, z components

$$a_x = a \sin \theta \cos \phi$$

$$a_y = a \sin \theta \sin \phi$$

$$a_z = a \cos \theta$$

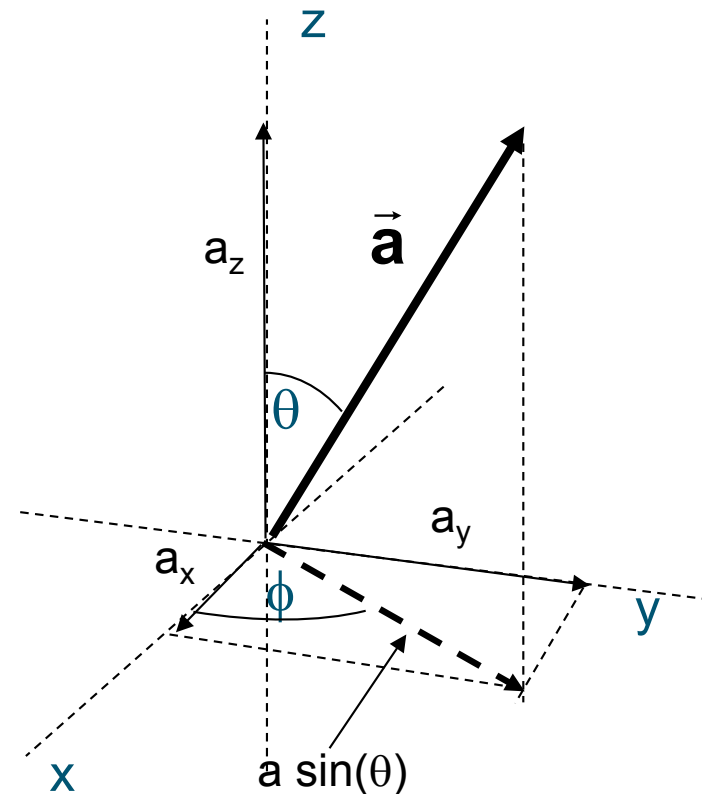
Conversion from x, y, z components

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

$$\theta = \cos^{-1} a_z / a$$

$$\phi = \tan^{-1} a_y / a_x$$

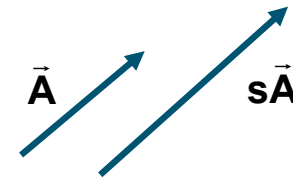
Rene Descartes
1596 - 1650



There are 3 Forms of Vector Multiplication

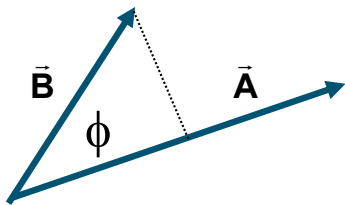
Multiplication of a vector by a scalar:

$$s\vec{A} = sA_x\hat{i} + sA_y\hat{j}$$



vector **times** scalar $\rightarrow$ vector whose length is multiplied by the scalar

Dot product (or Scalar product or Inner product):



- vector **times** vector $\rightarrow$ scalar
- projection of A on B or B on A
- commutative

$$\vec{A} \circ \vec{B} \equiv AB\cos(\phi) = \vec{B} \circ \vec{A} = A_xB_x + A_yB_y + A_zB_z$$

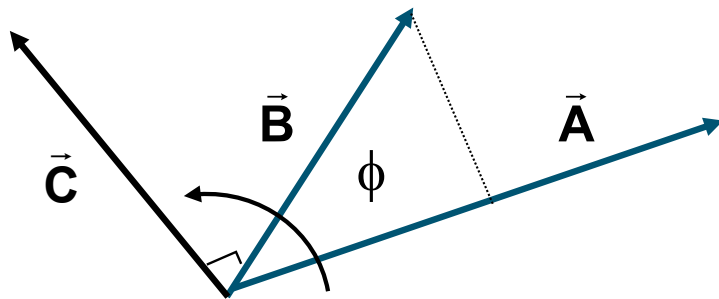
unit vectors measure
perpendicularity:

$$\begin{aligned}\hat{i} \circ \hat{j} &= 0, & \hat{j} \circ \hat{k} &= 0, & \hat{i} \circ \hat{k} &= 0 \\ \hat{i} \circ \hat{i} &= 1, & \hat{j} \circ \hat{j} &= 1, & \hat{k} \circ \hat{k} &= 1\end{aligned}$$

Vector multiplication, continued

Cross product (or Vector product or Outer product):

