Abstract

Many researchers and managers agree that small workgroups tend to lose productivity and speed as they grow large and that decision-making tends to become sluggish in very large organizations. These effects are attributed to collaborative coordination costs that grow in importance as functions of an organization’s scale. Understanding of these effects has however been primarily empirical and qualitative.

This paper presents a first-principles mathematical model that quantitatively predicts such productivity variations. It is based on a fundamental cost mechanism that has been overlooked in the past but which nonetheless dramatically limits knowledge workers' productivity and timely response: one must account for the information associated with each actor’s range of collaborator choices when trying to understand organizations as systems of human knowledge processors. A Shannon-like “collaborative entropy” is quantified and introduced to model the extra decision information that must be generated when an organization distributes its functions among collaborating internal actors.

The implied coordination cost is a fundamental limit on the per capita productivity for knowledge work. The productivity limit would apply even if management made perfect resource allocations. Information is lost if actors try to exceed the maximum rates for exchanging decision information. The model compensates for value gained by accessing specialized expertise and hiding complexity. The productivity metric used assumes constant decision quality.

In a single growing workgroup the per capita productivity increases while the group is small and lightly loaded, but it falls off logarithmically rather than remaining constant once the group size exceeds a saturation value at which raw decision capacity is all in use. The productivity fall-off is due entirely to collaborative entropy.

For organizations as a whole two additional scale effects may apply: the fraction of an actor’s total effort spent in collaboration versus individual work may grow, and the average number of collaborators per actor may also grow owing to increasing specialization. Productivity is then strongly peaked around an optimum organization size and varies rapidly above or below the peak by a factor in the range of 2 - 5.

The productivity impact of collaborative entropy is large enough to strongly affect competitive advantage. Impact is maximized when the ratio of collaborative to individual effort is large. Large organizations may thus be inherently disadvantaged versus small ones wherever fast decision-making or high knowledge-worker productivity are key drivers. Even a modest amount of collaboration significantly decreases the productivity of actors functioning primarily as individual contributors.

Keywords

collaboration costs, collaborative entropy, systems theory, management, theory of the firm, coordination theory, collaborative costs, operations research, optimization, information theory, Shannon entropy, command and control, C3I, entropy-based warfare, decision networks, decision complexity, economics, knowledge work, productivity metrics
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Rich Janow earned an A. B. at Columbia College and a Ph. D. at The City University of New York, both in Physics. His interest in systems, management, technology strategy and assessment, technological forecasting, and strategic marketing developed during 18 years at Bell Laboratories and 5 years as chief technology executive in a technology company. He spent several years working on defensive systems and C3I while at Bell Labs. He has published academic research papers in computer science, condensed matter theory, and surface physics, and has been awarded 9 U. S. patents.

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1. Introduction and technical approach

This paper presents a quantitative, first-principles model of collaboration costs in organizations, built on a proposed basic cost mechanism that implies a fundamental productivity limit and is central to understanding organizations as systems of human knowledge processors. The theory incorporates elements of systems theory, management, information theory, industrial engineering, and physics, echoing suggestions [e.g., OMS2001 & Ma1990] that a successful coordination theory requires a multi-disciplinary approach.

The work that follows addresses only a subset of the many possible coordination and transaction costs: those traceable to collaboration between individual knowledge-workers. These measure the productivity lost when sub-tasks and responsibilities are partitioned among groups of people working together on sizable tasks. Calculations presented below predict that the impact of a “collaborative entropy” mechanism can be important and sometimes decisive in determining competitive success.

This work in not intended to be a substitute for full “theories of the firm” [GaSa2005, CoPr1996, CyMa1992, KoZa1992, KoZa1996] discussed by many researchers. Put succinctly, such theories principally seek answers to the following two questions [Coase1937, CoPr1996]:

- “Why does a firm exist rather than a set of independent contractors covering the same markets in toto?”
- “What determines the firms’ size and scope?”

While this paper is clearly related to those questions in their broadest sense it addresses only a small subset of the important factors. As result, many questions implicit in those above are too broad to answer here.

Organizations are represented as informal functional networks of actors working in mixed collaborative and individual modes. Coordination costs do not stop at the formal organization boundaries, which are represented only in an indirect sense in this model. The model does not consider “contractual” influences (e.g., property rights, incentives, opportunism, moral hazard) that are known to directly influence where firm boundaries fall, given a set of markets and contributors [Foss1996]. The models below deal with functional rather than formal relationships; formal organization boundaries are acknowledged only by arguing that an average actor’s collaborative activity often grows at the expense of individual contributions when the (formal or informal) organization’s total population grows.

So-called “knowledge-based” or “resource-based” theories of the firm are most compatible with the models of organizations used below. Kogut and Zander [KoZa1992, KoZa1996] for example view a firm as a “repository of capabilities as determined by the social knowledge embedded in individual relationships structured by organizing principles”. Knowledge may consist of “codified” flows of new or existing explicit knowledge. Knowledge may also be “tacit” knowledge shared by the collaborative community, over and above that residing in individuals. Tacit knowledge is a collective (holistic) property of the network structure that is embedded in collaborative relationships [Kogut2000]; it is represented below by collaborative linkages between players with complementary capabilities that are assumed to have reached an equilibrium state after a period of organizational learning.

Flows of “knowledge” are represented by flows of application level “decision” information, themselves composed in principle of binary choices. The limiting “decision rate” is used as a surrogate for flows of compressed knowledge; it is quantified below and used as the principal productivity metric.

Coordination costs are not a new discovery. The impossibility of achieving perfect task management and functional partitioning for complex efforts is sufficient to guarantee non-zero coordination costs, even if truly fundamental cost mechanisms were absent. But this paper is concerned with something quite different. It identifies and quantifies a fundamental upper limit on collaborative productivity that would be present even in the ideally managed case: this basic mechanism has not been considered until now.

The proposed fundamental per capita productivity limit is due to “collaborative entropy”, which is given a precise definition below using information theory (“Shannon entropy”). The productivity limit falls off essentially logarithmically with the growth of workgroups and organizations when some simple modeling assumptions are made - including accounting for the compensating “value” that collaboration between experts has on productivity. Frederick Brooks [Br1995] discussed coordination costs in “The Mythical Man-month” in 1975 in the context of software development, seemingly having in mind a sequential-logic-driven mechanism distinct from the one developed here. Both mechanisms coexist in general.

The central idea developed below is the following: when activities are partitioned onto a number of actors who must work together, they must create additional information – that is, they must pay a quantifiable coordination cost – as the price of dividing up the work [Ma1988]. The motives for picking specific collaborators can include the following: the need for an enlarged labor pool, the need to share decisions with stakeholders, and/or the need to work with experts...
having specialized abilities. In the latter case expert collaborations bring “value” by hiding complexity from the non-experts in those fields, thereby reducing the amount of information transfer needed.

The term “collaborative entropy” was defined as a measure of decision information: not as an entropy model harking back to physical thermodynamics. The phrase “organizational entropy” is already identified with some “annealing” concepts that rely on analogies between organizational structure and phase changes in physical systems, but which are not related to this work. Some researchers invoke a form of information entropy known as “mutual information” [Fr2006, Th1972, Pa2004] that measures deviations from randomness in sets of choices, but those applications and definitions are likewise unrelated to the focus of interest here.

The paper is divided into 5 sections plus an appendix (Section 6). This Section 1 lays down the concepts and approach. Section 2 presents a highly simplified model of individuals’ coordination costs for the limiting case in which the work of an organization is “collaboration dominated” – i.e., almost entirely collaborative rather than composed of individual contributions. Section 3 connects the productivity findings for individual collaboration-dominated nets (workgroups) to the total population of an organization using the notion of an evolution operator. Section 4 redevelops the theory more generally and presents the most general results: it adds models of the benefits as well as costs for both collaboration and individual contributions, parameterizes the mix of individual versus collaborative tasks, and shows the impact of organization size on productivity. The most important numerical results are in Sections 4.8 and 4.10; they clearly show that the effects can have significant magnitude. Section 5 summarizes results, discusses some of the implications, and indicates directions for further work. Appendix Section 6 contains supplementary discussions and formal development too long and detailed for the main text.

The model proposed is quantitative and may lead to some useful tools for improving per capita productivity. It will clearly require field work to determine numerical parameters and a period of detailed model development will be needed to make such tools a reality. But even without this add-on development, the model presented below allows us to parametrically demonstrate the trends and clearly show some of the expected impacts.

### 1.1. Coordination costs and organization size

There is some debate about the relationship between organization size and coordination costs (see for example [BCK2001] for a brief review). Some analysts stress the importance of market size rather than organization size. Others attribute much of small-firms’ relative effectiveness to more highly motivated individual contributors. But the consensus view is that coordination costs become a performance driver when organizations reach some critical size that depends strongly on typical complexities of the problems to be solved. For example, Macher [JM2006], Becker & Murphy [BM1992], Henderson & Cockburn [HC1996], and Zenger [Ze1994]) concur that coordination costs can grow to dominance and produce overall dis-economies of scale correlated to firm size. The need to match organization size (corresponding roughly with the breadth and depth of available expertise) to problem complexity is known [NiZe2004]; the task partitioning that minimizes coordination costs is most “efficient” and drives the organization size [Ta2003].

Knowledge workers who have spent time in very large firms also often conclude that size-related productivity impairment is real; i.e., that decision-making inevitably becomes sluggish as a function of workgroup and firm population. Small collaborative workgroups and start-up firms are lionized for productivity and innovation speed but they seem to lose their competitive speed and productivity edge as they grow. When a small firm is acquired by a large one, knowledge workers often experience a sharp culture shock, lament low responsiveness in the parent entity to the decisions and information they generate, and adjust to lowered per capita productivity or leave.

The models for decision channel costs developed below do in fact quantitatively predict such effects for knowledge work: productivity declines above some critical workgroup size. Groups of collaborating actors tend to become broader on the average in large organizations and individuals may on the average spend a larger fraction of their time working collaboratively. It may consequently be reasonable to expect productivity to tend to decline as a function of total organization size. Increased specialization may provide compensatory reductions in task complexity. But if the number of collaborations (span) grows without limit added “value” will eventually stop accruing while costs keep growing. Individual productivity will then fall off. The logarithmic productivity deterioration with group size that is found may seem to be a mild effect and it may sometimes be hidden by more familiar coordination and transaction costs. If actors’ work becomes more collaborative (at the expense of individual roles) as a function of organization size the productivity impact is greatly amplified (Section 4.10).

Organizations may thus develop dis-economies of scale - impaired competitiveness - in industries for which knowledge-worker productivity is a key competitive driver. Over time the implied bias should have a decisive impact, favoring small over large organizations competing in knowledge-based markets in much the same way that a very small bias in gambling odds results in the “house” almost inevitably bankrupting gamblers who continue to play.

Traditional scale economies often decide dominance among firms, with knowledge-worker productivity having only a minor impact on competitive outcomes. But competition is driving products generally toward greater knowledge- and
service-content, higher complexity of production, and more reliance on networks of interacting specialists. As a result, knowledge-worker coordination costs will tend to grow in relative importance in upcoming years.

1.2. Sequential Constraints (Dependencies)

In “The Mythical Man-Month” [Br1995] Brooks noted that adding staff late in a project usually does not help to fix schedule slippage problems. Instead, huge coordination costs due to training, initialization, and repartitioning and reassigning of the work often arise and worsen the existing situation. Adding staff can retard project completion, especially if people are added when distress is already extreme and experienced people must spend much of their effort training and re-planning.

“Brooks’ Law” included a proposed quantitative rule arguing that growth in the number of collaborators generates only linear growth in capacity but increases coordination costs in proportion to the square of the number of collaborators (i.e., the costs are directly proportional to the potential number of collaboration linkages). Despite the apt warning about adding staff “Brooks’ Law” overestimates the coordination costs quantitatively: if it were correct, large organizations’ capacities for completing work would actually decrease linearly with population, contradicting experience.

The total capacity for knowledge work normally does grow as organizations grow. But the capacity grows slower than linearly, meaning that per capita productivity – measured by decision throughput – must still actually decrease (Section 4.11). The quantitative form of “Brook’s Law” may apply to some centrally controlled projects in extremis but it probably does not adequately portray large organizations as a whole, which handle many projects concurrently. Brooks seems to acknowledge this in the 20th year anniversary edition [Br1995] of “The Mythical Man-Month”, agreeing also that information hiding (i.e., a combination of using experts and hierarchically structuring work) may be the best way to improve software project productivity (see the essay “David Parnas was right…” in [Br1995]).

Brooks’ work with software development organizations also appeared to depict coordination costs as the product of essentially sequential constraints – sequential dependencies that require actors to collect information and generate decisions in a prescribed order. Networks of parallel processors may have been the model for emphasizing sequential constraints in software development organizations: certain actors on the critical path become choke points while others are left waiting for inputs, thereby throttling the work schedule.

Sequential constraints (dependencies) would be absent in an (ideal) computer network executing computing algorithms that are completely parallelizable; in the ideal case (no other costs), total processing capacity would increase linearly with the number of processing nodes, allowing per processor efficiency to remain constant as the problem size increases. Computer scientists call this rule “Amdahl’s Law” [Am1967].

The analogous situation for organizations would mean that sequential decision (information-transfer) constraints do not affect group or firm productivity. Latency and critical path effects might be absent. More likely, sequential constraints might affect individual project schedules but with workgroups or whole firms handling many projects concurrently the individual actors’ work queues are never empty and overall productivity is not affected.

1.3. What if sequential constraint costs are not the driver?

If the sequential constraints become negligible, do the collaboration costs then approach zero? The novel model developed below says that a basic, inescapable coordination cost mechanism due to the division of functions and the implied need to collaborate would still be present; the mechanism is distinct from the sequential logical costs and is aptly described using information theory.

Returning for a moment to the parallel processor analogy, an algorithm that is completely parallelizable would still carry a coordination cost (over and above operating system context switching): extra computing needed to translate application level language, programs, and data into forms usable by cooperating processing nodes with heterogeneous capabilities. Costs are magnified if the individual application processors have mismatched application interfaces.

