Notes and Solved Problems on Time Dependent Circuits

1. Time dependent current in an RC circuit.

The RC circuit consists of the capacitor C, the resistor R, the battery E and the switch S.

When the switch S is closed in the RC circuit, the capacitor is charged. Application of the Kirchoff loop equation to the RC circuit at time t leads to a differential equation. The solution of the equation is:

\[ q(t) = EC(1 - e^{-t/RC}) \]

This evaluates the charge on the capacitor as a function of time. The current in the RC circuit is:

\[ i(t) = \frac{dq}{dt} = \frac{E}{R} e^{-t/RC} = \frac{E}{R} e^{-t/\tau_c} \]

where the time constant \( \tau_c = RC \) shows the characteristic time for the charging process.

The voltage across the capacitor can be found using:

\[ V(t) = \frac{q(t)}{C} = E(1 - e^{-t/RC}) \]

After the capacitor is fully charged the switch is opened. The capacitor now having charge \( q_0 \) gets discharged. These functions are the following:

\[ q(t) = q_0 e^{-t/RC} \]

\[ i(t) = \frac{dq}{dt} = -\frac{q_0}{RC} e^{-t/RC} \]

\[ V(t) = \frac{q(t)}{C} = \frac{q_0}{C} e^{-t/RC} \]

2. Time dependent current in an RL circuit

The RL circuit consists of the inductor L, resistor R, battery E and the switch S.

When the switch is closed in RL circuit, the current does not jump to its value \( E/R \) instantaneously, but increases smoothly from zero. The reason for that is that the inductor L responds on the current change (zero -> non-zero) by maintaining self-induced EMF.

Application of the Kirchoff loop equation to the RC circuit at time t leads to the differential equation. The solution of the equation is:
(2.1) \[ i(t) = \frac{E}{R} (1 - e^{-t/R}) = \frac{E}{R} (1 - e^{-t/\tau_L}) \]

where the constant \( \tau_L = L/R \) shows characteristic time of the process.

The induced EMF across the inductor can be found:

(2.2) \[ E_L(t) = -L \frac{di}{dt} = -E e^{-t/\tau_L} \]

At \( t = 0 \) the left hand side is just \(-E\), meaning that at time \( t = 0 \) the EMF of the battery will be cancelled completely by the self-induced EMF of the inductor. That is why at \( t = 0 \) there is still no current in the RL circuit.

In reverse situation, when the switch is opened, the current does not drop to zero instantaneously, but will decrease smoothly. Current behaves as

(2.3) \[ i(t) = \frac{E}{R} e^{-t/\tau_L} \]

and the self-induced EMF across the inductor behaves as

(2.4) \[ E_L(t) = -L \frac{di}{dt} = E e^{-t/\tau_L} \]

3. Time dependent current in LC circuit

The LC circuit consists of the capacitor \( C \) and the inductor \( L \). The capacitor first should be charged. Suppose we connect the charged capacitor with charge \( Q \) to the inductor. Then, there will be charge oscillations in the LC circuit. If resistances of all the elements can be neglected, these charge oscillations can continue infinitely long.

Application of the Kirchoff loop equation for time moment \( t \) to LC circuit leads to differential equation. The solution of the equation reads as follows:

(3.1) \[ q(t) = Q \cos \omega t \]

is the charge on the capacitor as a function of time. Frequency \( \omega = \frac{1}{\sqrt{LC}} \) is the frequency of the charge oscillations. Current can be found in the LC circuit

(3.2) \[ i(t) = \frac{dq}{dt} = -Q \omega \sin \omega t \]

Self-induced EMF across the inductor is

(3.3) \[ E_L(t) = -L \frac{di}{dt} = \frac{Q}{C} \cos \omega t \]

Voltage across the capacitor is

(3.4) \[ V(t) = \frac{q(t)}{C} = \frac{Q}{C} \cos \omega t \]
It is easy to see that at each time moment \( t \) voltage across the capacitor is minus EMF self-induced across the inductor.

4. Time dependent current in RLC circuit: Damping

An RLC circuit consists of a capacitor, inductor, and resistor. The capacitor first should be charged. Suppose we connect the charged capacitor with charge \( Q \) to the inductor. Then, there will be charge oscillations in the RLC circuit. Since the resistor is present, the charge oscillations will NOT continue infinitely long but will decay with time.

