Notes and Solved Problems for Common Exam 1

1. ELECTRIC CHARGE

Key concepts:

Charged particles attract or repel each other. If particles have charges of the same sign they repel each other. If their charges are of the opposite sign they attract each other.

Units of charge: Coulombs.

Coulomb’s Law states that the value of the force between two charged particles \( Q_1 \) and \( Q_2 \) is proportional to their charges and is inversely proportional to the square of the distance \( r \) between them:

\[
F = k \frac{Q_1 Q_2}{r^2}
\]

where coefficient \( k = 8.99 \times 10^9 \text{Nm}^2 / \text{C}^2 \) which is sometimes expressed via another constant

\[
\varepsilon_0 = \frac{1}{4 \pi k} = 8.85 \times 10^{-12} \text{C}^2 / \text{Nm}^2
\]

Concept of charge density: for charge \( Q \) uniformly distributed over some volume \( V \), the charge per unit volume or \textit{volume} charge density is the ratio \( \rho = \frac{Q}{V} \). If an object is flat (2 dimensional) the \textit{surface} charge density is the ratio of the charge \( Q \) to the surface area \( S \): \( \sigma = \frac{Q}{S} \). For linear, or 1 dimensional objects, the \textit{linear} charge density is the charge per unit length: \( \lambda = \frac{Q}{L} \). We therefore have:

\[
\begin{align*}
3D : \rho &= \frac{Q}{V} \\
2D : \sigma &= \frac{Q}{S} \\
1D : \lambda &= \frac{Q}{L}
\end{align*}
\]

These formulas define a uniform distribution.

Typical problems related to electric charge:

**Problem 1.** Two electrons are placed at a certain distance between each other? The force between them is:

A. Repulsion
B. Attraction
C. Zero
D. Not determined
E. None of the above

**Solution.**

Two electrons have negative charge, therefore the force is repulsion.
Problem 2. Find force between a proton and an electron placed at the distance 1 \( \mu \)m.

A. \( 2.3 \times 10^{-13} \) N, attraction
B. \( 2.3 \times 10^{-13} \) N, repulsion
C. \( 2.3 \times 10^{-16} \) N, attraction
D. \( 2.3 \times 10^{-16} \) N, repulsion
E. \( 2.3 \times 10^{-9} \) N, attraction

Solution.

The force between two oppositely charged particles is attraction and its value is

\[
F = k \frac{q_1 q_2}{r^2} = 8.99 \times 10^9 \frac{1.6 \times 10^{-19} \times 1.6 \times 10^{-19}}{(10^{-6})^2} = 2.3 \times 10^{-16}
\]

Answer C.

Problem 3. A positively charged particle with \( Q = 5 \) mC is placed between two negatively charged particles with \( q_1 = 1 \) mC (left) and \( q_2 = 9 \) mC (right). The distance between \( q_1 \) and \( q \) is 5 cm and the distance from \( q \) to \( q_2 \) is 9 cm. What is the total force acting on the middle particle? Find the value and the direction.

A) \( 3.2 \times 10^7 \) N, left
B) \( 3.2 \times 10^7 \) N, right
C) \( 2.8 \times 10^{-9} \) N, left
D) \( 2.8 \times 10^{-9} \) N, right
E) Zero

Solution.

The value of the force from particle 1 on the middle particle is

\[
F_1 = k \frac{q_1 Q}{r_1^2} = 8.99 \times 10^9 \frac{1 \times 10^{-3} \times 5 \times 10^{-3}}{(5 \times 10^{-2})^2} = 1.8 \times 10^7
\]

The direction of this force is to the left since particle 1 and the middle particle are oppositely charged and attract each other.