For an organization of human actors consider the following analogous scenario: suppose there are no critical path effects or schedule deadlines and there are enough independent tasks to always keep actors’ work queues filled. All of the tasks are small in comparison to organization size and have no sequential dependencies important enough to bring much of the work to a halt. But suppose the workload still requires a sizable group of collaborators in order to bring specialized skill sets to bear, or to involve stakeholders, or simply to supply enough labor to keep up with the total workload. Actors need never be idle while waiting for responses: they can switch tasks to use the time. No single overloaded actor (decision node) halts the work of others, although individual projects can still become stretched out in time due to capacity limits. In effect the work on any particular task could proceed asynchronously.

Large workgroups or firms handling many “small” projects concurrently may approximate this state. Another example
may be “open source software development” (Linux, for example). E. S. Raymond in “The Cathedral and the Bazaar” [Ra1999], identifies key success factors for “open source” as the relief of sequential dependencies by decoupling schedules, the use of information hiding (i.e., using high-level task formulations and expert collaborators), and allowing self-organized (rather than static) collaborations to form dynamically between individuals with complementary skills.

Given the idealized scenario: could decision capacity grow linearly (or faster) as a function of organization size? If so, per capita productivity could be constant (or growing) at heavy load levels and coordination costs would not grow in relative importance. A model for evaluating such questions should include productivity gains attributable to the “value” of using expert collaborators – matching a qualified person to the task and thereby lowering the task’s decision complexity. Can the decision-complexity fall fast enough to compensate for the increased coordination costs as workgroup size and specialization of roles increase?

While groups are still small the answer can be affirmative. But there will always be a practical limit to the number of expert collaborators that can add productivity (“value”) to a specific problem. If workgroups grow beyond that size limit the (non-sequential) coordination costs should continue to grow and become dominant, driving productivity down.

“Synergistic” growth in total capacity (i.e., capacity growth faster than linear with population) should not occur in knowledge-driven organizations unless radical new technology is inserted or other intangible influences not measured in this paper are brought to bear and produce holistic results. Odlyzko and Tilly [OT2005] [OT2006] critically discussed similar issues pertaining to the (economic) “value” created by the growth of telecommunication networks in their critique of “Metcalfe’s Law” [Me1995/6] - an overly aggressive “value” growth function (as the square of the number of actors connected) that was postulated some years ago.

1.4. Coordination costs due to decision channel constraints

The upper limit on actors’ own capacities for making elementary binary choices is set by biology and perhaps other factors. It is assumed to be fixed for a particular actor and is denoted by the symbol $D_{i}^{\text{tot}}$, with portions allocated to collaboration ($D_{i}^{\text{coll}}$) and to individual tasks ($D_{i}^{\text{sol}}$). At saturation, the collaborative capacity is capped by the total amount of cognitive effort that actor $i$ can assign to his own set of collaborative channels. Saturated actors do not necessarily have diminished per capita productivity unless collaborative entropy is included (see below): they do fall behind in completing their workload.

The proposed collaborative cost, distinct from saturation, represents the extra information transfer needed to operate a decision system when functions are distributed across actors [Ma1988]. If collaborations become broader, more and more extra information must be generated and assimilated to make the task partitioning work properly, whether or not there are sequential constraints embedded in the tasks. The originators of information packages must tailor them specifically to the intended receivers. One actor’s collaborators might each use a different high-level professional language and need different information items along with their translations. If collaborations are long-lived the amount of coordination information needed can be reduced by organizational learning. The collaborative cost should scale with the workgroup size and possibly the firm size (executing many projects). Brooks’ and later work does not differentiate this cost from one due to sequential constraints.

The additional information is needed when functionality is distributed because choice is involved. If one actor’s collaborators grow more numerous there is a greater and greater choice of partners and interfaces each time a piece of decision information is sent or received. Wherever there are choices some amount of “missing” information is needed to resolve them (see discussion in Appendix Section 6.1: “Entropy and choice”). In digital communications, for example, the amount of information associated with symbols in a set is related to the frequency of choice statistics, which measure the information needed to specify system states and are expressed using the same mathematical representation as physical entropy. The larger the range of choice for the states of a system, the more information must be supplied to resolve the ambiguity and specify the system’s actual state.

The suggested coordination cost is represented below by an information measure whose mathematical form is similar to “Shannon entropy”. Information theory was originally developed for and applied to data communication systems [Sh1998], wherein symbols such as the ASCII text characters carry information. Physical communication channels (even without noise) have a fundamental upper limit on their capacity that can be overrun if the symbol transmission rate is too high.

Some elements of information theory are adapted below to describe the flow of decision information in an organization. The “collaborative decision complexity” for actor $i$ (denoted by the symbol $A_{i}^{\text{coll}}$ below) is introduced to measure the average (i.e., the expected value) of the information content in collaborative decisions. For a single collaboration channel it consists of an overall factor that depends on the task (perhaps measuring the ‘translation’ information) plus a breadth of choice factor that has the mathematical form of information entropy. Additional models are introduced in Section 4 to account for productivity enhancements due to specialized expertise.
organizational "energy level" and perhaps regulate self-organizing processes. These concepts have no connection whatever to the current discussion.

Part of this capacity may be allocated to individual work and another portion reserved for inbound and outbound generalized workgroup structure. The results for sub-nets are then applied to complete organizations consisting of many such sub-nets. Each individual actor is viewed as having fixed capacity for making elementary binary decisions.

Imagine compressing the actual information exchange into its most compact form and representing it as a set of binary decision choices. The binary choices are called "dits" in the discussions below to stress that they are elementary decision quanta analogous to "bits" in digital communications and to avoid confusion with high level cognitive quanta. The maximum decision rate for an entire organization (denoted \( M_{\text{tot}} \)) is the sum of all the individual contributions. The decision rates measure only potentially decidable information in cognitive outputs and inputs, not the volumes of data (words or pictures) needed to represent the content to other people.

Despite earlier references to data communications, note clearly that we are not considering anything associated with physical transmission systems. Collaborations between individuals – referred to below as "links" – exist at the cognitive level not as a set of physical channels. They could for example be implemented by spoken conversations, handwritten documents, clay tablets, etc. and existed for millennia before digital communications appeared.

1.5. Decision channels and key metrics

All of the collaborative knowledge-transfer in organizations is assumed to be reducible to application-level decisions flowing between collaborators, analogous to symbol flows in communications channels. Conceptually, one might imagine compressing the actual information exchanged into its most compact form and representing it as a set of binary decision trees.

The productivity measure for each individual actor doing knowledge work is taken to be the maximum rate of decision flow (defined below and denoted symbolically by \( M_{\text{tot}} \)), possibly including both collaborative and individual contributor tasks. The maximum decision rate for an entire organization (denoted \( M_{\text{tot}} \)) is the sum of all the individual contributions. The decision rates measure only potentially decidable information in cognitive outputs and inputs, not the volumes of data (words or pictures) needed to represent the content to other people.

Despite earlier references to data communications, note clearly that we are not considering anything associated with physical transmission systems. Collaborations between individuals – referred to below as "links" – exist at the cognitive level not as a set of physical channels. They could for example be implemented by spoken conversations, handwritten documents, clay tablets, etc. and existed for millennia before digital communications appeared.

1.6. Collaborative sub-networks, span, and decision complexity

In previous work [Ja2003, Ja2004] the author incorporated the preceding notions into a simple model that treated a complete organization as a single, common, shared decision channel. This work proceeds somewhat differently; it first develops the dynamics of comparatively small collaborative sub-networks (called sub-nets for brevity), which are a generalized workgroup structure. The results for sub-nets are then applied to complete organizations consisting of many such sub-nets. Each individual actor is viewed as having fixed capacity for making elementary binary decisions. Part of this capacity may be allocated to individual work and another portion reserved for inbound and outbound collaborative decision channels.

An application-level decision can be represented in principle by a maximally compact binary tree and therefore by a string of binary decision choices. The binary choices are called "dits" in the discussions below to stress that they are elementary decision quanta analogous to "bits" in digital communications and to avoid confusion with high level decisions. The average number of "dits" per application-level decision is the measure of the decision complexity.

An actor (subscripted "i") has a sub-net of outbound collaboration links that is regarded as a single shared channel and also an inbound sub-bit that forms a second shared channel. The linkages in the two sub-nets may not all be the same. The total raw "dit" capacity (\( R_{\text{cell}} \)) actually assigned to all of the collaborative links must not exceed the actors' total cognitive capacity (\( D_{\text{coll}} \)) available for collaboration. The collaboration capacity actually assigned is \( R_{\text{cell}} < D_{\text{coll}} \), analogous to the total bit capacity assigned to a collection of communication sub-channels.

The "collaborative span" for inbound and outbound links (or simply the "span") is defined as the average number of collaborators per actor: it measures the size of an actor's own sub-net (workgroup).

Actors' roles can become increasingly specialized at the cost of increasing the collaborative span. This may have benefits that partially or completely offset the increased coordination costs, but only up to some limit. As long as specialization is not "excessive" for the tasks, the average complexity of the work may decrease by enough to compensate for the increased collaborative entropy. Beyond some point however adding collaborators does not lower complexity while collaborative costs continue to grow along with the group population. The maximum per capita collaborative decision rate (the analog of the symbol transmission rate in communication channels) then declines if collaborative span continues to grow, and it becomes increasingly likely that individuals will over-run the collaboration channels' ability to keep up with them.
2. The maximum decision rate for collaboration-dominated organizations

This section develops an illustrative formula for the maximum decision rate, applicable when individuals in an organization spend essentially all of their time in collaborative as opposed to individual activities. This "collaboration-dominated" limiting case allows us to produce an analytically solvable model with few parameters. Appendix Sections 6.3 and 6.4 provide a more mathematically detailed development of the theory leading to the same result when the simplifying conditions introduced here are applied.

Section 4 considers the more general case when individual contributor productivity is included and the mix of individual versus collaborative activities is allowed to evolve with time. The span then takes on additional meaning as the number of contributors working individually on parts of a problem, and the model predicts strong productivity variation when organizations evolve to be heavily collaborative.

Organizations are represented as networks of collaborative human decision makers [Be1972], who are referred to interchangeably below as actors or nodes. The internal collaboration links between actors are a kind of glue that distinguishes an organization from a collection of autonomous individuals. Some external linkages are obviously needed if the enterprise is to have a purpose other than its own perpetuation, but these are assumed to be few enough to support a well-defined organization boundary and to have negligible perturbative impact on the overall work flow.

2.1. Definition of decision information

Intellectual property creation and knowledge transfer is viewed as a set of "decisions" [Be1972]. This model may seem mechanistic when applied to highly creative work but it has the advantage of allowing quantification. Knowledge work is represented in principle as a complex sequence of many basic independent binary choices, and each high level decision can be represented in principle by a binary decision tree representing a sequence of binary choices [Be1972].

The quanta of binary decision information are analogous to "bits" in classical information theory and they are accordingly called "dits" below to clearly distinguish them from more complex application-level decisions and to suggest a limited parallelism with communication applications. An "application-level" decision pertaining to some practical task corresponds to a string of "dits" whose length is a measure of its complexity. Constructing the binary representations (e.g., via binary trees) would be tedious to say the least, and so decision complexity parameters would probably be determined in practice by empirical comparisons rather than absolute measurements.

The average amount of decision information carried by one of these application level decisions will be called the "decision complexity" – equated to the average number of dits per decision which is in turn assumed to be proportional to the amount of cognitive effort needed to arrive at the decision.

This two-level information structure echo's the one in digital communication systems. Claude Shannon [Sh1998] recognized that the amount of information in symbols (such as letters of the alphabet) is measured by the number of elementary binary “bits” per symbol, assuming the most compressed representation. The "Fundamental Theorem for a Noiseless Channel" showed that the maximum symbol transmission rate (symbols/unit time) through conventional communication channels is a quotient in which the numerator is the physical channel capacity (in bits/unit time). The denominator is an entropy-like function representing the amount of information per symbol; that is, the number of bits per symbol needed using the most compact coding (see Appendix Section 6.2 for more discussion).

If the symbols to be transmitted are predictable they carry very little choice information and can be represented by a short string of bits. Conversely, if they are unpredictable (all equally probable, for example) the amount of choice information is a maximum and a longer string of bits is needed. The connection between information, entropy functions, and choice has been conceptually and mathematically appreciated in statistical physics for a long time (see Appendix 6.1).

The maximum decision rate $M_i$ for an individual actor – viewed as a single cognitive channel - is likewise defined as the ratio of the raw (binary “dit”) decision capacity $R_i$ to the decision complexity $A_i$. The latter includes the "collaborative entropy" – a coordination cost that includes the information overhead of having to choose and interact with multiple collaborators. $M_i$ measures application level decisions per unit time, which are composed of binary “dits”.

There is no relationship between a string of “dits” that represents a decision and any of the ways that the same decision might be coded by a communication system, using ordinary text or graphics symbols. Dits measure an amount of choice related to problem-solving and potential actions, but not the way a decision is displayed or stored by information systems hardware or software, or for that matter by handwritten documents. Additionally, there is emphatically no relationship between collaborative decision channel capacities and traffic levels on physical data networks that may be used to convey the information. Collaborative coordination costs exist independently of the
physical means used for communication and have been present as long as humans have divided up tasks and formed workgroups.

2.2. Sub-nets

The elementary workgroup structures will be called “collaborative sub-networks”. Each actor is the terminus of a unique set of collaboration links, each of which is inbound or outbound. The links map out the functional (and sometimes also the formal) structure. One actor makes decisions and communicates them using the outbound sub-net of links (viewed as shared capacity among links) and also accepts decision information back from the same or other collaborators through an inbound sub-net (also shared among links). The inbound and outbound links may or may not be to the same collaborators; hence, sub-nets can potentially have a broader range of topologies (structures) than the simplest workgroups.