Application of the Kirchoff loop equation for time moment \( t \) to LC circuit leads to differential equation. The solution of the equation reads as follows:

\[
(4.1) \quad q(t) = Q e^{-Rt/2L} \cos \omega' t
\]

where frequency of decaying oscillations is given by

\[
(4.2) \quad \omega' = \sqrt{1/LC - (R/2L)^2} = \sqrt{\omega^2 - (R/2L)^2}
\]

It is easy to see that this frequency is different from the frequency \( \omega = 1/\sqrt{LC} \) of oscillations in a simple LC circuit. The exponent \( e^{-Rt/2L} \) shows that the charge oscillations will decay with time. The characteristic time is given by the parameter \( \tau = 2L/R \) since \( e^{-Rt/2L} = e^{-t/\tau} \)

Current in the RLC circuit is:

\[
(4.3) \quad i(t) = \frac{dq(t)}{dt} = -Q \omega' e^{-Rt/2L} \sin \omega' t - \frac{Q}{2L} e^{-Rt/2L} \cos \omega' t
\]

Self-induced EMF across the inductor can be also found as the derivative:

\[
(4.4) \quad E_L(t) = -L \frac{di}{dt}
\]

Voltage across the capacitor is

\[
(4.5) \quad V(t) = \frac{q(t)}{C} = \frac{Q}{C} e^{-Rt/2L} \cos \omega' t
\]

5. Time dependent current with applied sinusoidal EMF and resistive load

Consider the situation when a resistor is connected to the AC source with time dependent EMF of the following form: \( E(t) = E_m \sin \omega_d t \) where \( E_m \) is the maximum amplitude of the EMF and \( \omega_d \) is the “driving” frequency of the EMF oscillations. Note that this is not the frequency \( \omega = \frac{1}{\sqrt{LC}} \) discussed before! The current in such circuit can be found by applying Kirchhoff loop equation. It is given by
It is clear that the current oscillates in the same way as the external EMF; they both behave as $\sin \omega_d t$.

6. Time dependent current with applied sinusoidal EMF and capacitive load

Consider the situation when a capacitor is connected to an AC source with time dependent EMF of the following form: $E(t) = E_m \sin \omega_d t$ where $E_m$ is the maximum amplitude of the EMF and $\omega_d$ is the frequency of the EMF oscillations. Note that this “driving frequency” is not the same as the natural oscillation frequency $\omega = \frac{1}{\sqrt{LC}}$ discussed before! The current in this circuit can be found by applying the Kirchoff loop equation. It is given by

$$i(t) = E_m C \omega_d \cos \omega_d t$$

and the charge on the capacitor is the following function of time:

$$q(t) = \int i(t) dt = E_m C \sin \omega_d t$$

The voltage across the capacitor at any time moment $t$ is:

$$V(t) = \frac{q(t)}{C} = E_m \sin \omega_d t$$

The expression for the current is usually rewritten in the form

$$i(t) = E_m C \omega_d \cos \omega_d t = \frac{E_m}{X_C} \cos \omega_d t$$

where constant $X_C = 1/\omega_d C$ has a dimension of the resistance and it is called capacitive reactance.

Note that current oscillates out-of-phase with the external EMF.

7. Time dependent current with applied sinusoidal EMF and inductive load

Consider the situation when inductor is connected to the AC source with a time dependent EMF of the following form: $E(t) = E_m \sin \omega_d t$ where $E_m$ is the maximum amplitude of the EMF and $\omega_d$ is the frequency of the EMF oscillations. Note that this is not the frequency $\omega = \frac{1}{\sqrt{LC}}$ discussed before! The current in such circuit can be found by applying the Kirchoff loop equation. It is given by
(7.1) \( i(t) = -\frac{E_m}{\omega_d L} \cos \omega_d t \)

Self-induced EMF across the inductor can be found:

(7.2) \( E_L(t) = -L \frac{di}{dt} = -E_m \sin \omega_d t \)

The expression for the current is usually rewritten in the form

(7.3) \( i(t) = -\frac{E_m}{\omega_d L} \cos \omega_d t = -\frac{E_m}{X_L} \cos \omega_d t \)

where constant \( X_L = \omega_d L \) has dimensions of resistance and it is called **inductive reactance**. Note that current oscillates out-of-phase with external EMF.

### 8. Time dependent current in an LCR circuit with applied AC EMF

Consider the situation when a resistor, capacitor and inductor are connected to the AC source with a time dependent EMF of the following form:

\( E(t) = E_m \sin \omega_d t \)

where \( E_m \) is the maximum amplitude of the EMF and \( \omega_d \) is the frequency of the EMF oscillations. Note that this is not the frequency \( \omega = \frac{1}{\sqrt{LC}} \) discussed before! The current in such circuit can be found by applying the Kirchoff loop equation. It is given by

(8.1) \( i(t) = I \sin(\omega_d t - \varphi) \)

which shows oscillating behavior with the same frequency as the frequency of external EMF.

