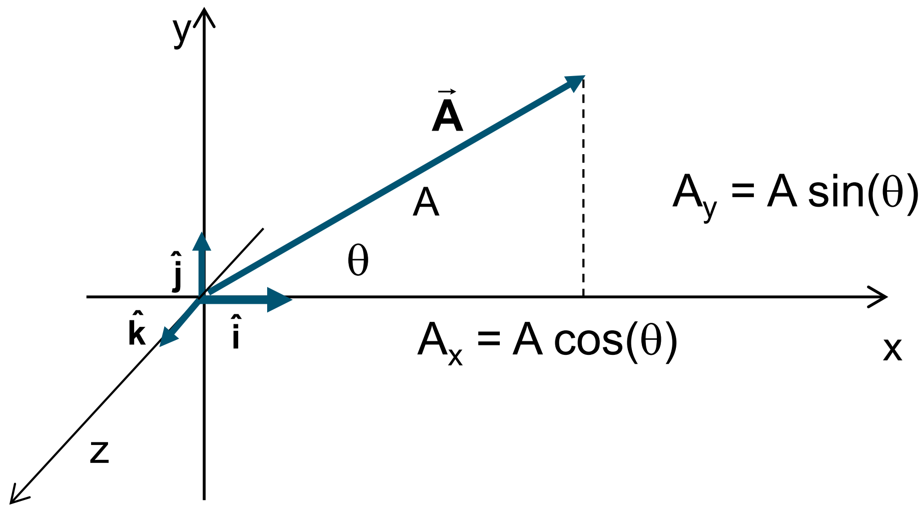
Vector Definitions

- Experiments tell us which physical quantities are scalars and vectors
- E&M uses vectors for fields, vector products for magnetic field and force

Representations in 2 Dimensions:



- Cartesian (x,y) coordinates

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$

- Magnitude & direction

$$A = \sqrt{A_x^2 + A_y^2} \qquad \theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$

- Addition and subtraction of vectors:

$$\vec{C} = \vec{A} + \vec{B} \text{ means } C_x = A_x + B_x \text{ and } C_y = A_y + B_y$$

$$\vec{C} = -\vec{A} \text{ means } C_x = -A_x \text{ and } C_y = -A_y$$

- Notation for vectors

$$\mathbf{F} = m\mathbf{a}$$

$$\vec{F} = m\vec{a}$$

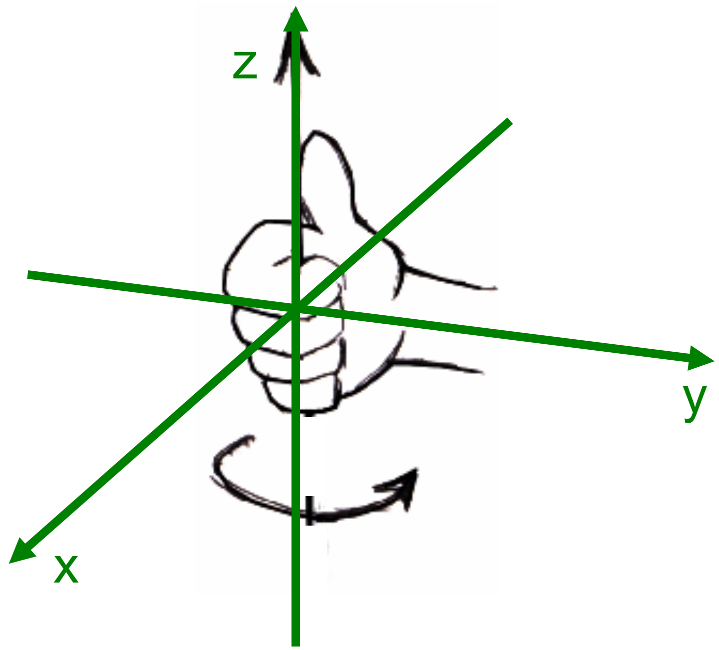
$$\underline{F} = m\underline{a}$$

Definition: Right-Handed Coordinate Systems

We always use right-handed coordinate systems.

In three-dimensions the right-hand rule determines which way the positive axes point.

Curl the fingers of your RIGHT HAND so they go from *x* to *y*. Your thumb will point in the positive *z* direction.