For simplicity, this section assumes that the inbound and outbound collaborative spans – the numbers of collaborations - are equal to each other (numerically symbolized by \( m \)) and are much less than \( n \), the total population of the organization.

In order for the collaborations to have impact, actors must create and assimilate decisions cognitively. A sub-net’s maximum total throughput \( R_{i}^{\text{coll}} \) - measured in “dits” - is thus bounded above by the capacity \( D_{i}^{\text{coll}} \) of the person where the collaboration links come together. A sub-net behaves like a single channel with a hard upper limit on its “dit” capacity – a maximum rate for handling elementary binary decisions. There is thus a related maximum rate for making and sharing application-level decisions. If these rates are exceeded the information to be exchanged may be lost or arrive too late.

2.3. Decision rate expressions in the collaboration-dominated limit

The maximum decision rate for an entire organization is the sum of the maximum decision rates of each of the \( n \) sub-nets (i.e. terminating on the \( n \) actors) in it:

\[
M_{\text{tot}}^{\text{(n)}} = \sum_{i=1}^{n} M_{i}^{\text{tot}} \quad (2.1) \quad (6.14)
\]

The superscript is a reminder that most complex tasks have both solitary and collaborative components. The average per capita maximum decision rate \( \mu_{\text{tot}}^{\text{(n)}} \) is simply the expression above divided by the number of actors \( n \) in the entire organization viz.:

\[
\mu_{\text{tot}}^{\text{(n)}} = \frac{M_{\text{tot}}^{\text{(n)}}}{n} \quad (2.2) \quad (6.15)
\]

This quantity is a productivity cap on the use of intellectual capital and should correlate strongly with organizations’ competitiveness in markets where knowledge-based efficiency is a factor. It is the principal metric considered below. It is far from being merely a constant, but varies with workgroup and/or organization scale.

For simplicity, this section of the paper assumes that collaborative activity is dominant and that the individual contributor (“solo”) components of \( M^{\text{tot}} \) are negligible. Appendix Section 6.3 presents a more general model that incorporates individual contributors – see especially Equation 6.13 which and serves as the starting point for the more sophisticated model developed in Section 4.

The maximum collaborative decision rate (application-level decisions/unit time) for a single actor is the ratio of the raw dit capacity usable for collaboration to the collaborative decision complexity:

\[
M_{i}^{\text{coll}} = \frac{R_{i}^{\text{coll}}}{A_{i}^{\text{coll}}} \quad (2.3) \quad (6.16a)
\]

This expression is the analog of the maximum symbol transmission rate stated by Shannon’s Fundamental Theorem for a Noiseless Channel (see appendix Section 6.2 and [Sh1998]). The individual contributions workload was assumed to be zero.

The dit capacity \( R_{i}^{\text{coll}} \) is the sum over all inbound and outbound collaboration links and can not exceed the \( i^{\text{th}} \) individual’s hard decision capacity limit \( D_{i}^{\text{coll}} \); that is:

\[
R_{i}^{\text{coll}} \leq D_{i}^{\text{coll}} \leq D_{i}^{\text{tot}} \quad (2.4) \quad (6.10b)
\]
In the above, \( D_i^{\text{tot}} \) may also include some capacity allocated to individual contributions (see Appendix 6.3.1) or other activities that are ignored in this collaborative limit, letting:

\[
R_i^{\text{coll}} = \frac{D_i^{\text{coll}}}{D_i^{\text{tot}}} \quad \text{and} \quad D_i^{\text{coll}} = D_i^{\text{tot}}
\]

The relation between \( D_i^{\text{col}} \) and \( D_i^{\text{coll}} \) is defined further by Equation (6.2) of the Appendix, viz.:

\[
D_i^{\text{coll}} = D_i^{\text{tot}} \left( \frac{\eta_i}{1 + \eta_i} \right) \quad \text{where} \quad \eta_i = \frac{D_i^{\text{coll}}}{D_i^{\text{sol}}} \quad (6.2)
\]

Here we assume that the ratio \( \eta_i \) is >> 1, signifying collaboration dominance, but in general that condition need not hold (See Section 4).

### 2.4. Approximate solution for the maximum per capita decision rate

Three additional assumptions lead to an illustrative but approximate formula for evaluating Equation (2.3) above:

- All of the collaborations within a sub-net (i.e. entering or leaving a particular node \( i \)) have indistinguishable capacity requirements.
- All of the actors (nodes) are indistinguishable as well. The subscript \( i \) can be dropped.
- Each node has the same total number \( 2m \) of collaborations, with \( m \) of them outbound and \( m \) of them inbound.

These simplifications are adopted also in Section 4.2 and in Appendix Sections 6.3 and 6.4 to make all of the summations that appear in the defining equations trivial to perform.

The assigned collaborative dit capacity \( R^{\text{coll}} \) has the same value for any node, and can be written as

\[
R^{\text{coll}} = 2R_0 \min[m, m_{\text{crit}}] \quad (2.6)
\]

The factor \( \min[m, m_{\text{crit}}] \) is less than or equal to the span value that brings on saturation. The constant \( R_0 \) is just the average allocated “dit” capacity per uni-directional collaboration link – the analog of a communication sub-channel bandwidth in bits/second. The summation over outbound and inbound links produces the factor \( 2m \). The more general expression (Equation (6.10a) of Appendix Section 6.3) reduces to this one when the assumptions above are made in Sections 6.4.2 and 6.4.3.

The saturation condition in Equation (2.4) triggers if the collaborative span \( m \) reaches a critical value \( m_{\text{crit}} \) defined by:

\[
D^{\text{coll}} = 2R_0 m_{\text{crit}} \quad (2.7) \quad (6.3b)
\]

\( m_{\text{crit}} \) is the number of (uni-directional) collaboration links that saturate an individual’s binary decision capacity. Its value depends on the nature of the tasks to be performed at that node. Simple tasks would require a low bandwidth reserve \( R_0 \) and thus allow \( m_{\text{crit}} \) to become large. The reverse would hold for complex, teamwork-intensive collaborations such as design and analysis, or military combat.

If \( m \) exceeds \( m_{\text{crit}} \) then \( m_{\text{crit}} \) takes over in Equation (2.6) to limit “dit” capacity and throttle the average capacity available per link. The capacity available to individual collaborations in the sub-net would then be less than \( R_0 \): if the collaboration processes do not slow down they will lose content to errors and missed messages.

Many of these comments are summarized by defining the saturation fraction \( \beta^{\text{coll}}(m) \) as:

\[
\beta^{\text{coll}} = \frac{R^{\text{coll}}}{D^{\text{coll}}} = \min[m, m_{\text{crit}}] \quad \text{subject to} \quad m_{\text{crit}} \leq 1 \quad (2.8) \quad (\text{see 6.6})
\]

The collaborative decision complexity \( A^{\text{coll}} \) appears in the denominator of Equation (2.3). It contains the coordination cost and is assumed to have the same value for each of the \( 2m \) links at each node. For any single link the decision complexity is taken to be the product of an overall coefficient \( A_0 \) (that measures the intrinsic complexity for a one-way collaborative task) and a factor \( h \) that measures the range of collaborator choices. The latter has the logarithmic form of Shannon information entropy, now applied to decisions:

\[
h = -p \log_2(p)
\]
Appendix Section 6.6 shows why this logarithmic factor makes sense, following arguments in Hamming’s [Ha1980] text or Shannon’s original paper [Sh1998]. In the above “p” is the probability of choosing a particular collaborator rather than the probability of choosing a particular symbol. The (base 2) logarithmic factor measures the information per collaboration in “dits” where p is just 1/m for this simplified case. h is thus proportional to the expected value of the information for one link, including choice. After summing over all 2m identical links, the collaborative decision complexity \( A^{coll} \) for any single node is just the overall complexity coefficient \( A_0 \) multiplied by the choice factor for the one node, i.e.:

\[
A^{coll} = 2A_0 \log_2(m) \quad (2.9)
\]

Equation (6.12) of Appendix Section 6.3 reduces to this result when the assumptions above are made in Sections 6.4.3 and 6.4.4.

There is an additional argument for logarithmic scaling of the decision complexity. Consider a sub-net to be an information-processing machine that sorts the collaborators in order to locate information. The decision complexity \( A^{coll} \) should be proportional to the minimum sorting time. When \( m \) items are to be sorted using comparison (i.e., when there is no special property of the sort key we can depend on), the number of steps (proportional to the sort time) in the most efficient sorting algorithm is proportional to \( m \log_2(m) \) [Ah1974], where \( m \gg 1 \). If all the collaborators are equivalent the per capita value sort time needed scales as \( \sim \log_2(m) \).

Equations (2.1), (2.3), (2.6), (2.7), and (2.9) are used to yield the expression for the per capita decision rate \( \mu^{coll} \). It is written for clarity as the product of an overall constant \( M_0 \) and a dimensionless structure factor \( S^{coll} \):

\[
\mu^{coll} = M_0 S^{coll} \quad (2.10a) (6.20)
\]

\[
\text{where } M_0 = \frac{2R_0}{A_0} = \frac{D^{coll}}{m^{crit}A_0} \quad (2.10b)
\]

The normalization constant \( M_0 \) has a simple interpretation: it is the maximum per capita application-level decision rate for an elementary organization of three collaborative actors, each of whom is connected to the others by bi-directional collaborative channels (span \( m = 2 \)). This benchmark organization is the smallest one that it is meaningful to analyze. \( m^{crit} \) can be much greater than 2 in general, so such a group can be operating far below its saturation load level.

A structure factor \( S^{coll} \) is defined to measure productivity referenced to that small group as a benchmark. It is given by:

\[
S^{coll}(m) = \frac{1}{2} \frac{\min[m,m^{crit}]}{\log_2(m)} \quad \text{for } m \geq 2 \quad (2.11) (6.19)
\]

The numerator in the above becomes constant when \( m \) exceeds \( m^{crit} \). Appendix Section 6.5 offers a more detailed discussion and derivation of this result.

As \( m^{crit} \) is varied in Equation (2.11), series’ of structure factor values can be compared, by plotting them for example on the same scale. \( S^{coll} \) for different values of \( m^{crit} \) therefore ought to be normalized to the same value of \( M_0 \). However, the normalization factor \( M_0 \) defined in Equation (2.10b) appears superficially to vary inversely with \( m^{crit} \); or equivalently, it appears to be directly proportional to the assigned “dit” capacity \( R_0 \) for a link. At first thought, that would seem to require inserting ratios of the appropriate \( m^{crit} \) values into Equation (2.10b) to correct the normalization.

In actuality however \( M_0 \) should be approximately constant as the decision complexity changes. The independent variable in Equation (2.10b) is the decision complexity coefficient \( A_0 \). The collaborative bandwidth \( R_0 \) (and therefore \( m^{crit} \)) that is needed to service it should depend most strongly on \( A_0 \). For simplicity, assume roughly proportionality:

\[
R_0 = CA_0, \quad C = \text{a constant}
\]

Equation (2.7) and the above thus imply that re-scaling of the structure factor is not needed when the decision complexity changes since:

\[
M_0 = \frac{2m^{crit}R_0}{m^{crit}A_0} = 2C \quad \text{(constant)}
\]
The critical span $m_{\text{crit}}$ is thus inversely related to the decision complexity and it is used below as a dimensionless measure of an application’s complexity: complex tasks require collaboration links that can carry higher information levels, and an actor’s “dit” capacity is saturated by a few of them (small $m_{\text{crit}}$). Relatively simple collaborative tasks require links with smaller capacity and therefore allow $m_{\text{crit}}$ to be larger.

2.5. Discussion of the solution: the optimal collaborative span

At any value of the collaborative span the per capita productivity is reduced by the logarithmic cost factor, compared to what it would be if collaborative entropy were to be neglected (replaced by unity). In the latter case, $S_{\text{coll}}$ would grow linearly as a function of the collaborative span $m$, saturate at $m = m_{\text{crit}}$, and then remain constant as $m$ continues to grow.

The per capita maximum decision rate grows with the overall scaling factor $m / \log_2(m)$ until the collaborative span $m$ reaches $m_{\text{crit}}$. The net productivity is improving but the logarithmic factor in the denominator is nevertheless imposing a collaborative coordination cost by forcing productivity growth to be slower than linear in $m$ even though the actors have not yet reached their capacity saturation levels.

Once the collaborative span reaches and exceeds the saturation value $m_{\text{crit}}$ productivity begins to actually fall off as $1 / \log_2(m)$ due to increasing coordination costs.

The structure factor $S_{\text{coll}}(m)$ and therefore $\mu_{\text{coll}}(m)$ have maxima where $m$ equals $m_{\text{crit}}$. As a result, per capita productivity peaks just as actors’ capacity becomes saturated: $m_{\text{crit}}$ is therefore the optimal collaborative span insofar as productivity is concerned.

The disclosure that there is an optimal sub-net size means that there is a significant productivity penalty associated with operating collaborative workgroups that are overly large and thus operate in an over-saturated mode with overloaded people. The coordination penalty is smaller and partially masked below saturation, where net productivity is growing with group size as an increasing fraction of the decision capacity is put to use. The presence of a peak rather than a plateau is due to the collaborative entropy. Empirical experience suggests that the optimal size for tightly linked workgroups doing complex, collaborative teamwork is often in the range of 8 – 12 members.

2.6. Parametric study of span dependence

Figure (1) is a parametric plot of the structure factor $S_{\text{coll}}(m)$ versus collaborative span $m$ for several values of $m_{\text{crit}}$ between 8 and 24. As before, $m_{\text{crit}}$ is inversely related to the average task complexity.

The capacity limit $D_{\text{coll}}$ remains fixed by assumption while $m_{\text{crit}}$ progressively increases. The decision capacity $R_S$ allocated per link must decrease in response. More application-level decisions can be supported but each one has progressively decreased information content.