The amplitude of the current oscillations is given by

(8.2) \( I = \frac{E_m}{\sqrt{R^2 + (X_L - X_C)^2}} \)

where \( X_L = \omega_d L \) and \( X_C = 1/\omega_d C \). This expression is sometimes rewritten in the form

(8.3) \( I = \frac{E_m}{Z} \)

where

(8.4) \( Z = \sqrt{R^2 + (X_L - X_C)^2} \)

is called **impedance**.

The phase constant \( \varphi \) is determined as follows

(8.5) \( \tan \varphi = \frac{X_L - X_C}{R} \)

The self-induced EMF across the inductor at any time moment \( t \) can be found:
9. Resonance in LCR circuit with applied driving EMF

What happens if we tune the external frequency $\omega_d$ of the EMF oscillations to the frequency $\omega = 1/\sqrt{LC}$ of the LC oscillations? In this case, $X_L = X_C$ and impedance $Z = \sqrt{R^2 + (X_L - X_C)^2} = R$. The amplitude of the current oscillations reaches maximum possible value for given R,L,C. This phenomenon is called resonance. The condition for the resonance is thus the external frequency applied to the circuit is equal to the internal frequency of the LC oscillations, i.e $\omega_d = \omega$

10. Root-mean-square current

Root-mean-square (rms) current is defined as the square root of the average square of the current in RLC circuit:

\begin{equation}
I_{\text{rms}} = \sqrt{\overline{i^2(t)}} = \sqrt{\overline{I^2 \sin^2 (\omega_d t - \varphi)} \sqrt{I^2 / 2}} = \frac{I}{\sqrt{2}}
\end{equation}

Therefore rms value for the time dependent current is its maximum value divided by square root of 2:

\begin{equation}
I_{\text{rms}} = \frac{I}{\sqrt{2}} = \frac{E_m}{\sqrt{2 \sqrt{R^2 + (X_L - X_C)^2}}}
\end{equation}

Analogously, for any function of the form $f(t) = F \sin \omega t$ we can consider its rms value:

\begin{equation}
F_{\text{rms}} = \sqrt{\overline{f^2(t)}} = \sqrt{\overline{F^2 \sin^2 (\omega t)} \sqrt{F^2 / 2}} = \frac{F}{\sqrt{2}}
\end{equation}
which is given by its maximum value divided by square root of 2:

\[
F_{\text{rms}} = \frac{F}{\sqrt{2}}
\]

This applies to voltages, EMF's, currents, etc.

11. Power in LCR circuits with applied AC EMF

Power in these circuits depends on time

\[
P(t) = i^2(t) * R = I^2 R \sin^2(\omega_d t - \varphi)
\]

We can consider rms power by performing time averaging of the square of the power:

\[
P_{\text{rms}} = \sqrt{P^2(t)} = \sqrt{I^2 R \sin^2(\omega_d t - \varphi) \sqrt{I^2 R / 2} = \frac{I^2 R}{2} = I_{\text{rms}}^2 R}
\]

i.e it is given by the product of the rms values of the current squared time \( R \)

12. Frequency and period

Frequency \( \omega \) is measured in rad/s. Another frequency \( \nu \) is frequently used:

\[
\nu = \frac{\omega}{2\pi}
\]

and it is measured in 1/s, the unit is called Hertz.

Period of oscillations \( T \) is related to both frequencies \( \omega \) and \( \nu \) as follows

\[
T = \frac{1}{\nu} = \frac{2\pi}{\omega}
\]
SAMPLE PROBLEMS

Problem 1. What is the voltage across the capacitor in the RC circuit with R= 3 Ohm , C= 5 pF and E=9V just after the switch is closed.

A. 0 V
B. 1 V
C. 3 V
D. 5 V
E. 9 V

Solution. After the switch is closed the capacitor is beginning to charge. The charge on the plates is the function of time:

\[ q(t) = EC(1 - e^{-t/RC}) \]

Voltage across the capacitor is thus

\[ V(t) = \frac{q(t)}{C} = E(1 - e^{-t/RC}) \]

Therefore, just after the switch is closed, t=0, voltage V(t=0) = 0.

Answer A.

Problem 2. What is the charge accumulated by the capacitor in the RC circuit with R= 3 Ohm , C= 5 pF and E=9V after 5 seconds the switch is closed.