This course uses several right hand rules related to this one!

Vectors in 3 dimensions

Rene Descartes
1596 - 1650

Unit vector (Cartesian) notation:

$$\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

Spherical polar coordinate representation:

1 magnitude and 2 directions

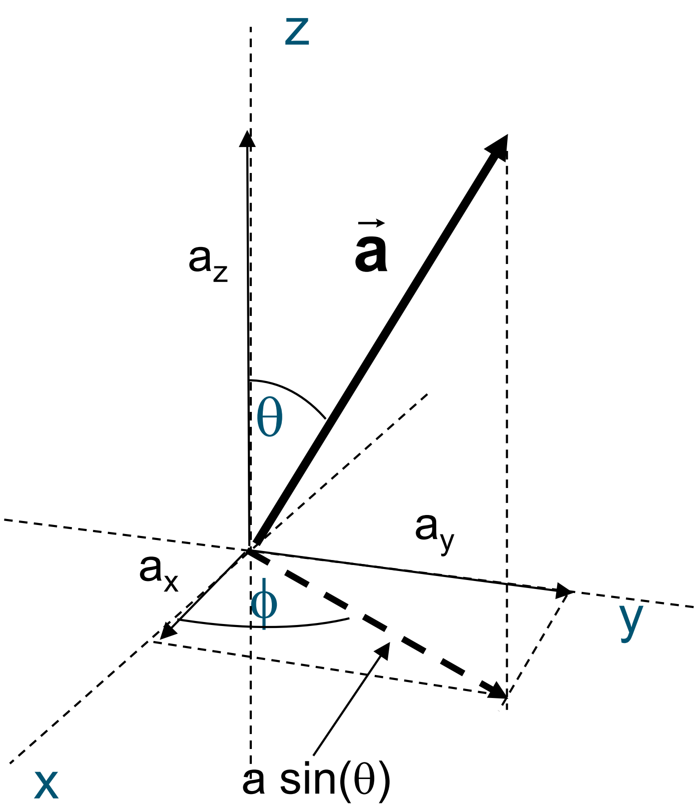
$$\vec{a} \equiv (a, \theta, \phi)$$

Conversion into x, y, z components

$$a_x = a \sin\theta \cos\phi$$
$$a_y = a \sin\theta \sin\phi$$
$$a_z = a \cos\theta$$

Conversion from x, y, z components

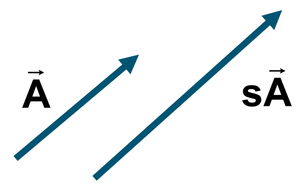
$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$
$$\theta = \cos^{-1} a_z / a$$
$$\phi = \tan^{-1} a_y / a_x$$



There are 3 Forms of Vector Multiplication

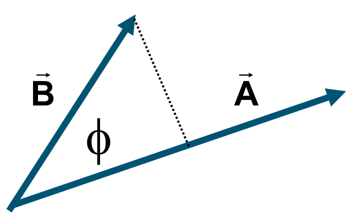
Multiplication of a vector by a scalar:

$$s\vec{A} = sA_x\hat{i} + sA_y\hat{j}$$



vector times scalar → vector whose length is multiplied by the scalar

Dot product (or Scalar product or Inner product):



- vector times vector → scalar
- projection of A on B or B on A
- commutative

$$\vec{A} \circ \vec{B} \equiv AB\cos(\varphi) = \vec{B} \circ \vec{A} = A_xB_x + A_yB_y + A_zB_z$$

unit vectors measure perpendicularity:

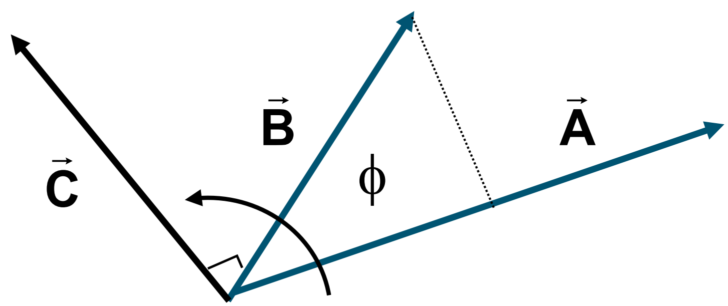
$$\hat{i} \circ \hat{j} = 0, \quad \hat{j} \circ \hat{k} = 0, \quad \hat{i} \circ \hat{k} = 0$$

$$\hat{i} \circ \hat{i} = 1, \quad \hat{j} \circ \hat{j} = 1, \quad \hat{k} \circ \hat{k} = 1$$