Each curve has a peak at its own value of $m_{\text{crit}}$ and the magnitude of the structure factor at that peak grows with $m_{\text{crit}}$. The curves are modulated above and below their peaks by the entropy cost factor $1 / \log_2(m)$ as in Equation (2.11). The structure factor (referenced to productivity in a three person workgroup with $M = 2$) never grows larger than about 2.5 in Figure 1

Collaborative coordination cost substantially lowers productivity by a factor of $\log_2(m)$ even at the optimal point where $m = m_{\text{crit}}$. Figure (2) shows how the optimal value of the structure factor varies as a function of $m_{\text{crit}}$ (the optimal collaborative span), and also plots the value the structure factor would have if collaborative entropy were not considered; i.e., if the logarithmic term in Equation (2.11) is set equal to one. In small groups with $m_{\text{crit}} = 8$ for example, about 2/3 of the potential (no entropy) peak productivity ($m = m_{\text{crit}}$) is consumed by coordination costs.

Inasmuch as the cognitive limit $D_{\text{coll}}$ is fixed, the optimal collaborative span $m_{\text{crit}}$ is larger for simple tasks than for complex ones in Figure 2, as expected intuitively. If the tasks are made simpler the peak productivity grows but the entropy cost consumes a higher percentage of the effort than for more complex tasks. If $m_{\text{crit}}$ is 32 for example, about 80% of the potential (no-entropy) productivity is lost to collaboration costs.

As remarked in Section 4, the coordination costs will be smaller than this study would suggest when the value of expert collaborations that reduce decision complexity is taken into account.
Figure 1: The structure factor in the collaborative limit. The peaks are at $m^{\text{crit}}$.

Figure 2: Coordination cost impact on optimal collaborative spans.
3. Collaborative productivity as a function of organization size

The productivity limit derived above is a function of the generalized workgroup size (collaborative span). But a wealth of anecdotal evidence suggests that there is also a productivity penalty that depends on the total population \( n \) of an entire organization. This section couples the *per capita* productivity to the total organization size, arguing that the collaborative span tends to grow as a function of the total population and thereby increases the collaborative entropy costs at the workgroup level.

The complexity and inter-relatedness of tasks that firms can credibly address grows as the firms themselves grow, stimulating wider distribution of functions over knowledge workers who offer specialized experience and skills. The number of available collaborative system states increases as the population \( n \) increases. If an entropic increase principle holds for organizations, as it does for some other systems in nature, then an organizations’ evolution may favor populating more and more of the available collaborative system states. This means that actors’ spans may expand to fill the collaborative state space available – sometimes beyond the point where adding collaborators improves productivity.

Managers might control such coupling by closely monitoring workgroup structure, i.e., by “sparsifying” collaborative linkages as is already done intuitively in process re-engineering. It is probably not possible though to completely uncouple collaborative span from organization size, owing to the growing task complexity and possibly to an entropic principle.

3.1. A model for the span as a function of an organization’s population

This section formulates the model used to predict deteriorating *per capita* productivity as an organization’s population grows. A “propagator” function is used to describe the detailed evolution path of the collaborative span \( m \) as a function of the total population \( n \). Once constructed, the propagator is substituted into Equation (2.11). The function \( m(n) \) should increase at most linearly with \( n \), otherwise the total number of possible collaborations \( (nm) \) would grow faster than \( n^2 \)- in conflict with expectations for the most extreme case.

The simple model adopted assumes that a *fractional* change in collaborative span \( m(n) \) is caused by a *fractional* change in an organization’s total population \( n \). The function \( \gamma(n) \) describes the detailed response of the span to population changes. These notions are expressed by the following differential equation:

\[
\frac{dm}{m} = \gamma(n) \frac{dn}{n}
\]

The formal solution of the equation above (a so-called “propagator”) describes the growth path for the span as a function of the total population growth:

\[
m(n) = m_0 \exp \left\{ \int_{n_0}^{n} \frac{\gamma(n')}{n'} dn' \right\} \quad (3.1)
\]

The solution depends on two arbitrary constants \( m_0 \) and \( n_0 \) that are the lower integration limits. The integral in the exponent on the right hand side above vanishes as \( n \to n_0 \) for any well-behaved choice of the function \( \gamma(n) \), causing the exponential factor to approach unity and \( m(n) \) to approach \( m_0 \). The constants are advantageously chosen to be the saturation values \( m_{\text{crit}} \) and \( n_{\text{crit}} \). As the total population \( n \) approaches the value \( n_{\text{crit}} \) the average actor becomes saturated as defined previously and the average span \( m \) approaches \( m_{\text{crit}} \).

The functional form of \( \gamma(n) \) can in principle be used to model the growth path in detail, reflecting resource allocation decisions, re-assignments, etc. For simplicity though, assume that \( \gamma(n) \) is just a constant \( \gamma \) in which case the propagator above simplifies to a power law:

\[
m(n) \equiv m_{\text{crit}} \left( \frac{n}{n_{\text{crit}}} \right)^\gamma \quad \gamma \leq 1, \ m \geq 2 \quad (3.2)
\]

The magnitude of \( \gamma \) must be smaller than \( 1 \) or else the number of collaborations would grow faster than \( n^2 \), which is absurd. Equation (3.2) is invertible so that one can also write \( n \) or \( n_{\text{crit}} \) as functions of \( m \). The values of \( n/n_{\text{crit}} \) that result in \( m < 2 \) must be excluded. A negative value for \( \gamma \) would signify that span decreases as the population grows.
This possibility would run counter to the usual trend and is not considered further at this time, although it is potentially applicable to backing away from an over-saturated staff.

The integration constant \( n_{\text{crit}} \) represents the population an organization has when the average actor in it is at the point of saturation; its value must be determined empirically, perhaps with the aid of additional models. It will vary for different organizations. Equations (3.1) and (3.2) are not a definition of \( n_{\text{crit}} \) and do not contain enough information to determine it.

Managers responsible for controlling productivity during growth periods in effect need to modulate \( \gamma \) (although few would describe their actions quite so abstractly). In an ideal case \( \gamma \) might become zero; the coupling would be broken and collaborative span \( m(n) \) would become independent of organization size \( n \) and might remain constant. If \( \gamma \) should happen to equal 1 the collaborative span increase would be exactly proportional to the growth in organization size and productivity would fall precipitously. The fraction \( m/n \) of the total population in the average sub-net would then remain constant and equal to \( m_{\text{crit}}/n_{\text{crit}} \). If that ratio should happen also to initially equal 1 (with \( \gamma \) also equal to 1) everyone would be collaborating with everyone else; this defines a “fully connected organization” as discussed in earlier papers by the author [Ja2003, Ja2004].

The most reasonable range of values for \( \gamma \) is probably from 0.1 to 0.5. Suppose for example that collaborative span \( m(n) \) doubles whenever \( n \) increases by an order of magnitude: \( \gamma \) would then be 0.301. Empirical experience is needed to validate the model and parameter range.

### 3.2. Structure factor as a function of population

The next step is to substitute Equation (3.2) into Equation (2.11). The logarithm in the denominator becomes:

\[
\log_2(m) = \log_2(m_{\text{crit}}) + \gamma \log_2(n) - \gamma \log_2(n_{\text{crit}})
\]

This expression is then expanded and approximated to lowest order, yielding simplified expressions for the structure factor \( S_{\text{coll}} \) and the per capita maximum decision rate \( \mu_{\text{coll}} \):

- An organization with \( n/n_{\text{crit}} >> 1 \) is far above saturation since this implies that \( m >> m_{\text{crit}} \). The term \( \gamma \log_2(n) \) is then the dominant one in the denominator. The structure factor falls off logarithmically as a function of the organization’s total population.

- When \( n < n_{\text{crit}} \) or \( n = n_{\text{crit}} \) an organization is below or near saturation. The first term in the denominator (above) can be shown to be dominant by expanding to lowest order. The structure factor then grows as \( n^{\gamma} \); i.e. slower than linearly (since \( \gamma < 1 \)) when viewed as a function of \( n \).

The approximations for the structure factor as a function of total population size are:

\[
S_{\text{coll}}(n) = \begin{cases} 
\frac{1/2 m_{\text{crit}}}{\log_2(m_{\text{crit}})} \left( \frac{n}{n_{\text{crit}}} \right)^\gamma & \text{for } n \leq n_{\text{crit}} \quad (3.3a) \\
\frac{1/2 m_{\text{crit}}}{\gamma \log_2(n)} & \text{for } n >> n_{\text{crit}} \quad (3.3b)
\end{cases}
\]

The per capita average maximum decision rate \( \mu_{\text{coll}} \) is simply either of the above approximations multiplied by \( M_0 \), as per Equation (2.10a).

Wherever the premise of this section applies - if collaborative span grows with the total organization size - the point will be reached where Equation (3.3b) is applicable; that is, the average collaborative span will be driven past saturation. The per capita productivity \( \mu_{\text{coll}} \) will then fall off as \( 1/\log_2(n) \) with further growth.

As firms trafficking in knowledge work become large they may therefore develop a dis-economy of scale that can have a decisive effect on their competitiveness. When there is compensating “value” that reduces the collaborative or solitary decision complexities (see Section 4 of this paper), the logarithmic decline may begin at a shifted point but it still dominates behavior in the large organization limit.

### 3.3. Parametric study of population dependence

Figure 3 plots the structure factor parametrically as a function of the organization size for two organizations performing tasks with very different values of \( m_{\text{crit}} \); that is, with very different decision complexities.
Equations (2.11) and (3.2) were evaluated in their exact forms. The population ratio \( \tau = n/n^{\text{crit}} \) appears in Equation (3.2) rather than the separate magnitudes of \( n \) and \( n^{\text{crit}} \). The average actor’s dit capacity becomes saturated (\( m = m^{\text{crit}} \)) when \( \tau = 1 \). Small values of \( \tau \) may correspond to spans less than two and were therefore excluded, as they violate the validity limits for this theory.

The overall features discussed earlier in connection with Figure 1 appear again, but now mapped onto a total population scale. For example, the curves peak when \( \tau = 1 \) (\( n = n^{\text{crit}} \)). The curves are increasing functions immediately below that point and fall off logarithmically above it. As noted earlier, the collaborative entropy cost contributes an attenuation factor of \( \log_2(n) \).

The lower family of curves with \( m^{\text{crit}} = 8 \) in Figure 3 (open symbols) might represent an organization engaged in high decision complexity applications (e.g., product development). The upper curve family (\( m^{\text{crit}} \) equals 24) might represent an organization whose decision tasks are much less complex and therefore allow the span to become much larger before bringing actors to saturation. Each of these examples is plotted for three bracketing values of the power law exponent \( \gamma \) (equal to 0.15, 0.30, and 0.45). The qualitative behavior is the same in all of the cases but with differences as noted below.

![Figure 3: Structure factor as a function of population using the power law model](image)

As either organization starts with a small population and grows, the collaborative productivity grows slower than linearly. It peaks when the population ratio \( \tau = 1 \). Further growth drives the collaborative span beyond its optimal (saturation) value and \textit{per capita} productivity falls off logarithmically. If the collaborative entropy cost had been neglected the productivity for \( \tau > 1 \) would have remained constant at a much higher peak value.

The structure factor (productivity measure) is roughly twice as large in the simple-task organization for a given value of \( \tau \) (saturation level). The normalization factor \( M_0 = 2R_0/A_0 \) in Equation (2.10b) is essentially the same for both curve families, since the decision complexity coefficient \( A_0 \) and the required collaboration channel bandwidth \( R_0 \) scale together (discussed toward the end of Section 2.4).

There is a significant productivity penalty both above and below saturation due to collaborative entropy, but it is most apparent in Figure 3 for \( \tau > 1 \) when collaborators’ workloads exceed their saturation levels. At the optimal population size (\( \tau = 1, \ m = m^{\text{crit}} \)) the coordination cost penalty is a factor of \( \log_2(m^{\text{crit}}) \), referenced to the no-entropy case.

- For the \( m^{\text{crit}} = 8 \) application that factor is \( \log_2(8) = 3 \). Thus, the coordination cost is 67% at the optimal span.
- For the \( m^{\text{crit}} = 24 \) case the no-entropy productivity peak (\( n^{\text{crit}} \)) would be 12 rather than 2.62 as shown. The corresponding coordination cost in this case is about 78%.
Away from $\tau = 1$ the coordination cost is largest for the strongest coupling of span to organization size: that is, for $\gamma = 0.45$ in Figure 3. The *per capita* productivity would fall by about 50% if the organization’s population were to grow from $\tau = 1$ to the extreme value of about $\tau = 100$. For $\gamma = 0.3$ the fall-off is slower: productivity falls by about 40% of its peak value as the organization’s population grows through the same range.

The productivity varies much more strongly in Section 4.9, when we add the assumption that the *fraction* of actors’ effort spent collaboratively also grows as a function of total organization size.
4. Productivity models that include “value” and the collaborative/solo mix ratio

Previous discussions acknowledged that “value” added by expert collaborations may offset coordination cost, but for simplicity this effect was neglected. The impact of individual versus collaborative work on per capita productivity was likewise neglected. This section fills those gaps. Each actor is now viewed as splitting his/her effort between collaboration and individual work. The modified model incorporates some of the broader formal development presented in the Appendices as well as on the preceding work. The overall results found in Sections 2 and 3 change in detail, gain generality, and become identified as the limiting case when collaborative effort is dominant.

4.1. Assigning “value” to collaborations and expertise

Collaborative and individual tasks can often be made more productive by broadening the set of expert participants. High “value” produces large reductions of the decision complexities for collaborative and/or individual tasks, as defined quantitatively by Appendix Equations (6.11) and (6.12). “Value” and decision complexity are inversely related to each other: increased value improves the productivity measure (see Equation (4.1) below).

Expert actors are all assumed to be appropriate to their tasks and indistinguishable from one another in terms of their modeling parameters (obviously not in their functions). In real organizations optimal staffing choices are not likely, so this assumption tends to over-estimate productivities.