- Vector **times** vector $\rightarrow$ another vector perpendicular to the plane of $\underline{A}$ and $\underline{B}$
- Draw $\underline{A}$ & $\underline{B}$ tail to tail: right hand rule shows direction of $\underline{C}$



$$\vec{C} = \vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \quad (\text{not commutative})$$

$$\text{magnitude : } |\vec{C}| = AB \sin(\phi)$$

where ϕ is the smaller angle from $\vec{A}$ to $\vec{B}$

- If $\underline{A}$ and $\underline{B}$ are parallel or the same, $\underline{A} \times \underline{B} = 0$
- If $\underline{A}$ and $\underline{B}$ are perpendicular, $\underline{A} \times \underline{B} = AB$ (**max**)

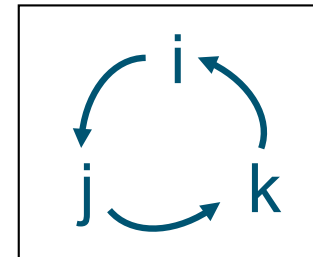
Algebra:

$$\begin{aligned} \text{distributive rule : } \vec{A} \times (\vec{B} + \vec{C}) &= \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \\ \text{associative rules : } s\vec{A} \times \vec{B} &= (s\vec{A}) \times \vec{B} = \vec{A} \times (s\vec{B}) \\ (\vec{A} \times \vec{B}) \times \vec{C} &= \vec{A} \times (\vec{B} \times \vec{C}) \end{aligned}$$

$$\begin{aligned} \text{Unit vector representation: } \hat{i} \times \hat{j} &= \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{i} \times \hat{k} = -\hat{j} \\ \hat{i} \times \hat{i} &= 0, \quad \hat{j} \times \hat{j} = 0, \quad \hat{k} \times \hat{k} = 0 \end{aligned}$$

$$\vec{A} \times \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$

$$= (A_y B_z - A_z B_y) \hat{i} + (A_z B_x - A_x B_z) \hat{j} + (A_x B_y - A_y B_x) \hat{k}$$



Applications: $\vec{\tau} = \vec{r} \times \vec{F}$ $\vec{L} = \vec{r} \times \vec{p}$ $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$

Example:

A force $\mathbf{F} = -8\mathbf{i} + 6\mathbf{j}$ Newtons acts on a particle with position vector $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j}$ meters relative to the coordinate origin. What are a) the torque on the particle about the origin and b) the angle between the directions of $\mathbf{r}$ and $\mathbf{F}$.

a) Use: $\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$

$$\begin{aligned}\vec{\tau} &= \vec{\mathbf{r}} \times \vec{\mathbf{F}} = (3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}) \times (-8\hat{\mathbf{i}} + 6\hat{\mathbf{j}}) \\ &= -(3 \times 8)\hat{\mathbf{i}} \times \hat{\mathbf{i}} + (3 \times 6)\hat{\mathbf{i}} \times \hat{\mathbf{j}} - (4 \times 8)\hat{\mathbf{j}} \times \hat{\mathbf{i}} + (4 \times 6)\hat{\mathbf{j}} \times \hat{\mathbf{j}} \\ &= 18\hat{\mathbf{k}} + 32\hat{\mathbf{k}}\end{aligned}$$

$$\therefore \hat{\tau} = 50\hat{\mathbf{k}} \text{ N.m} \quad |\hat{\tau}| = 50$$

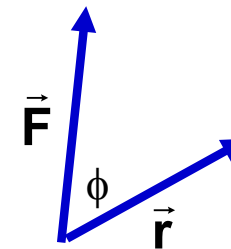
along z axis

b) Can Use: $|\vec{\tau}| = r F \sin(\phi)$

$$r = [3^2 + 4^2]^{1/2} = 5 \quad F = [8^2 + 6^2]^{1/2} = 10$$

$$r F \sin(\phi) = 50 \sin(\phi) \quad \therefore \sin(\phi) = 1$$

$$\therefore \phi = 90^\circ \text{ that is } \vec{\mathbf{F}} \perp \vec{\mathbf{r}}$$



Better to Use: $\vec{\mathbf{r}} \cdot \vec{\mathbf{F}} = r F \cos(\phi) = 50 \cos(\phi)$

$$\begin{aligned}\vec{\mathbf{r}} \cdot \vec{\mathbf{F}} &= -(3 \times 8)\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} + (3 \times 6)\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} - (4 \times 8)\hat{\mathbf{j}} \cdot \hat{\mathbf{i}} + (4 \times 6)\hat{\mathbf{j}} \cdot \hat{\mathbf{j}} \\ &= -24 + 24 = 0\end{aligned}$$

$$\therefore 50 \cos(\phi) = 0 \text{ so } \phi = 90^\circ \text{ that is } \vec{\mathbf{F}} \perp \vec{\mathbf{r}}$$

Why?