The value of the force from particle 2 on the middle particle is

\[
F_2 = k \frac{q_2 Q}{r_2^2} = 8.99 \times 10^9 \frac{9 \times 10^{-3} \times 5 \times 10^{-3}}{(9 \times 10^{-2})^2} = 5.0 \times 10^7
\]

The direction of this force is to the right since particle 2 and the middle particle are oppositely charged, and this force is attractive too, but points in the opposite direction.
If we choose the x-axis from the left to the right, the force F1 has a minus sign (since its direction is opposite to the x-axis) and the force F2 has a plus sign (since its direction is along the x-axis). The total force is therefore:

\[ F_{\text{tot}}^x = F_1^x + F_2^x = -1.8 \times 10^7 + 5.0 \times 10^7 = 3.2 \times 10^7 \, N \]

since it is positive, the total force is pointed to the right.

Answer B.

**Problem 4.** A negatively charged particle with \( q = -1 \, \mu C \) is placed at the center of the uniformly charged cube with side \( a = 1 \, \text{mm} \) and total charge \( Q = 3 \, C \) spread out over all six faces of the cube. What is the total force acting on the particle in the center? Find the magnitude and the direction.

A) 0.75 \times 10^6 \, \text{N, left}
B) 0.75 \times 10^6 \, \text{N, right}
C) 2.8 \times 10^{-9} \, \text{N, top}
D) 2.8 \times 10^{-9} \, \text{N, bottom}
E) Zero

**Solution.** Since the cube is uniformly charged and it has a central symmetry. The force from any infinitesimal charge located at any chosen point \( r \) will be cancelled by the force from the same amount of infinitesimal charge located at the point \( -r \). Therefore, the total force acting on any charge placed at the center is equal zero by symmetry.

Answer E.

**Problem 5.** What is the surface charge density on a sphere with \( R = 1 \, \text{cm} \) and \( Q = 1 \, \mu C \) uniformly distributed on the sphere’s surface:

A) 0.01 \, \text{C/m}^2
B) 124 \, \text{C/m}^2
C) 0.45 \times 10^{-7} \, \text{C/m}^2
D) 7.96 \times 10^{-4} \, \text{C/m}^2
E) 0.0034 \, \text{C/m}^2

**Solution.**

The surface charge density is the charge per unit area of the sphere surface. It is therefore:

\[ \sigma = \frac{Q}{S} = \frac{Q}{4 \pi R^2} = 7.96 \times 10^{-4} \, \text{C/m}^2 \]

Answer D.
2. ELECTRIC FIELDS

Key concepts:

A charged particle creates an electric field around it. If the particle is positively charged, the electric field lines point away from the charge; if the particle is negatively charged, the electric field lines point inward toward the charge.

The value of the electric field created by a charged particle located at the origin at some point a distance \( r \) from it is given by:

\[
E(r) = k \frac{q}{r^2}
\]

Compare this formula with Coulomb’s law. Note that the electric field is a vector field, i.e. at any point of space the field has both magnitude and direction, so we can write a more general formula:

\[
\vec{E}(\vec{r}) = k \frac{q}{r^2} \frac{\vec{r}}{r}
\]

where the vector \( \frac{\vec{r}}{r} \) is a unit vector pointing in the direction of the field.

If another (test) charged particle is placed somewhere in the electric field, it will experience the Coulomb force:

\[
\vec{F} = q_{\text{test}} \vec{E}
\]

Note that the direction of this force depends on the direction of the field and the sign of the charge of the test particle: if the test particle is positively charged, it experiences the force along the field, if the test particle is negatively charged its force is opposite to the field direction. If the particle is not bound to anything it will accelerate due to the electric field according to Newton’s second law:

\[
\vec{F} = q_{\text{test}} \vec{E} = m \vec{a}
\]

Units of electric field: N/C

Useful formulae for electric fields created by various objects:

A uniformly charged infinite non-conducting plate creates an electric field, which is constant in all of space and it is given by

\[
E = \frac{\sigma}{2\varepsilon_0}
\]

where \( \sigma \) is the surface charge density on the plate. A uniformly charged infinite conducting plate also creates electric field constant in all of space, but the magnitude is twice that above:

\[
E = \frac{\sigma}{\varepsilon_0}
\]

A uniformly charged line of charge with linear density \( \lambda \) on it creates an electric field, given by:
Typical problems related to electric fields:

Problem 6. A negatively charged particle is placed in a uniform electric field directed from South to North. In which direction will the particle move after it is released?