The motives for collaboration include increasing the total amount of labor available, sharing responsibility and information among stakeholders, and/or bringing specialized expertise to bear on problems. These objectives frequently co-exist but the first two do not necessarily produce a near-term reduction in decision complexity - they may actually increase it. For example, a product development program with strong collaborative hooks to users, suppliers, and marketing communities may increase coordination costs and thereby reduce individual productivity as measured here. But collaboration may nonetheless produce in very much better quality decisions and benefits, such as improved marketability of the final product. The metrics adopted in this paper do not measure intangible and delayed benefits.

The average decision complexity of a project should decrease as the number of collaborators and solo contributors grows, so long as the added specialized expertise and throughput fills an actual gap rather than duplicating that which is already available. This is modeled by replacing constant decision complexities (e.g., replacing \( A_0 \) in Section 2 above) with functions that decrease as the span \( m \) grows in order to represent complexity-reducing “value”. The functions should stop decreasing when the number of actors becomes large enough to meet all the expertise and capacity needs. Adding more collaborators or individual contributors then produces no further complexity reduction but continues to increase coordination costs. The decision complexities for each actor are all considered identical here for simplicity, but in practice they would vary link-by-link as indicated in Equation (6.12) of Appendix Section 6.3.

4.2. The per capita productivity limit

Equations (6.13), (6.14), and (6.15) of Appendix 6.3 prescribe how to include individual (“solo”) as well as collaborative contributions in the application-level per capita decision rate. Equation (6.13) is a generalization of Equation (2.3) of Section 2 but \( R^{\text{tot}} \) and \( A^{\text{tot}} \) are now broadened to include individual contributor contributions. Appendix Sections 6.3.1 through 6.3.4 provide a full discussion of the parameters controlling “dit” capacity allocation, collaboration-dominance, solo-dominance, and collaborative or solo capacity saturation. Section 6.3.6 discusses the decision complexity function and generalizes the expression for \( A^{\text{tot}} \) to include solo activity (see Equation (6.11))

The three modeling assumptions listed at the beginning of Section 2.4 are applied again to Equation (6.13), resulting in the following starting point expression for the maximum per capita decision rate:

\[
\mu^{\text{tot}}(m,k) = \frac{R^{\text{tot}}(m,k)}{A^{\text{tot}}(m)} = \frac{R^{\text{sol}}(k) + R^{\text{coll}}(m)}{A^{\text{sol}}(m) + A^{\text{coll}}(m)} \quad (4.1) (6.13)
\]

The total assigned “dit” capacity \( R^{\text{tot}} \) is given by \( R^{\text{tot}} = R^{\text{sol}} + R^{\text{coll}} \). The collaborative portion \( R^{\text{coll}} = 2R_{\text{m}}\min(m, m^{\text{crit}}) \) is a function of the span \( m \) as in Section 2 and Section 6.3.1. When \( m = m^{\text{crit}} \) the collaborative decision capacity is saturated and equals \( D^{\text{coll}} \) (see Equations (6.3a), (6.3b)). \( R_{\text{m}} \) is the average “dit” capacity needed per unidirectional collaboration linkage.

An individual contributor’s “dit” capacity \( R^{\text{sol}} \) is similarly proportional to the average number \( k \) of solo tasks that he is working on concurrently. When \( k \) reaches \( k^{\text{crit}} \) the allocated individual workload becomes saturated at the value \( D^{\text{sol}} \) (see Equations (6.4a) & (6.4b)). Individual contributors are also characterized by a solo task partitioning span that
measures the number of other solo contributors working on associated tasks.

The decision complexity functions \( A_{\text{coll}}(m) \) and \( A_{\text{solo}}(m) \) each include models for the “value” (discussion to follow). They are functions respectively of the average collaborative and solo spans, which are assumed for simplicity to have the same value \( m \) although this is not necessarily so. Individual tasks may have different inherent complexities that justify different spans than collaborative tasks do. But actors’ collaborative and individual work are often part of the same problems hence the simplifying approximation that the “spans” for task partitioning and for collaboration are the same is reasonable.

Note that \( m \) and \( k \) are independent of each other: \( k \) represents the average number of “solo” tasks an actor is working on, while \( m \) represents the total number of other individual contributors participating in the \( k \) tasks, as well as the number of overt collaborators.

The total decision complexity \( A_{\text{tot}} \) was defined in Equation (4.1) as a weighted average of \( A_{\text{solo}} \) and \( A_{\text{coll}} \), wherein the weighting coefficients \( \alpha_{\text{coll}} \) and \( \alpha_{\text{solo}} \) measure the relative effort allocated to collaborative versus individual activity for the average actor. These coefficients are the “attention” coefficients discussed in Section 6.3.5 and defined by:

\[
\alpha_{\text{coll}}(k,m) = \frac{R_{\text{coll}}(m)}{R_{\text{tot}}(k,m)} = \frac{\xi}{1 + \xi}
\]

\[
\alpha_{\text{solo}}(k,m) = \frac{R_{\text{solo}}(k)}{R_{\text{tot}}(k,m)} = \frac{1}{1 + \xi}
\]

(4.2) (see 6.9)

\[ \alpha_{\text{coll}} + \alpha_{\text{solo}} = 1 \quad \text{independent of } m, k \]

The collaborative/solo index \( \xi \) is defined as the following important ratio and is discussed in Section 6.3.2:

\[
\xi = \frac{R_{\text{coll}}(m)}{R_{\text{solo}}(k)}
\]

(4.3) (6.5)

In principle \( \xi \) depends on the spans \( m \) and \( k \) or equivalently on the saturation fractions \( \beta_{\text{coll}} \) and \( \beta_{\text{solo}} \) (see Section 6.3.3, Equation (6.6)). But when the ratio \( \beta_{\text{coll}}/\beta_{\text{solo}} \) of saturation fractions is held constant \( \xi \) is also constant and it measures the balance between collaboration and solitary work for all values of \( m \) and \( k \) (see Equation (6.7)). The ratio of saturation fractions is always constant under saturated conditions (\( m > m_{\text{crit}} \) and \( k > k_{\text{crit}} \)).

The “attention” coefficients \( \alpha_{\text{coll}} \) and \( \alpha_{\text{solo}} \) are used below as power series expansion parameters to evaluate Equation (4.1) near the collaboration-dominated and solo-dominated limiting cases.

- When activity is collaboration-dominated, \( \xi \) becomes very large, approaching infinity. \( R_{\text{tot}} \) then approaches \( R_{\text{coll}} \), and \( \alpha_{\text{coll}} \) is approximately 1. Both \( R_{\text{solo}} \) and \( \alpha_{\text{solo}} \) become small, first order quantities approaching zero.
- The solo-dominated limit is the reverse. \( \xi \) becomes very small, approaching zero. \( R_{\text{tot}} \) then approaches \( R_{\text{solo}} \), and \( \alpha_{\text{solo}} \) is approximately 1. Both \( R_{\text{coll}} \) and \( \alpha_{\text{coll}} \) become small, first order quantities approaching zero.

Equations (4.4) and (4.5) developed below represent the decision complexity; they are substituted into the denominator of Equation (4.1).

The collaboration links are all regarded as identical in all of the models. That would not be so in real workgroups – especially large and heterogeneous ones - as indicated in the Appendices. The effect of differentiating the links would be to smear out the fairly crisp features to be seen in figures presented below.

4.3. A model for the “value” of networks of collaborators

The approach for modeling “value” draws on work by Odlyzo and Tilly [OT2005], who considered how to estimate “value” for growing networks carrying information and entertainment. Those authors debunked an exaggerated and widely hyped model known as “Metcalfe’s Law” [Me1995/6]. They concluded via three distinct arguments that the economic value per user (one of \( N \) in the network) is proportional to \( \log_2(N) \); that is, the incremental value of adding users is diminishing (see Appendix Section 6.7 for more discussion).

The logarithmic form should also apply to organizations, but with modifications: the value remains constant once the
span reaches some maximum value beyond which there is no further gain. For organizations, the reciprocal of the value model appears in decision complexity expressions. The collaborative decision complexity also contains the coordination cost factor.

The logarithmic value function inherently assigns “diminishing returns” to the value of one additional collaborator when \( m \) of them are already present. The incremental value is proportional to \( \log_2(\frac{(m+1)}{m}) \), which approaches zero as \( m \) becomes large, so the incremental contributions when single individuals are added become smaller. However, the total contribution of collaborators \( m+1 \) to infinity would be infinite. As a practical matter infinite human populations can never be approached. In fact, the value due to the first ten collaborative partners can never increase by even one order of magnitude inasmuch as that would require on the order of \( 10^{10} \) collaborators - exceeding the world’s current population.

The rules for estimating per capita value added therefore have implied caps and can be represented by expressions of the following form:

\[
\text{value} \propto \log_2(\min[m, m_{\text{cap}}])
\]

The expression “\( \min[m, m_{\text{cap}}] \)” is an instruction to choose the smaller of the quantities in brackets, thereby holding the value of the function constant when the argument \( m \) exceeds the “cap” value.

The “value” of communication networks (e.g., the Internet) may possibly increase until most of the global population is incorporated. But task-oriented work groups - the focus of this paper – should behave somewhat differently. The problems to be solved and the range of specialized skills and expertise needed can often be enumerated (at least in principle) at a given time for all but the most technically aggressive (e.g., R&D) tasks. A set of well-chosen experts who provide the full range of needed skills is needed. Expanding the set of collaborators beyond the “cap” level would duplicate expertise already available and/or unnecessarily enlarge capacity - thereby generating additional coordination cost without adding “value”.

R&D problems requiring extreme creativity may not neatly fit this picture because each expert has a very small probability of making extremely high value contributions. Replication of expertise in such cases may increase the (small) odds of success. In effect, the “caps” would be much larger for the R&D-like cases.

“Value” caps will be called \( m_{\text{sol}} \) for solitary work and \( m_{\text{col}} \) for collaborations; they are in principle independent but may often be equal when the collaborative and solo workloads are focused on common problems.

The collaborative entropy function is logarithmic as well as the “value” functions; a cap should most properly be applied to it as well, inasmuch as there is a limit to the potential size of each actor’s sub-net. However, this implicit cap is very much larger than the value caps – potentially including the entire organization or profession - and so it is not stated explicitly in the formulas above and below. Productivity would approach zero if the span \( m \) could approach infinity, but that limit is not reachable.

4.4. Models for the decision complexity that incorporate “value”

The decision complexity functions \( A_{\text{coll}}(m) \) and \( A_{\text{solo}}(m) \) should both be inversely proportional to their respective “value” functions. The collaborative entropy appears also in \( A_{\text{coll}}(m) \). No explicit coordination cost is associated with the individual contributor value function \( A_{\text{solo}}(m) \).

4.4.1 A solo (individual contributor) “value” function

The solo decision complexity \( A_{\text{solo}} \) measures the “value” of partitioning work into sub-tasks performed by autonomous, individual contributors who may be subject matter experts. An actor can support the individual work of others not only through collaboration (represented by the value/cost function in \( A_{\text{coll}} \)) but also by working autonomously using complementary expert skills in the solo tasks. If the functional partitioning is suitable the need for collaborative information exchange might be reduced.

The cutoff for the number of individual contributor sub-tasks that adds value is \( m_{\text{sol}} \), meaning that partitioning over a larger group brings no added advantage. The decision complexity function is:

\[
A_{\text{solo}}(m) = A_{\text{solo}}^1 \frac{1}{\log_2(\min[m, m_{\text{sol}}])} \quad (4.4)
\]

The constant \( A_{\text{solo}}^1 \) must be empirically determined. The right hand side is constant when \( m > m_{\text{sol}} \).
No coordination costs appear in the “solo” model: only value enhancement. A task switching coordination cost might exist, due to actors’ concurrently switching their attention among $k$ solo tasks, but that possibility is not considered in this work.

4.4.2 A collaborative “value” function

The collaborative decision complexity $A_{\text{coll}}$ was approximated by $2A_0 \log_2(m)$ in Section 2, with the logarithmic function attributable to coordination costs. It is now replaced by a function that measures the collaborative “value” as well as the coordination entropy. The collaborative decision complexity function $A_{\text{coll}}(m)$ decreases as appropriate members join the team until there are $m_{\text{coll}}$ of them, beyond which there is no further improvement. The model is:

$$A_{\text{coll}}(m) = 2A_1^{\text{coll}} \frac{\log_2(m)}{\log_2(\min[m, m_{\text{coll}}])}$$

(4.5)

The constant $A_1^{\text{coll}}$ is empirical and defined differently from the constant $A_0$ in Section 2. The factor of 2 in the definition is nonetheless inserted for consistency with Section 2 (Equation (2.9)).

The collaborative “value” and entropy cost tend to offset each other in Equation (4.5) above. While a collaborative sub-net remains small (i.e., while $m < m_{\text{coll}}$) the logarithmic terms cancel and the decision complexity $A_{\text{coll}}(m)$ is a constant proportional to $2A_1^{\text{coll}}$ and independent of span $m$. As the span grows, the benefits continue to offset the coordination costs until $m$ equals $m_{\text{coll}}$, beyond which adding to the team adds no further gain. But the coordination cost (entropy) factor continues to grow for $m > m_{\text{coll}}$ and drives down per capita productivity.

4.5. Productivity in the collaboration-dominated limit

The collaboration-dominated limit means that actors spend almost all their time exchanging information with others, who may be subject matter specialists and other stakeholders. There is essentially no individual work. Some examples of inherently collaboration-intensive tasks may be broad systems integration, planning, intelligence analysis, financial ventures and analysis, and complex negotiations in general.

