Physics 121 Practice Problem Solutions 04
Gauss’ Law

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Prob 121P04 - 2P: The square surface shown in Fig. 24-25 measures 3.2 mm on each side. It is immersed in a uniform electric field with magnitude $E = 1800 \text{ N/C}$. The field lines make an angle of 35° with a normal to the surface, as shown. Take that normal to be directed "outward," as though the surface were one face of a box. Calculate the electric flux through the surface.

$$\Delta \Phi = \vec{E} \cdot \hat{n} \Delta A$$

$$= 1E/ \Delta A \cos(35°)$$

$$= 1.8 \times 10^3 \times 9.02 \times 10^5 \times 0.82$$

$$\Delta \Phi = 1.57 \times 10^{-2} \text{ Nm}^2$$

$$\hat{n} \cdot \vec{E} = 1E/ \cos(35°)$$

$$\Delta A = (3.2 \text{ mm})^2$$

$$= 1.02 \times 10^{-5} \text{ m}^2$$
Prob 121P04 - 4P: You have four point charges, $2q$, $q$, $-q$, and $-2q$. If possible, describe how you would place a closed surface that encloses at least the charge $2q$ (and perhaps other charges) and through which the net electric flux is (a) 0, (b) $+3q/\varepsilon_0$, and (c) $-2q/\varepsilon_0$.

- Use: \[ \Phi = \oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enclosed}}}{\varepsilon_0} \]

- Must use $+2q$

(a) $+2q, +q, -q, -2q$

or $+q, +2q, -q, -2q$

(b) $+2q, +q, -q, -2q$

- If must use $+2q$, then cannot enclose

(c) $+2q, +q, -q, -2q$

If must use $+2q$, then cannot enclose

\[ \frac{-2q}{\varepsilon_0} \]

is no solution
Prob 121P04 - I-5P*: A point charge of 1.8 \( \mu \)C is at the center of a cubical Gaussian surface 55 cm on edge. What is the net electric flux through the surface?

\[
\Phi = \int \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0} \\
\text{cube faces}
\]

1. The hard way - integrate over 1 face, multiply by 6. Evaluate coulomb force on each face

2. The general law way

\[
\Phi = \frac{q_{enc}}{\varepsilon_0} \text{ for any surface}
\]

\[
\Phi = \frac{1.8 \times 10^{-6}}{8.85 \times 10^{-12}} = 2.0 \times 10^5 \text{ Nm}^2/\text{C}
\]
Prob 121P04 - 13P: A uniformly charged conducting sphere of 1.2 m diameter has a surface charge density of 8.1 \( \mu \text{C/m}^2 \). (a) Find the net charge on the sphere. (b) What is the total electric flux leaving the surface of the sphere?

\[ q_{\text{tot}} = \int \sigma dA \]

\[ V = 8.1 \times 10^{-6} \text{ m}^3 \]

\[ r = 0.6 \text{ m} \]

\[ S' = \text{Spherical Gaussian Surface Just Outside Sphere} \]

(a) Uniform

\[ E = 0 \text{ inside sphere, so net charge can only be due to } E \]

\[ q_{\text{tot}} = \sigma \times \text{Area} = \sigma \times 4\pi r^2 \]

\[ q_{\text{tot}} = 8.1 \times 10^{-6} \times 4\pi (0.6)^2 \]

\[ q_{\text{tot}} = 3.66 \times 10^{-5} \text{ C} \]

(b) Final \( \Phi \), use spherical Gaussian surface outside sphere

\[ \Phi = \frac{q_{\text{tot}}}{\varepsilon_0} = \frac{3.66 \times 10^{-5}}{8.85 \times 10^{-12}} \]

\[ \Phi = 4.1 \times 10^{-6} \text{ Nm}^2/\text{C} \]
Prob 121P04 - 15P: An isolated conductor of arbitrary shape has a net charge of $+10 \times 10^{-6}$ C.
Inside the conductor is a cavity within which is a point charge $q = +3.0 \times 10^{-6}$ C. What is the charge (a) on the cavity wall and (b) on the outer surface of the conductor?

- $q = \text{charge on conductor on outside surface}.$
  
  $= 10 \mu\text{C}$.

- $S' = \text{Gaussian surface, enclosing cavity.} \ \Phi = 0 \ \text{and} \ \vec{E} = 0$
  everywhere on $S'$ inside conductor.

$\oint \vec{E} \cdot d\vec{S} = \Phi = 0$.

On outside surface, have total $q$ enclosed, which is $9 \text{enc} = q + q = 13 \mu\text{C}$.