A) West  
B) East  
C) South  
D) North  
E) North-West

Solution. A negatively charged particle placed into the electric field will move in the direction opposite to the field. Therefore, it is South.

Answer C.

Problem 7. A negatively charged particle accelerates from East to West in a uniform electric field. What are the direction and the value of the electric field if the particle has charge $q=3 \, \mu C$, mass $m=1 \, mg$, and if the value of its acceleration is $3 \, mm/s^2$? Select the closest answer:

A) West, 0.001 N/C  
B) East, 0.001 N/C  
C) West, 1 N/C  
D) East, 1 N/C  
E) North, 1000 N/C

Solution. If the particle is moving from East to West and it is negatively charged, the electric field is opposite to the particle movement, therefore it is directed from West to East. Its value can be found from the second Newton law: $qE = ma$ and it is given by $E = ma/q = 10^{-6} \times 3 \times 10^{-3} / 3 \times 10^{-6} = 0.001$

Answer B.

Problem 8. Consider a dipole placed in a constant electric field. The field is directed from the top to the bottom. The dipole is initially positioned with the positive charge on the left and the negative charge on the right. What happens after we release the dipole?

A) The dipole will be rotated by 90 degrees, positive charge is on the top, negative charge is on the bottom.  
B) The dipole will be rotated by 90 degrees, positive charge is on the bottom, negative charge is on the top.  
C) The dipole will be rotated by 180 degrees, positive charge is on the right, negative charge is on the left.  
D) The dipole will not be rotated, positive charge remains on the left, negative charge remains on the right.  
E) The dipole will be rotated by 45 degrees, positive charge is on the top-left, negative charge is on the bottom-right.
Solution. The negative charge of the dipole will move in the direction opposite to the field while the positive charge will move along the field. Remember that the distance between negative and positive charge in the dipole remains fixed. Therefore, the dipole will rotate by 90 degrees so that the negative charge ends up on the top and positive charge on the bottom. After that, the dipole will not move for two reasons: the force on the negative charge is equal and opposite to the force on the positive charge, and the torque on the dipole is zero in that position. If the dipole is rotated away from the equilibrium position (clockwise or counterclockwise) a restoring torque will appear tending to push it back toward equilibrium. \( \text{Torque} = d \times F \).

Problem 9. A small particle with mass \( m=1 \text{ mg} \) and positive charge \( Q=1 \text{ mC} \) is placed just near the ground. What should be the surface charge density on the ground to keep the particle above it in a stationary position? Assume that the ground is a non-conductor.

A) \( 1.75 \times 10^{-19} \text{ C/m}^2 \)
B) \( 1.75 \times 10^{-16} \text{ C/m}^2 \)
C) \( 1.75 \times 10^{-13} \text{ C/m}^2 \)
D) \( 1.75 \times 10^{-10} \text{ C/m}^2 \)
E) \( 1.75 \text{ C/m}^2 \)

Solution. Since the particle is small comparing to the ground, we can assume the ground to be a flat infinite plate. Distributing the positive charge on the plate will create the electric field \( E = \sigma / 2 \varepsilon_0 \) pointing up. Therefore, the particle will feel a force \( F_c = eE = e\sigma / 2 \varepsilon_0 \) pointing up and the gravitational force \( F = mg \) pointing down. To keep the particle above the ground, the total force should be equal zero (equilibrium), i.e. \( mg = e\sigma / 2 \varepsilon_0 \) from which we can find the value of \( \sigma = 2e\varepsilon_0 mg / e = 2 \times 8.85 \times 10^{-12} \times 1 \times 10^{-6} \times 9.89 / 1 \times 10^{-3} = 1.75 \times 10^{-13} \text{ C/m}^2 \)
Answer C.