The conditions defining collaboration-dominance are the following:

$$\xi \gg 1 \quad \text{or} \quad \alpha_{\text{coll}} = 1 \quad \text{and} \quad \alpha_{\text{solo}} \ll 1$$

$$\text{or} \quad R_{\text{solo}} << R_{\text{coll}}$$

(4.6)

where

$$R_{\text{coll}} = 2R_0 \min[m, m_{\text{crit}}]$$

To find the limiting case expression, Equation (4.1) was expanded as a power series in the small parameter $\alpha_{\text{solo}}$ to first order. The leading term (zero$^\text{th}$ order) is larger than the first order correction term (proportional to $\alpha_{\text{solo}}$) by a comfortable margin (see discussion below). The zero$^\text{th}$ order approximation to the productivity upper bound $\mu_{0_{\text{coll}}}$ is independent of $\alpha_{\text{solo}}$:

$$\mu_{0_{\text{coll}}}(m) = \lim_{\alpha_{\text{coll}} \to 0} \mu_{\text{tot}}(m,k) = \frac{R_{\text{coll}}(m)}{A_{\text{coll}}(m)} = \frac{R_0 \min[m, m_{\text{crit}}]}{A_1^{\text{coll}} \log_2(\min[m, m_{\text{coll}}])}$$

(4.7a)

A comparable formula appeared in Section 2 (Equation (2.11)), but the constant decision complexity $A_0$ in the earlier model has been replaced here by the decision complexity function $A_1^{\text{coll}} / \log_2(\min[m, m_{\text{coll}}])$. An alternative form for $\mu_{0_{\text{coll}}}$ uses the saturation fraction $\beta_{\text{coll}}$ (see Equations (6.6), (2.8), and (6.2)).
\[ \mu_0^{\text{coll}}(m) = \beta^{\text{coll}}(m) \frac{D^{\text{coll}}(m)}{A^{\text{coll}}(m)} = \beta^{\text{coll}}(m) \frac{\eta D^{\text{tot}}}{(1 + \eta)A^{\text{coll}}(m)} \log_2(\min[m,m^{\text{coll}}]) \]

(4.7b)

where \( \beta^{\text{coll}} = \frac{\min[m,m^{\text{crit}}]}{m^{\text{crit}}} \leq 1 \)

In small collaborative nets (i.e., where \( m < m^{\text{coll}} \)) the collaborative “value” and coordination entropy cancel each other in Equation (4.7a), masking both effects although they are present nonetheless. This masking may help explain why collaborative entropy was not clearly identified long ago.

If \( m \) is also less than \( m^{\text{crit}} \) (as in a small group below saturation) the per capita productivity grows linearly with \( m \), essentially because individual actors’ raw collaboration capacity is underutilized on the average and can accommodate more partners without saturating. Small groups may thus seem to have collaborative “synergies” as per capita productivity grows along with the sub-nets. If the productivity growth trend were to continue, total organization output would scale as \( nm \) - clearly not realistic. But this scaling shuts off when the sub-net reaches “dit” capacity saturation (\( m = m^{\text{crit}} \)), halting further productivity growth. Genuine synergies can accompany truly complementary collaborator skills and task partitioning.

If the span \( m \) is greater than \( m^{\text{coll}} \) the “value” factor is constant while the collaborative entropy cost factor in the denominator continues to increase. If \( m \) also exceeds \( m^{\text{crit}} \), the per capita productivity deteriorates logarithmically as \( 1/\log_2(m) \).

For intermediate values of \( m \) the span-dependence depends on which of \( m^{\text{coll}} \) and \( m^{\text{crit}} \) is larger. The top sketch in Figure 4 plots Equation (4.7a) for the condition \( m^{\text{coll}} < m^{\text{crit}} \). The lower sketch shows the reverse condition \( m^{\text{crit}} < m^{\text{coll}} \) below it. In both sketches productivity growth is linear in \( m \) for very small sub-nets. For very large sub-nets \( m \) greater than both \( m^{\text{crit}} \) and \( m^{\text{coll}} \) both sketches also predict logarithmic deterioration due to collaborative entropy.

- The top sketch \( (m^{\text{coll}} < m^{\text{crit}}) \) might correspond to comparatively simple tasks for which actors can acquire access to all the important collaborators and their skills before their collaborative capacity saturates. The per capita productivity \( \mu_0^{\text{coll}} \) continues to grow from \( m^{\text{coll}} \) to \( m^{\text{crit}} \) but it has below-linear scaling \( (m/\log_2(m)) \) until saturation, as did Equation (2.11).

- The lower sketch \( (m^{\text{crit}} < m^{\text{coll}}) \) might apply to a more stressing application that requires a broader range of skills; the actors reach capacity saturation while there is still added value to be found by expanding the team. Productivity remains constant in this range \( (m^{\text{crit}} < m < m^{\text{coll}}) \) with value growth and entropy costs offsetting each other until the span reaches \( m^{\text{coll}} \), above which the productivity deteriorates logarithmically.

The first order correction \( \mu_1^{\text{coll}} \) to Equation (4.7a) is proportional to the small parameter \( \alpha^{\text{solo}} \) and to span-dependent factors. It can always be neglected for all values of \( m \) – even in the worst case - as it is smaller than the leading term \( \mu_0^{\text{coll}} \) by a factor of \( \alpha^{\text{solo}} \). For example, suppose that the complexity coefficients \( A_{1,\text{solo}}^{\text{solo}} \) and \( 2A_1^{\text{coll}} \) are comparable in magnitude and also that the value cutoffs \( m^{\text{coll}} \) and \( m^{\text{solo}} \) are roughly equal. The convergence criterion for the series expansion becomes:

\[ \alpha^{\text{solo}}(k,m) \frac{A^{\text{solo}}(m)}{A^{\text{coll}}(m)} \leq \alpha^{\text{solo}}(k,m) \ll 1 \]

It is then straightforward to show that the first order correction is negligible, i.e., that the ratio below is small:

\[ \frac{\mu_1^{\text{coll}}}{\mu_0^{\text{coll}}} = \alpha^{\text{solo}}(k,m) \left\{ 1 - \frac{1}{\log_2(m)} \right\} \]

The right hand side above is positive for all \( m \) and the quantity in brackets is always smaller than unity. Hence, the ratio of terms never exceeds the small quantity \( \alpha^{\text{solo}} \).
Equation (4.7b) is re-written as the product of an overall normalization constant $M_{1}^{\text{coll}}$ and a dimensionless structure factor $S_{0}^{\text{coll}}$ (as before). First define:

$$M_{1}^{\text{coll}} = \frac{D^{\text{coll}}}{m_{\text{crit}}A_{1}^{\text{coll}}} = \frac{2R_{0}}{A_{1}^{\text{coll}}} \quad (4.8a)$$

for which

$$\mu_{0}^{\text{coll}} = M_{1}^{\text{coll}} S_{0}^{\text{coll}} \quad (4.8b)$$

The constant $M_{1}^{\text{coll}}$ is the per capita decision rate for a collaboration-dominated workgroup of three actors, each collaborating with the two others ($m = 2$). This is the smallest organization that can be analyzed meaningfully (see Appendix 6.5). The normalization constant $M_{1}^{\text{coll}}$ is independent of $m_{\text{crit}}$ because the “dit” capacity $R_{0}$ required is itself proportional to the complexity coefficient $A_{1}^{\text{coll}}$ (see Section 2.4). The structure factor below is therefore correctly normalized so that it can represent intrinsic differences in the decision complexity (differing values of $m_{\text{crit}}$) as shown:

$$S_{0}^{\text{coll}}(m) = \frac{1}{2} \min[m, m_{\text{crit}}] \frac{\log_{2}(\min[m, m_{\text{coll}}])}{\log_{2}(m)} \quad (4.9)$$

for $m \geq 2$, $m_{\text{coll}} \geq 2$, $m_{\text{crit}} \geq 2$

When $m = 2$ the function $S_{0}^{\text{coll}}$ equals unity, confirming that the structure factor measures productivity relative to a benchmark three-actor group.
4.6. Productivity in the individual contributions (solo) limit

In the opposite limiting case the dominant part of an average actor’s effort is individual work. Complex tasks are partitioned onto specialists who can execute them with almost no collaboration, working essentially autonomously on pieces of a larger task that has been structured to make good use of their specialized skills. Many kinds of high level technical and analytical work can fit this pattern - large, technically aggressive engineering projects, for example.

The solo-dominance conditions applied to Equation (4.1) are the following:

\[
\xi << 1 \quad \text{or} \quad \alpha^{\text{solo}} = 1 \quad \text{and} \quad \alpha^{\text{coll}} << 1
\]

where

\[
R^{\text{solo}} = T_0 \min[k, k^{\text{crit}}]
\]

\(T_0\) is the average decision capacity in “dits”/second required for each of \(k\) tasks assigned on the average per actor (see Section 6.3.1). Equation (4.1) will once again be expanded as a power series, now with \(\alpha^{\text{coll}}\) as the small expansion parameter and \(\alpha^{\text{solo}} = 1\). With the substitutions above Equation (4.1) becomes:

\[
\mu^{\text{solo}}(k,m) = \mu_0^{\text{solo}}(k,m) \frac{1 + \alpha^{\text{coll}}}{1 + \alpha^{\text{coll}} \frac{A^{\text{coll}}(m)}{A^{\text{solo}}(m)}}
\]

(4.11)

In the above \(\mu_0^{\text{solo}}\) is the leading (zero order) term of the power series expansion in \(\alpha^{\text{coll}}\) (to be performed below). It is given by the following:

\[
\mu_0^{\text{solo}}(m,k) \equiv \lim_{\alpha^{\text{solo}} \to 0} \mu^{\text{tot}}(m,k) = \frac{R^{\text{solo}}(k)}{A^{\text{solo}}(m)} = \frac{T_0 \min[k, k^{\text{crit}}]}{A_1^{\text{solo}}} \log_2(\min[m, m^{\text{solo}}])
\]

(4.12a)

The “span” parameter \(m\) has been given an additional interpretation for “solo” activity: it represents the total number of other “solo” contributors working on tasks related to an actor’s own \(k\) tasks, as well as representing the average number of collaboration links between the actors (used infrequently in this limiting case).

The per capita productivity grows as actors take on additional tasks until their “dit” capacity saturates when there are \(k^{\text{crit}}\) tasks. There is also logarithmic growth in the productivity due to the partitioning of tasks among individual contributors until there are \(m^{\text{solo}}\) of them, at which point the “value” stops increasing.

Productivity then remains constant rather than declining, inasmuch as coordination costs affecting “solo” tasks are absent. Increasing the staffing \((m)\) beyond this point \((m^{\text{solo}})\) adds total capacity but leaves the per capita productivity limit unchanged – a marked difference from the collaborative limit. Measures of productivity that include financial cost would deteriorate due to the over-staffing.

Alternatively, \(\mu_0^{\text{solo}}\) can be written in terms of the saturation fraction \(\beta^{\text{solo}}\) (see Equations (6.6) and (6.2)):

\[
\mu_0^{\text{solo}}(m,k) = \beta^{\text{solo}}(k) \frac{D^{\text{solo}}}{A^{\text{solo}}(m)} = \beta^{\text{solo}}(k) \frac{D^{\text{tot}}}{(1 + \eta)A_1^{\text{solo}}} \log_2(\min[m, m^{\text{solo}}])
\]

(4.12b)

where \(\beta^{\text{solo}} = \frac{\min[k, k^{\text{crit}}]}{k^{\text{crit}}} \leq 1\)

\(\mu_0^{\text{solo}}\) is independent of the collaborative parameters inasmuch as coordination costs are absent (the span \(m\) is tracking the functional partitioning for solo work). Collaborative costs appear in the terms of Equation (4.11) that are proportional to \(\alpha^{\text{coll}}\).
The second term in the denominator of Equation (4.11) must be small for the expansion to convergence quickly. The overall condition \( \alpha_{\text{coll}} \ll 1 \) is not by itself sufficient to ensure that, so it is necessary to impose and satisfy the following somewhat stronger criterion:

\[
\alpha_{\text{coll}}(k,m) \ll \frac{A_{\text{solo}}(m)}{A_{\text{coll}}(m)} = \frac{A_{\text{solo}}(k,m)}{2A_{\text{coll}}(k,m)} = \frac{\log_2(\min[m,m_{\text{coll}}])}{\log_2(\min[m,m_{\text{solo}}])} \frac{1}{\log_2(m)} \tag{4.13}
\]

When the inequality above holds strictly, only the leading term in the expansion (\( \mu_{\text{solo}}^0 \)) is important and coordination costs are negligible. When a weaker or reversed inequality holds, terms proportional to the first and possibly higher powers of \( \alpha_{\text{coll}} \) must be retained consistently, signifying that coordination costs may not be completely neglected. For example, suppose as before that the decision complexity coefficients are roughly equal (i.e., \( A_{\text{solo}} \sim 2A_{\text{coll}} \)) as are the “value cutoffs” (i.e., \( m_{\text{coll}} \sim m_{\text{solo}} \)). Convergence condition (4.13) then simplifies and can be approximated by:

\[
\alpha_{\text{coll}}(k,m) < \frac{1}{\log_2(m)}
\]

With \( \alpha_{\text{coll}} < 1 \) this condition can be violated in principle if the span \( m \) becomes very large, approaching infinity. But that situation never arises for reasons discussed earlier. Hence it is unlikely that terms proportional to \( \alpha_{\text{coll}} \) can become large enough to threaten convergence of the power series expansion.

But the first order correction term in the expansion of Equation (4.11) may still not be negligible when \( \alpha_{\text{coll}} \) is less than unity (say in the range \( \alpha_{\text{coll}} \sim 0.1 \)) if the collaborative span is fairly large. The following standard power series expansion formula is used to find \( \mu_{\text{solo}}^1 \) the first order correction term proportional to the first power of \( \alpha_{\text{coll}} \).

\[
\frac{1}{1 \pm x} = 1 \mp x + x^2 \mp \ldots = 1 \mp x \quad \text{for} \quad x \ll 1
\]

Equation (4.11) is then approximated by the leading (zeroth order) and first order terms:

\[
\mu_{\text{solo}}(k,m) = \mu_{\text{solo}}^0(k,m) + \mu_{\text{solo}}^1(k,m) \tag{4.13a}
\]

The ratio of the terms on the right is:

\[
\frac{\mu_{\text{solo}}^1}{\mu_{\text{solo}}^0} = \alpha_{\text{coll}} \left(1 - \frac{A_{\text{coll}}}{A_{\text{solo}}} \right) = -\alpha_{\text{coll}} [\log_2(m) - 1] \tag{4.13b}
\]

The approximate expression on the extreme right follows by setting \( A_{\text{solo}} \sim 2A_{\text{coll}} \) and \( m_{\text{coll}} \sim m_{\text{solo}} \) once again. It is always negative – signifying a productivity decrease - and its magnitude grows as a function of the span \( m \). \( \mu_{\text{solo}}^1 \) contains the coordination cost due to collaborative entropy, surviving as the logarithmic term on the extreme right.