$\oint \vec{E} \cdot d\vec{S} = \Phi = \frac{q_{\text{enc}}}{\epsilon_0}$.
Prob 121P04 - 17P: An infinite line of charge produces a field of $4.5 \times 10^4$ N/C at a distance of 2.0 m. Calculate the linear charge density.

- $\vec{E}$ is radial, for infinite line.
  - No edge effects
- $S'$ = Gaussian surface (with caps)
- $\Delta q_{\text{enclosed}} = \lambda \Delta z$
- Zero flux through ends of $S$
- $\Phi = \frac{\lambda \Delta z}{\varepsilon_0} = \Phi_{\text{enclosed}}$
  - $\Phi = E \oint_{\Delta S} d\vec{A} = E \int \Delta z$
  - $\frac{\lambda \Delta z}{\varepsilon_0} = 2\pi r E \Delta z$
  - $\frac{\lambda}{\varepsilon_0} = 2\pi r E$
  - $\lambda = 2\pi r \varepsilon_0 E = 2\pi \times 2 \times 8.85 \times 10^{-12} \times 4.5 \times 10^4$
  - $\lambda = 5 \times 10^{-6}$ C/m

$E = \frac{\lambda}{2\pi \varepsilon_0 r}$

Line of Charge
Prob 121P04 - 18P*: Figure 24-28 shows a section of a long, thin-walled metal tube of radius \( R \), with a charge per unit length \( \lambda \) on its surface. Derive expressions for \( E \) in terms of the distance \( r \) from the tube axis, considering both (a) \( r > R \) and (b) \( r < R \). Plot your results for the range \( r = 0 \) to \( r = 5.0 \) cm, assuming that \( \lambda = 2.0 \times 10^{-8} \) C/m and \( R = 3.0 \) cm. (Hint: Use cylindrical Gaussian surfaces, coaxial with the metal tube.)

\[
\oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\varepsilon_0}
\]

- \( \vec{E} \) MUST BE RADIAL BY SYMMETRY
- FLOW THROUGH END CAPS = 0
- \( q_{\text{enc}} = \lambda \Delta Z \) [NOTE \( \lambda = \frac{\sigma}{2\pi R} \)]

FOR \( r > R \)

\[
\vec{E} \cdot d\vec{A} = E \cdot 2\pi r \Delta Z
\]

\[
E = \frac{\lambda}{2\pi r \varepsilon_0}
\]

\( E_{\text{max}} = 1.2 \times 10^{-4} \) N/C.

FOR \( r < R \), \( q_{\text{enc}} = 0 \)

\[
\Rightarrow \vec{E} = 0
\]
Prob 121P04 - I-19P: A very long conducting cylindrical rod of length $L$ with a total charge $+q$ is surrounded by a conducting cylindrical shell (also of length $L$) with total charge $-2q$, as shown in Fig. 24-29. Use Gauss' law to find (a) the electric field at points outside the conducting shell, (b) the distribution of charge on the shell, and (c) the electric field in the region between the shell and rod.

- By Symmetry, $E$ is radial.
- For $S_1$ (Gaussian Surface Outside)
  \[ \Phi = E \times 2\pi r L = \frac{9\pi q}{\epsilon_0} \]
  \[ \implies E = -\frac{\lambda_1}{2\pi \epsilon_0 r} \]
  Same as line of charge.
- For $S_2$, $E = 0$ everywhere.
  So $\Phi = 0$ on inner surface.
  Hence, $\Phi = -9\pi q$. Charge density is $-9\pi q$.
- Between shells:
  \[ \Phi = E \times 2\pi r' = \frac{\lambda_1}{\epsilon_0} \]
  \[ E = \frac{2L}{2\pi \epsilon_0 r'} \]
  Same as line of charge.
Prob 121P04 - 26P: Figure 24-31 shows cross-sections through two large, parallel, nonconducting sheets with identical distributions of positive charge with surface charge density $\sigma$. What is $E$ at points (a) above the sheets, (b) between them, and (c) below them?

**Approximate Both Sheets by Infinite Sheets**

(a) **Region I**

$$ E_{\text{tot}} = E_0 + E_\infty = \frac{2V}{2\varepsilon_0} = \frac{V}{\varepsilon_0} $$

(b) **Region II**

$$ E_{\text{tot}} = -1(E_1 + E_2) = 0 $$

(c) **Region III**

$$ E_{\text{tot}} = -\frac{\sigma}{\varepsilon_0} $$

$E = \frac{V}{2\varepsilon_0}$ for non-conducting sheets.
Prob 121P04 – 29P  In Fig. 24-33, a small, nonconducting ball of mass \( m = 1.0 \) mg and charge \( q = 2.0 \times 10^{-8} \) C (distributed uniformly through its volume) hangs from an insulating thread that makes an angle \( \theta = 30^\circ \) with a vertical, uniformly charged nonconducting sheet (shown in cross section). Considering the gravitational force on the ball and assuming that the sheet extends far vertically and into and out of the page, calculate the surface charge density \( \sigma \) of the sheet.