Vector multiplication, continued

Cross product (or Vector product or Outer product):

- Vector times vector → another vector perpendicular to the plane of A and B
- Draw A & B tail to tail: right hand rule shows direction of C



$$\vec{C} = \vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \quad \text{(not commutative)}$$

$$\text{magnitude: } |C| = AB\sin(\phi)$$

where ϕ is the smaller angle from $\vec{A}$ to $\vec{B}$

- If A and B are parallel or the same, A x B = 0
- If A and B are perpendicular, A x B = AB (max)

Algebra:

distributive rule: $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

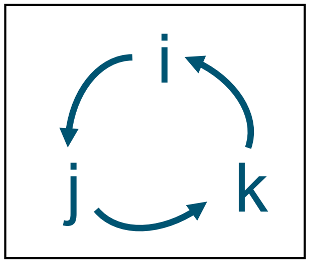
associative rules: $s\vec{A} \times \vec{B} = (s\vec{A}) \times \vec{B} = \vec{A} \times (s\vec{B})$

$$(\vec{A} \times \vec{B}) \times \vec{C} = \vec{A} \times (\vec{B} \times \vec{C})$$

Unit vector representation:

$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{i} \times \hat{k} = -\hat{j}$$

$$\hat{i} \times \hat{i} = 0, \quad \hat{j} \times \hat{j} = 0, \quad \hat{k} \times \hat{k} = 0$$



$$\vec{A} \times \vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \times (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$= (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$$

Applications: $\vec{\tau} = \vec{r} \times \vec{F} \qquad \vec{L} = \vec{r} \times \vec{p} \qquad \vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$

Example:

A force F = -8i + 6j Newtons acts on a particle with position vector r = 3i + 4j meters relative to the coordinate origin. What are a) the torque on the particle about the origin and b) the angle between the directions of r and F.

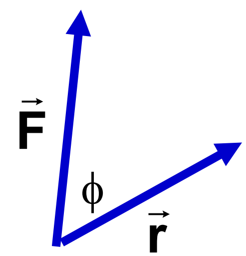
a) Use: $\vec{\tau} = \vec{r} \times \vec{F}$

$$\vec{\tau} = \vec{r} \times \vec{F} = (3\hat{i} + 4\hat{j}) \times (-8\hat{i} + 6\hat{j})$$
$$= -(3{\times}8)\hat{i} \times \hat{i} + (3{\times}6)\hat{i} \times \hat{j} - (4{\times}8)\hat{j} \times \hat{i} + (4{\times}6)\hat{j} \times \hat{j}$$
$$= 18\hat{k} + 32\hat{k}$$

$$\therefore \hat{\tau} = 50\hat{k} \text{ N.m} \quad |\hat{\tau}| = 50$$ along z axis

b) Can Use: $|\vec{\tau}| = r F \sin(\phi)$

$$r = [3^2 + 4^2]^{1/2} = 5 \qquad F = [8^2 + 6^2]^{1/2} = 10$$



$$r F \sin(\phi) = 50 \sin(\phi) \quad \therefore \sin(\phi) = 1$$

$$\therefore \phi = 90^\circ \text{ that is } \vec{F} \perp \vec{r}$$

Better to Use: $\vec{r} \circ \vec{F} = r F \cos(\varphi) = 50 \cos(\varphi)$

Why?

$$\vec{r} \circ \vec{F} = -(3{\times}8)\hat{i} \circ \hat{i} + (3{\times}6)\hat{i} \circ \hat{j} - (4{\times}8)\hat{j} \circ \hat{i} + (4{\times}6)\hat{j} \circ \hat{j}$$
$$= -24 + 24 = 0$$

$$\therefore 50 \cos(\varphi) = 0 \text{ so } \phi = 90^\circ \text{ that is } \vec{F} \perp \vec{r}$$