To verify this interpretation note that if one eliminates collaborative entropy by setting \( \log_2(m) = 1 \) in Equation (4.5), the value/cost functions \( A_{\text{solo}}(m) \) and \( A_{\text{coll}}(m) \) cancel in the denominator of Equation (4.11). The first order correction above, as well as higher order terms, would then vanish identically and Equation (4.12a) would be the exact result.

The coordination cost \( \mu_{\text{solo}}^1 \) can be negligible even for large teams (large spans) provided that \( \alpha_{\text{coll}} \ll 1 \) holds (very little collaboration). This is so because \( \alpha_{\text{coll}} \) is bounded above by \( 2R_{\text{cm}}(m)/R_{\text{solo}} \) even for very large spans, and the factor in brackets (Equation (4.13b)) is less than about 10, say, for sub-nets up to the unwieldy size of \( \sim 2048 \).

However, \( \mu_{\text{solo}}^1 \) can predict a significant productivity decrease even when tasks are primarily (but not totally) solitary; i.e., when only a modest fraction of the actors’ time is spent in collaboration. Suppose \( \alpha_{\text{coll}} \) is on the order of, say, 0.1 or larger; that is, if the collaborative share of attention is small but not negligible. Consider an organization performing large, highly specialized efforts with each actor supported by about 16 - 32 other skilled individual contributors, spending about 10% of their time on collaboration with the other experts (\( \alpha_{\text{coll}} \sim 0.1 \)), and spending the remaining time on solitary tasks. For this situation the first order coordination cost \( \mu_{\text{solo}}^1 \) would be about 30% - 40% of the solo contributor productivity. This may overstate the costs somewhat, inasmuch as the second order term in \( \alpha_{\text{coll}} \) (about +8%) and perhaps higher order terms should be included.

Clearly, even modest amounts of collaboration can produce a significant impact on productivity in large teams. The actors must create a substantial amount of extra information (entropy) to feed the collaborative effort.
Returning to $\mu_{\text{coll}}$, Equation (4.12b) is re-written as the product of an overall normalization constant $M_{1\text{solo}}$ and a dimensionless structure factor $S_{0\text{solo}}$. First define:

$$M_{1\text{solo}} = \frac{D_{\text{solo}}}{k_{\text{crit}} A_{1\text{solo}}} = \frac{T_0}{A_{1\text{solo}}}$$

(4.14a)

for which

$$\mu_{0\text{solo}} = M_{1\text{solo}} S_{0\text{solo}}$$

(4.14b)

The constant $M_{1\text{solo}}$ once again represents the simplest non-trivial solo-dominated organization, where $M_{1\text{solo}}$ is the per capita (application-level) decision rate for an individual contributor working on one task. As in the collaborative case, $M_{1\text{solo}}$ is approximately constant despite its seeming dependence on $k_{\text{crit}}$, and $k_{\text{crit}}$ thus correlates inversely to the solo task decision complexity (see discussion, Section 2.4). Related task segments partitioned to other actors use no additional capacity. The structure factor is:

$$S_{0\text{solo}}(k, m) = \min[k, k_{\text{crit}}] \log_2(\min[m, m_{\text{solo}}])$$

(4.15)

The function $S_{0\text{solo}}(k = 1, m = 2) = 1$ in accordance with the definition of $M_{1\text{solo}}$.

4.7. Productivity as a function of the collaborative/solo index ratio $\xi$

The productivity bound $\mu_{\text{tot}}$ has more intricate and more revealing behavior when neither of the previous limiting cases applies; it predicts a blend of the features described for the limiting cases including logarithmic productivity deterioration at large spans when the collaborative/solo index $\xi$ (see Equation (4.3)) is held constant. The productivity bound depends very strongly on the ratio $\xi$ leading to dramatic productivity variations if the collaborative/solo balance of effort changes during the course of an organization’s evolution.

The equation needed is derived from Equation (4.1). Neither of the limiting case conditions in Equations (4.6) or (4.10) is satisfied, so the collaborative/solo index $\xi$ defined in Equation (4.3) is far from either zero or infinity. Equivalently, neither of the “attention” parameters $\alpha_{\text{coll}}$ or $\alpha_{\text{solo}}$ is permitted to be dominant or to vanish.

The first step is to re-write Equation (4.1) in terms of the saturation fractions $\beta_{\text{solo}} = R_{\text{solo}}/D_{\text{solo}}$ and $\beta_{\text{coll}} = R_{\text{coll}}/D_{\text{coll}}$ that were introduced earlier and in Appendix Equation (6.6). Equations (6.1) and (6.2) prescribe how to represent $D_{\text{solo}}$ and $D_{\text{coll}}$ in terms of a common factor $D_{\text{tot}}$. Equation (4.2) defines the “attention” coefficients $\alpha_{\text{coll}}$ and $\alpha_{\text{solo}}$ as functions of the collaborative/solo index $\xi$. The result is:

$$\mu_{\text{tot}}(m, k) = \frac{1 + \xi}{1 + \eta} D_{\text{tot}} \frac{\beta_{\text{solo}}(k) + \eta \beta_{\text{coll}}(m)}{A_{\text{solo}}(m) + \xi A_{\text{coll}}(m)}$$

(4.16a)

In the above $\eta$ is a constant representing the ratio of capacities $D_{\text{coll}}/D_{\text{solo}}$. The variables are not all independent but are connected (see Section 6.3.3) by:

$$\frac{\beta_{\text{coll}}(m)}{\beta_{\text{solo}}(k)} = \frac{\xi(m, k)}{\eta}$$

(6.7)

The collaborative/solo index $\xi(m, k)$ on the right hand side depends in principle on $k$ and $m$, inasmuch as $\beta_{\text{solo}}$ or $\beta_{\text{coll}}$ have this dependence; it is used as the measure of collaborative versus solo dominance.

Either $\beta_{\text{solo}}$ or $\beta_{\text{coll}}$ can be eliminated using Equation (6.7) and the expression for $\mu_{\text{tot}}$ can be put into either of the following alternative forms:

$$\mu_{\text{tot}}(m, k) = \frac{(1 + \xi)^2 \eta}{1 + \eta \xi} A_{\text{solo}}(m) + \xi A_{\text{coll}}(m)$$

(4.16a)

or
\[
\mu^{\text{tot}}(m,k) = \frac{(1+\xi)^2}{1+\eta} \frac{D^{\text{tot}}}{A^{\text{solo}}(m) + \xi A^{\text{coll}}(m)} \tag{4.16b}
\]

In Equation (4.16a) the explicit dependence on the solo task parameter \(k\) has been eliminated but it is still implicitly present through \(\xi(m,k)\); the span parameter \(m\) describes both the “value” and collaboration cost scaling.

As a simplification, assume that the saturation fractions \(\beta^{\text{coll}}\) and \(\beta^{\text{solo}}\) are always equal. This couples the average collaborative workload (described by span \(m\)) to the average solo task load (measured by \(k\) - the number of solo tasks per actor). It ensures that “solo” and collaborative saturations occur simultaneously. In principle, team managers might make this load balancing a goal. Equation (6.7) then implies that \(\xi = \eta\) and Equations (4.16a) and (4.16b) both simplify to become:

\[
\mu^{\text{tot}}(m,\xi) = (1+\xi) \frac{\min[m,m^{\text{crit}}]}{m^{\text{crit}}} \frac{D^{\text{tot}}}{A^{\text{solo}}(m) + \xi A^{\text{coll}}(m)} \tag{4.17}
\]

The definition of \(\beta^{\text{coll}}\) in terms of the span was also substituted into the above (see Equations (4.7b) and (6.6)).

The decision complexity functions \(A^{\text{solo}}(m)\) and \(A^{\text{coll}}(m)\) (see Equations (4.4) and (4.5)) may be simplified as well at the price of some generality. As suggested earlier, it is often reasonable to assume that the decision complexities for solo and collaborative work are comparable to each other – especially within a big project. The “value” caps on the span are assumed to be comparable to each other as well. The following replacements, used previously, are applied again:

\[
A_1^{\text{solo}} = 2A_1^{\text{coll}} = 2A_1 \quad (4.18a)
\]

and

\[
m^{\text{solo}} = m^{\text{coll}} = m^{\text{cap}} \geq 2 \quad (4.18b)
\]

Following earlier sections, \(\mu^{\text{tot}}\) is split into the product of an overall normalization constant \(M_1^{\text{tot}}\) and a dimensionless structure factor \(S^{\text{tot}}\) where:

\[
S^{\text{tot}}(m,\xi) = \frac{1+\xi}{2} \frac{\min[m,m^{\text{crit}}]}{1+\xi \log_2(m)} \quad m \geq 2 \quad (4.19a)
\]

\[
\mu^{\text{tot}}(m,\xi) = M_1^{\text{tot}} S^{\text{tot}}(m,\xi) \quad \text{and} \quad M_1^{\text{tot}} = \frac{D^{\text{tot}}}{m^{\text{crit}} A_1} \quad (4.19b)
\]

The structure factor \(S^{\text{tot}}(m,\xi)\) depends on the collaborative/solo index ratio \(\xi\). \(S^{\text{tot}}\) again references productivity to the smallest organization that can be meaningfully analyzed (see Appendix 6.5). When \(m = 2\), \(S^{\text{tot}}\) equals unity for any value of the collaborative/solo index \(\xi\). For reasons discussed above, \(M_1^{\text{tot}}\) is a constant and is independent of \(m^{\text{crit}}\), despite the appearance to the contrary. \(M_1^{\text{tot}}\) is again the per capita decision rate for a three actor reference organization.

Equation (4.19a) reduces to the results found previously (Equations (4.9) and (4.15)) in the collaboration-dominated and solo-dominated limits. The “1” now appearing in the denominator remains at least about 10% of the logarithmic term unless \(\xi \gg 1\) and \(m\) is extremely large.

As before, when \(\xi \gg 1\) (collaboration dominance) with \(m < m^{\text{cap}}\) the collaborative entropy cost in the denominator is partially masked by a compensating gain in “value”. Once the set of contributors/collaborators grows beyond \(m^{\text{cap}}\) there is no further “value” increase but the collaborative costs continue to rise. Per capita productivity \(\mu^{\text{tot}}\) then falls off as \((1 + \xi \log_2(m))^{-1}\).

Productivity is minimized as a function of \(\xi\), when an organization becomes collaboration-dominated, leaving all other parameters unchanged. This may be expected intuitively but it is also straightforward to show that it is the case by setting \(dS^{\text{tot}}/d\xi = 0\) using Equation (4.19a) while holding all other parameters constant. The solution requires that \(\xi\) be infinite - synonymous with the extreme collaborative limit. The second derivative \(d^2S^{\text{tot}}/d\xi^2\) is positive for infinite \(\xi\), verifying that the collaborative limit minimizes rather than maximizes productivity.
Productivity as a function of span $m$ is maximized when $m$ is the greater of $m^{\text{crit}}$ or $m^{\text{cap}}$. The structure factor is continuous at these points but its slope changes abruptly; the maxima thus stand out clearly on the figures discussed below.

### 4.8 Parametric study of productivity versus span and collaborative/solo index

The structure factor Equation (4.19a) was evaluated numerically for a small range of parameters. An illustrative value ($m^{\text{cap}} = 16$) was chosen for the “value” cap. Two values of the saturation limit $m^{\text{crit}}$ were considered: $m^{\text{crit}} = 8$ describes an application with comparatively high decision complexity while $m^{\text{crit}} = 24$ connotes an application whose decision complexity is much lower. The index $\xi$ was varied between the collaborative and solo limits. The normalization constant $M_{\text{tot}}$ depends on quantities yet to be measured but this gap is not an obstacle as the numerical productivity estimates use the structure factor $S_{\text{tot}}$ exclusively.

First consider Figures (5a) and (5b) which plot the productivity bound for the two illustrative values of $m^{\text{crit}}$. Within each curve the collaborative/solo balance index $\xi$ is held constant while the per capita span grows from 2 to 64. Individual curves in each family correspond to values of $\xi$ between the limiting cases, approximated by $\xi \sim 0.0001$ for solo-dominance and $\xi \sim 10,000$ for collaboration dominance. The highest productivity (optimal span) in each case occurs at the larger of $m^{\text{crit}}$ (in Figure 5b) or $m^{\text{cap}}$ (in Figure 5a, where “value” continues to increase above $m^{\text{crit}}$ and thereby sustains increasing productivity to $m^{\text{cap}}$).

- For the solo-dominated curves coordination costs are absent. One can confirm this by letting $\xi \to 0$ in the numerator and denominator of Equation (4.19a). Once the peak of $S_{\text{tot}}$ is reached at the larger of $m^{\text{crit}}$ or $m^{\text{cap}}$, productivity remains constant as the span increases further.

- However, about 22 to 25% of the solo-dominated productivity near the optimal span is absorbed by coordination costs, when even as little as 10% of the effort is collaborative.

- All of the curves excepting those for complete solo dominance predict an approximately logarithmic fall-off in productivity for large span values.

- Even a 50-50 mix of effort ($\xi = 1$) yields productivity values close to the collaborative limit. The curves for $\xi > 1$ (more than about 50% collaborative effort) closely resemble the limiting case results and trends described and sketched earlier in Section 4.5.

The collaborative/solo index $\xi$ may sometimes remain constant as organizations or projects evolve. But it is more likely that tasks tend to become more heavily collaborative, on the average, as organizations evolve from small to large. Such trends are independent of and in addition to evolutionary change in the average span $m$. Ideally, one might attempt to manage growth so that the actors’ span values remain close to their optima, even as the character of the work changes.