Problem 10. Three particles are placed at the corners of the equilateral triangle with the side \( a=1 \text{ cm} \). Top particle has a positive charge of \( 3 \text{ C} \) and two bottom particles have a negative charge of \( -3 \text{ C} \). What are the value and the direction of the field exactly at the center of the triangle?

A) \( 0.17 \times 10^{-16} \text{ N/C}, \text{ top} \)
B) \( 2.35 \times 10^{-6} \text{ N/C}, \text{ top} \)
C) \( 8.55 \times 10^{-1} \text{ N/C}, \text{ bottom} \)
D) \( 1.6 \times 10^{15} \text{ N/C}, \text{ bottom} \)
E) Zero

Solution. Top particle has a positive charge, therefore it will create an electric field pointing down. Each bottom particle will create an electric field which points towards it, therefore the sum of the two fields will also point down. The total field which is a sum of three fields will point down. Let us choose our x axis from the left to the right and y-axis from the top to the bottom with the origin exactly at the center of the triangle. We need to find a projection of each electric field onto y-axis. The value of electric field from the first particle is \( E_i = k |q_i| / d^2 \) and its projection onto y axis \( E_{i_y} = E_i \times k |q_i| / d^2 \) where
d is the distance between the center of the triangle and its corner which is \( d = a / \sqrt{3} \). The value of electric field from each bottom particle is \( E_{2,3} = k |q_{2,3}| / d^2 \), and its projection onto y axis is \( E_{2,3}^y = E_{2,3} \sin 60 = E_{2,3} / 2 \). The total electric field projected onto y axis is
\[
E_{\text{tot}}^y = E_{1}^y + E_{2}^y + E_{3}^y = k |q_1| / d^2 + k |q_2| / 2d^2 + k |q_3| / 2d^2 = 8.99 \times 10^9 \times 3 / (10^{-2} / \sqrt{3})^2 \times (1 + 0.5 + 0.5) = 1.6 \times 10^{15} \text{N/C}
\]
Answer D

**Problem 11.** A positively charged particle with \( q = 1 \text{mC} \) and \( m = 10 \text{mg} \) is hung on a wire at some distance from a large positively charged plate with some surface charge density. Find the surface density if the angle is known to be 45 degrees.

A) \( 1.76 \times 10^{-16} \text{C/m}^2 \)
B) \( 1.76 \times 10^{-12} \text{C/m}^2 \)
C) \( 1.76 \times 10^{-8} \text{C/m}^2 \)
D) \( 1.76 \times 10^{-4} \text{C/m}^2 \)
E) \( 1.76 \text{C/m}^2 \)

**Solution.** The plate creates an electric field pointing to the left, therefore the particle experiences the Coulomb force \( F_c = qE = q\sigma / 2\varepsilon_0 \) pointing to the left. The particle also experiences the gravitational force \( F_g = mg \) pointing to the bottom and the reaction force \( N \) pointing to the connection point. Since the particle is at the equilibrium, the sum of all three forces should be equal zero, i.e. \( \vec{F}_c + m\vec{g} + \vec{N} = 0 \). Let us choose x axis from the left to the right, and the y axis from the bottom to the top with the origin at the center of the particle. Projection of the equation \( \vec{F}_c + m\vec{g} + \vec{N} = 0 \) on to y axis gives \( N_y = N \cos \theta = mg \) while projection on to x-axis gives \( q\sigma / 2\varepsilon_0 = N_x = N \sin \theta \), from which we can find \( \sigma = 2\varepsilon_0 N \sin \theta / q = 2\varepsilon_0 mg \sin \theta / q \cos \theta = 2\varepsilon_0 mg \tan \theta / q \) which is
\[
2 \times 8.85 \times 10^{-12} \times 10 \times 10^{-6} \times 9.95 \times 1/10^{-3} = 1.76 \times 10^{-12} \text{C/m}^2
\]
Answer B.