A. 45 pC
B. 45 nC
C. 45 µC
D. 45 mC
E. 45 C

Solution. After the switch is closed the capacitor is beginning to charge. The charge on the plates is the function of time:

\[ q(t) = EC(1 - e^{-t/RC}) \]

Evaluate this expression at t=5 s

\[ q(5) = 9 \times 5 \times 10^{-12} (1 - e^{-5/3 \times 5 	imes 10^{-12}}) = 45 \text{ pC} \]

Answer A.
**Problem 3.** Consider two RL circuits with the same inductances \( L \), the same batteries \( E \) and two different resistors \( R_1 \) and \( R_2 \). If \( R_1 \) is larger than \( R_2 \), at which circuit current will reach its maximum value faster?

A. Current at circuit with resistor \( R_1 \).
B. Current at circuit with resistor \( R_2 \).
C. The time needed is the same for both circuits
D. The current in the circuits will not flow.
E. None of the above.

**Solution.** The current in RL circuit increases as a function of time as
\[
i(t) = \frac{E}{R} \left[1 - \exp\left(-\frac{t}{\tau}\right)\right]
\]
If time constant \( \tau \) is smaller in one circuit compared to another, this would mean that the current in the first circuit reaches its maximum value faster. In other words, the smaller time constant, the shorter the time period needed to establish a steady current in the RL circuit. Time constant \( \tau = \frac{L}{R} \), therefore if \( R_1 > R_2 \), it follows that \( \tau_1 = \frac{L}{R_1} \) will be smaller than \( \tau_2 = \frac{L}{R_2} \). If time constant for the circuit with \( R_1 \) is smaller than that for the second circuit, the period of time needed to establish steady current in the circuit 1 is smaller than for the current \( R_2 \). Therefore the current at the circuit 1 reaches its maximum value faster.

Answer A.

**Problem 4.** Evaluate the current at RL circuit with \( L=2 \) Henry, \( R=4 \) Ohm, \( E=9 \) V after 3 seconds?

A. 0.224 A
B. 2.24 A
C. 22.4 A
D. 224.0 A
E. 2240.0 A

**Solution.** The formula to use is: 
\[
i(t) = \frac{E}{R} \left[1 - \exp\left(-\frac{t}{\tau}\right)\right]
\]
with \( \tau = \frac{L}{R} = \frac{2}{4} = 0.5 \) s.
\[
i(3 \text{ s}) = \frac{9}{4} \left(1 - 2.71^{-3/0.5}\right) = 2.24 \text{ A}
\]
Answer B.

**Problem 5.** What is the self-induced EMF across the inductor at RL circuit with \( L=2 \) Henry, \( R=4 \) Ohm, \( E=9 \) V after 3 seconds?

A. 20 V
B. 2 V
C. 0.2 V
D. 0.02 V
E. 0.002 V

**Solution.** Self-induced EMF across the inductor is defined by the formula:
\[
E_L(t) = -L \frac{di}{dt} = -E e^{-tR/L}
\]
Therefore \( E_L(3) = -9e^{-3\sqrt{2}/2} = -0.02V \)

Answer D

**Problem 6.** Find period of current oscillations in the LC circuit with \( L = 3 \) H and \( C = 3 \) pF.

A. 18.8 s  
B. 18.8 ms  
C. 18.8 \( \mu \)s  
D. 18.8 ns  
E. 18.8 ps

Solution. Current in the LC circuit oscillates as

\[
i(t) = \frac{dq}{dt} = -Q\omega \sin \omega t
\]

with the frequency \( \omega = \frac{1}{\sqrt{LC}} \). To get the period

\[
T = \frac{1}{\nu} = \frac{2\pi}{\omega} = 2\pi \sqrt{LC} = 6.28 \sqrt{3 \times 3 \times 10^{-12}} = 6.28 \times 3 \times 10^{-6} = 18.8 \mu s
\]

Answer C.

**Problem 7.** The 1 mF capacitor with charge \( Q = 5 \) C is connected to the inductor with \( L = 3 \) H. Find the voltage across the capacitor in this LC circuit just after the connection is established.

A. 5 nV  
B. 5 \( \mu \)V  
C. 5 mV  
D. 5 V  
E. 5 kV

Solution. Charge on the capacitor in LC circuit depends on time.

\[
q(t) = Q \cos \omega t
\]

Therefore, voltage across the capacitor is given by

\[
V(t) = \frac{q(t)}{C} = \frac{Q}{C} \cos \omega t
\]

After connection is just established, \( t=0, V(0)=5/0.001=5000 \) V = 5 kV

Answer E.

**Problem 8.** Find frequency \( \nu \) of decaying oscillations on the RLC circuit with \( R = 4 \) Ohm, \( L = 1 \) H, and \( C = 1 \) mF.