**EQUILIBRIUM PROBLEM**

\[
\begin{align*}
\sum F_x &= 0 = F_e - T \sin \theta \\
\sum F_y &= 0 = T \cos \theta - mg
\end{align*}
\]

Divided by \( \theta \)

\[
\tan \theta = \frac{F_e}{mg}
\]

For non-conducting sheet

\[
E = \frac{Q}{2\varepsilon_0} \quad F_e = 9E = \frac{9Q}{2\varepsilon_0}
\]

\[
\tan \theta = \frac{9Q}{2\varepsilon_0 mg}
\]

\[
Q = \frac{2\varepsilon_0 mg \tan \theta}{8}\]

\[
= 2 \times 8.85 \times 10^{-12} \times 10^6 \times 9.8 \times \tan 30^\circ
\]

\[
= 2 \times 10^{-8}
\]

\[
Q = 5.0 \times 10^{-9} C
\]

\[
\theta = 30^\circ \\
m = 10^{-3} \text{ g} = 10^{-6} \text{ kg} \\
q = 2 \times 10^{-8} \text{ C} \\
g = 9.8 \text{ m/s}^2
\]
A conducting sphere of radius 10 cm has an unknown charge. If the electric field 15 cm from the center of the sphere has the magnitude $3.0 \times 10^3$ N/C and is directed radially inward, what is the net charge on the sphere?

- Charge will distribute uniformly on surface. Ohm's current would flow in sphere.
- Shell theorem says $q$ will look like a point charge at origin, outside (for $r > R$).

Or,

$$\Phi_0 \text{(surface)} = E(r) \times A(r) = q_{\text{enc}}$$

$$E = \frac{q}{4\pi \varepsilon_0 r^2}$$

$$q = \frac{4\pi \varepsilon_0 E r^2}{4} = \frac{4\pi \times 8.85 \times 10^{-12} \times 3 \times 8 \times 10^{-3} 	imes (0.15)^2}{4}$$

$$q = 7.5 \times 10^{-9} \text{ C}$$
Prob 121P04 - 40P: A point charge $+q$ is placed at the center of an electrically neutral, spherical conducting shell with inner radius $a$ and outer radius $b$. What charge appears on (a) the inner surface of the shell and (b) the outer surface? What is the net electric field at a distance $r$ from the center of the shell if (c) $r < a$, (d) $b > r > a$, and (e) $r > b$? Sketch field lines for those three regions. For $r > b$, what is the net electric field due to (f) the central point charge plus the inner surface charge and (g) the outer surface charge? A point charge $-q$ is now placed outside the shell. Does this point charge change the charge distribution on (h) the outer surface and (i) the inner surface? Sketch the field lines now. (j) Is there an electrostatic force on the second point charge? (k) Is there a net electrostatic force on the first point charge? (l) Does this situation violate Newton's third law?

(a) $E = 0$ inside conductor, so flux through $S_1$ is zero: $q_1 + q_2 = 0$.  
\[ q_2 = \text{charge on inner surface} = -q_1 \]

(b) Conductor is neutral, so $q_3 = -q_2 = +q_1$

For $r < a$ use Gaussian $S_2$
\[ q_{\text{enclosed}} = q_1 = \text{flux through $S_2$} = \mathbf{E} \cdot \mathbf{A} \]
\[ \mathbf{E} = \frac{q_1}{4\pi\varepsilon_0 A} \]

(d) $a < r < b$ inside conductor, so $E = 0$
For \( r > b \): only \( q_3 \) produces \( E \). It behaves like point charge. So \( E = \frac{1}{4\pi\varepsilon_0} \frac{q_3}{r^2} \) for \( r \to b \) (or use Gauss law on surface outside \( b \)).

Thus:

9) For \( r > b \): \( E_1 + E_2 \) are screened out by conductor \( q_1 + q_2 = 0 \) as required by Gauss' law for conductor.

9) \( q_3 \) is spherically symmetric and \( q = q_1 \).

So for \( r > b \):

\[ E = \frac{q_1}{4\pi\varepsilon_0 r^2} \]

h) \(-q\) now outside. Yes, outer surface charge.

i) \( q_2 \) on inner surface unchanged. Conductor \( q_3 \) screens effect of \(-q\).

j) yes - due to induced charge separation.

k) NO. Screened by \( E = 0 \) region.

l) NO. Shell feels force, but not \( q_1 \).