A. 5 pHz  
B. 5 nHz
C. 5 mHz  
D. 5 Hz  
E. 5 kHz

**Solution.** In the RLC circuit oscillations of charge and current exist but decay with time exponentially. For example, charge oscillations are 

\[ q(t) = Qe^{-\frac{R}{2L}t} \cos \omega' t \]

Frequency of the oscillations is given by 

\[ \omega' = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} = \sqrt{\omega^2 - \left(\frac{R}{2L}\right)^2} \]

Frequency \( \nu \) of the oscillations is given by 

\[ \nu = \frac{\omega}{2\pi} = \frac{\sqrt{1000 - (4/2)^2}}{6.28} = 5 \text{ Hz} \]

Answer D.

**Problem 9.** What is the capacitive resistance of the circuit with capacitor \( C = 5000 \text{ pF} \) and applied ac voltage of 120 V and 50 Hz?

- A. 0.64 Ohm 
- B. 6.4 Ohm 
- C. 64 Ohm 
- D. 640 Ohm 
- E. 640 kOhm

**Solution.** Capacitive resistance is given by 

\[ X_C = \frac{1}{\omega_C C} \]

If the frequency of the ac source is given in Hertz, it is frequency \( \nu \) 

Therefore, 

\[ X_C = \frac{1}{\omega_C C} = \frac{1}{\sqrt{1/(2\pi C) - (1/2)^2}} = \frac{10^8}{(6.28 \times 25)} = 640 \text{ kOhm} \]

Answer E

**Problem 10.** What is the inductive resistance of the circuit with inductor \( L = 1 \text{ H} \) and applied ac voltage of 120 V and 50 Hz?

- A. 0.31 Ohm 
- B. 3.14 Ohm 
- C. 31.4 Ohm 
- D. 314 Ohm 
- E. 314 kOhm

**Solution.** Inductive resistance is given by 

\[ X_L = \omega_L L \]

If the frequency of the ac source is given in Hertz, it is frequency \( \nu \) 

Therefore, 

\[ X_L = \omega_L L = 2\pi\nu L = 6.28 \times 50 \times 1 = 314 \text{ Ohm} \]
Problem 11. What is the amplitude of the current oscillations in RLC circuit having impedance of 12 Ohm when ac voltage of 120 V and 50 Hz is applied?

A. 0.1 V  
B. 1 V  
C. 10 V  
D. 100 V  
E. 1000 V

Solution. Amplitude of the current oscillations is given by

\[ I = \frac{E_m}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{E_m}{Z} \]

where Z is the impedance. Therefore \( I = \frac{120}{12} = 10 \text{V} \)

Answer C

Problem 12. What is the rms-current in the RLC circuit having impedance of 12 Ohm when ac voltage of 120 V and 50 Hz is applied?

A. 1 V  
B. 3.5 V  
C. 7 V  
D. 10 V  
E. 14 V

Solution. Rms current is maximum current divided by square root of 2

\[ I_{rms} = \frac{I}{\sqrt{2}} = \frac{E_m}{\sqrt{2\sqrt{R^2 + (X_L - X_C)^2}}} = \frac{E_m}{\sqrt{2}Z} \]

Therefore, rms current is \( 120/12/1.41 = 7 \text{V} \)

Answer C

Problem 13. What is the rms current in the RLC circuit having resistor of 12 Ohm, capacitor of 30 pF and inductor of 3 H when ac voltage of 120 V is applied at resonance frequency?

A. 1 V  
B. 3.5 V  
C. 7 V  
D. 10 V
Solution. If the ac voltage is applied at the frequency of the resonance oscillations, i.e. \( \omega = 1/\sqrt{LC} = \omega \)

Then, \( X_L = X_C \) and impedance at resonance frequency \( Z \) is simply the resistance of the resistor

\[
Z = \sqrt{R^2 + (X_L - X_C)^2} = R
\]

Therefore, the rms-current at the resonance frequency is

\[
I_{rms} = I / \sqrt{2} = \frac{E_m}{\sqrt{2} \sqrt{R^2 + (X_L - X_C)^2}} = \frac{E_m}{\sqrt{2R}} = 120/12/1.41 = 7 \text{ V}
\]

Answer C.

Problem 14. An ideal transformer has 10 primary turns and 100 secondary turns. What is the secondary voltage if the primary voltage is 1 V?

A. 0.01 V  
B. 0.1 V  
C. 1 V  
D. 10 V  
E. 100 V

Solution. The secondary voltage of the transformer is calculated using the formula

\[
V_s = \frac{N_s}{N_p} V_p
\]

Therefore the secondary voltage is 100/10*1=10 V

Answer D.