Figures (6a) and (6b) portray this case using the same results as above, but now each curve in each family holds the span $m$ fixed and plots values of the structure factor $S_{\text{tot}}$ as functions of the collaborative/solo index $\xi$. Curves using dashed lines indicate span values below saturation; those at or above saturation are shown by solid lines.

- Families of markedly non-linear “S-curves” appear in each of the figures with productivity varying dramatically as a function of $\xi$ – the collaborative/solo index. This effect is attributable entirely to collaborative entropy. This claim can be checked by replacing the entropy term $\log_2(m)$ in the denominator of Equation (4.19a) by unity: this would remove the collaboration costs but leave intact all the other features built into the model, such as saturation, “value” contributions, and the probabilistic selection of collaborative partners. The resulting formula becomes independent of the index $\xi$ and reduces to the solo limit Equation (4.15) when the approximations (Equation 4.18) used above are applied.

As a result, if collaborative entropy were eliminated in the models, all of the S-curves in Figures (6a) and (6b) would become constant functions at the productivity levels calculated for their solo-dominated (left hand) ends, assuming the other variables to be held constant.

- For $\xi$ exceeding about 4.0 (80% collaborative) the slopes of the curves are quite flat and the organizations can be regarded as having substantially reached collaboration-dominated productivity levels. For $\xi$ less than about 0.05 organizations are likewise essentially solo-dominated.

- The productivity cost of even a small amount of collaboration is high. For the illustrative point $\xi \sim 0.1$ productivity is already reduced by about 23% to 25% as the price of spending only about 10% of actors’ effort in collaboration. The S-curves are very non-linear functions of the collaborative/solo index.

- The coordination cost paid when a team evolves from solo-limit behavior to collaborative-limit behavior is large and potentially decisive for organizational competitiveness, even if the span and other parameters remain
constant. For the peak-productivity spans in each figure ($m^{\text{cap}}$ for (6a) and $m^{\text{crit}}$ for (6b)) the productivity cost for making this transition would be about 75%. Discussions to follow will find coordination costs of this same order when entire organizations evolve to higher levels of collaboration.

Figure 5a: Relative productivity – high decision complexity

Figure 5b: Relative productivity – low decision complexity
Figure 6a: Coordination costs grow rapidly with increased collaboration – high decision complexity

Figure 6b: Coordination costs grow rapidly with increased collaboration – low decision complexity
4.9. Models for productivity variation as a function of an organization’s total population

The findings above are next applied to model the productivity impact when an entire organization (consisting of many sub-nets) grows or shrinks. The approach used in Section 3 is used again to turn the per capita decision rate expression above (Equation (4.19a)) into a function of an organization’s total population $n$: model functions that describe the evolution of the collaborative span $m(n)$ and now also the collaborative/solo index $\xi(n)$ are substituted into the sub-net expressions. Section 3 of this paper developed a simple, appropriate model of span variation as a function of total population and applied it to a simplified treatment of the collaborative limit. That model is re-applied here along with a mathematically similar one for variation of the collaborative/solo index as a function of population. The information to be built into these functions specifies organizations’ evolution paths as the population changes.

The power law model for the span was summarized by Equation (3.2). The total population is now represented by the dimensionless variable $\tau$ for which the power law becomes:

$$m(n) = m^{\text{crit}} \tau^\gamma \quad \text{where} \quad \tau = \frac{n}{n^{\text{crit}}} \quad \text{and} \quad \gamma \leq 1, \ m \geq 2 \quad (4.20)$$

As before, $n^{\text{crit}}$ is the population for which the average actor’s “dit” capacity reaches saturation; by definition $n = n^{\text{crit}}$ coincides with $\tau = 1$ and with $m = m^{\text{crit}}$. This model assumes that the span tends to change and typically grows as organizations grow, with a representative value $\gamma \sim 0.3$ for the exponent specifying a doubling of the span when the population grows by an order of magnitude. There may be instances where span decreases with growth (negative $\gamma$) perhaps in order to alleviate short term overloads. Experience suggests though that span growth typically accompanies organization growth and should be expected due to the enlarged range of choices in an enlarged organization. In numerical work a lower limit on $\tau$ was imposed to ensure that $m(\tau)$ remains greater than or equal to two.

Span evolution alone does not trigger the most dramatic effect; this was clearly evident in the “S-curves” in Figures (6a) and (6b). Productivity depends strongly on the collaborative/solo index $\xi$. As relatively small organizations grow into larger ones they take on more intricate assignments and also tend to fill up the enlarged collaborative space. An additional model is needed – one which assumes that total population growth results in actors’ work generally becoming more collaborative and less individual (increased index $\xi$), independently of any direct impact that population growth may have on the span.

The model for a collaborative/solo index “propagator” assumes that a fractional change in the population produces a fractional change in the index in response, i.e.,

$$\frac{d\xi}{d\tau} = \sigma(n) \frac{dn}{n} = \sigma(\tau) \frac{d\tau}{\tau^n}$$

The function $\sigma(n)$ contains information about the detailed path followed. As in Section 3.1, the formal solution of this type of differential equation is an evolution operator having the form:

$$\xi(n) = \xi_0 \exp \left\{ \int_{\tau_0}^{\tau} \frac{\sigma(\tau')}{\tau'} d\tau' \right\} \quad (4.21)$$

In the above, exp[x] is the exponential function.

Equations (3.1) and (4.21) are called evolution operators or propagators because they can describe the detailed path taken by the span and the collaborative/solo index as an organization evolves from its initial to its final size.

Equation (4.21) becomes especially simple if we assume $\sigma(\tau)$ is just a proportionality constant, resulting in a simple power law expression:

$$\xi(n) = \xi_0 \left( \frac{\tau}{\tau_0} \right)^\sigma \quad (4.22)$$

In the above, $\tau_0$ and $\xi_0$ are integration constants that will be chosen to match the conditions at a known point along an organization’s evolutionary path. For example, the choice $\xi_0 = 0.1$ with $\tau_0 = 1.0$ would mean that actors spend an average of 10% of their time on collaborations when the average actor is at the saturation point: coinciding with the
span equaling $m^{\text{crit}}$ and $n$ equaling $n^{\text{crit}}$ (points which are easily identifiable in most previous and upcoming figures). This parameter set is used as the baseline for numerical work below.

Choosing $\xi_0 = 0.1$ with $\tau_0 = 0.125$ instead would accelerate the onset of intense collaboration, causing the 10% collaboration point to coincide with a much smaller population that is at 1/8 of the saturation value; this would magnify the coordination costs for a given population level.

The exponent $\sigma$ should probably lie between about 0.75 and 1.33, with $\sigma = 1.0$ being a reasonable compromise value. Intuition suggests that the order-of-magnitude growth rates for population and for the collaborative/solo index are roughly the same. In principle, $\sigma$ can be negative if work becomes less collaborative as a function of population growth; but this seems unlikely to occur very often unless growth allows firms to afford capital-intensive tools that reduce collaboration needs.

Figure (7) plots Equation (4.21) for the parameters discussed above which are used in numerical work below.

4.10. Parametric study of productivity versus an organization’s total population

Figures (8a) and (8b) incorporate only the model for span evolution while holding the collaborative/solo index $\xi$ constant for each curve. Equation (4.20) was evaluated numerically using the exponent $\gamma = 0.3$, and the resulting span values were substituted into Equation (4.19a) to calculate the structure factor $S^{\text{tot}}$. As before, $S^{\text{tot}}$ is proportional to the per capita productivity measure $\mu^{\text{tot}}$. Figure (8a) represents a high decision complexity application ($m^{\text{crit}} = 8$) while Figure (8b) represents a lower complexity case ($m^{\text{crit}} = 24$).

The main features commented on in Section 4.8 appear again, mapped onto the normalized population variable $\tau = n/n^{\text{crit}}$. Collaborative entropy costs become progressively more severe for curves in each family as $\xi$ becomes larger (representing more heavily collaborative activity). Even when $\xi$ is as small as 0.1 (10% collaboration), there is a noticeable logarithmic productivity falloff above the productivity peak. For the high complexity example the peak is at $\tau(m^{\text{crit}}) = 10$. For the low complexity example $\tau(m^{\text{crit}}) = 0.26$ and the peak productivity is at $\tau = 1$.

As a real organization grows or shrinks in size the balance between solo and collaborative work will vary in a specific way that can be modeled using the evolution operator Equation (4.20) for $\xi$. The growth trajectory will then not simply follow a single one of the curves plotted in Figures (8a) & (8b) but it will cut across them instead, thereby amplifying the
impact of collaborative costs during the evolution process. Real organizations should therefore exhibit much stronger dependence of productivity on organization size than is evident in Figures (8a) & (8b).

Figures (9a) & (9b) illustrate growth paths that allow the collaborative/solo index $\xi$ to evolve with population $\tau$ via the power law model Equation (4.21). Each of the charts portrays two families of curves: one for $\tau_0 = 1.0$ (baseline) and another for $\tau_0 = 0.125$ (onset of high collaboration at a comparatively small population). The curves within each family bracket the three illustrative values of the power law exponent $\sigma$ mentioned earlier. Dashed lines repeat the solo- and collaborative limits from the previous figure for comparison.

- On all of the curves productivity depends very strongly on organization size $\tau$ and is strongly peaked. This is entirely traceable to collaborative entropy, as it was in the S-curve form of Figures (6a) & (6b). Most of the peak productivity points fall near the saturation point ($\tau = 1$). On all the curves productivity falls off above saturation ($\tau > 1$) much faster than logarithmically due to the growth of $\xi$ (collaborative index) with $\tau$ (total population).

- For the low complexity application (see Figure (9b) where $m_{\text{crit}} = 24$): the $\tau_0 = 1.0$ curve family has peaks as expected at $\tau = 1$. For the high complexity application (see Figure (9a) where $m_{\text{crit}} = 8$) the $\tau_0 = 1.0$ curve family also peaks at $\tau = 1$ but this is unexpected: the expected peak location based on the previous figures would lie at $\tau = 10$ (the “value” cutoff). The shift away from the expected optimal productivity point is due to much heavier attenuation of productivity above saturation for the high complexity application than for the low complexity application.

- The curves for $\tau_0 = 0.125$ (high collaboration at smaller population levels) have more diffuse peaks by comparison. The productivity levels are greatly reduced in the vicinity $\tau = 0.1$ to 10 and the structure factor values at the peaks no longer coincide for different values of the exponent $\sigma$.

This set of curves shows clearly that collaborative entropy costs can have a definitive impact on productivity. All of the features are due entirely to the inclusion of collaborative entropy in the model. If the collaborative entropy factor $\log_2(m)$ were to be replaced by unity in Equation (4.19a) all of the curves in these figures would approach the solo-dominated limit (shown on both figures).

### 4.11. Total decision capacity growth as a function of population growth

The results of the preceding section can be applied to a theme discussed in Section 1.2: Frederick Brooks’ observation’s [Br1995] that huge coordination costs may be incurred when an organization increases staff to depend on productivity than is evident in Figures (8a) & (8b).

- If we increase staff by a factor $F$ (e.g., double staff) by what factor does the organization’s capacity for completing knowledge-based tasks increase?

The model presented in this paper does not follow “Brook’s Law” scaling for reasons discussed earlier: the coordination cost model excluded sequential constraints in favor of collaborative entropy. But that is not a shortcoming inasmuch as the $N^2$ scaling that was proposed as “Brooks’ Law” may not be quantitatively correct.

Suppose for example that an organization is working at or above full capacity (saturated), causing some work to remain undone. The initial population variable $\tau_{\text{initial}}$ would be greater than or equal to unity. Management can attempt to increase capacity by adding staff, in which case the population goes from $\tau_{\text{initial}}$ to $\tau_{\text{final}}$ along some evolution path as discussed in Section 4.9.

The organizations’ capacity for knowledge work is represented by $M^{\text{tot}}(n)$ - the maximum rate for making and transmitting application level decisions - as defined by Equations (2.2) and (4.19b):

$$M^{\text{tot}}(n) = n \mu^{\text{tot}}(n) = n M_i^{\text{tot}} S^{\text{tot}}(\tau) \quad \text{where} \quad \tau = \frac{n}{n_{\text{crit}}} \quad (4.23)$$

The capacity gain $G(n_i, n_f)$ due to a staff increase is the quotient of the maximum decision rates for the final and initial population states:

$$G(n_i, n_f) = \frac{M^{\text{tot}}(n_f)}{M^{\text{tot}}(n_i)} = \frac{\tau_f}{\tau_i} \frac{S^{\text{tot}}(\tau_f)}{S^{\text{tot}}(\tau_i)} \quad \frac{\tau_f}{\tau_i} \equiv \frac{n_f}{n_i} \quad (4.24)$$
The population ratio \( \tau_i/\tau_0 = n_i/n_0 \). Before Brooks, the capacity gain might naively have been expected to be close to the population ratio; that is, capacity varying in proportion to the number of workers. But Equation (4.24) shows that that would only be the case if the structure factors in Equation (4.24) are equal, which is seldom so.

The clearest way to characterize capacity changes is to use the ratio of the structure factors for the final and initial populations (called a “form factor”) that appears on the right hand side of Equation (4.24) and is hereby defined as:

\[
g(\tau_i, F) = \frac{S^{\text{tot}}(\tau_f)}{S^{\text{tot}}(\tau_i)} \quad \text{where} \quad F = \frac{n_f}{n_i} = \frac{\tau_f}{\tau_i} \quad (4.25)
\]

The form factor \( g(\tau_i, F) \) measures the departure of the capacity gain function \( G(n_i, n_i) \) from simple proportionality to the population ratio \( F \). Simple proportionality would mean \( g(\tau_i, F) = 1 \). Actual values of \( g(\tau_i, F) \) depend on the initial and final values of \( S^{\text{tot}} \) that appear on curves such as those in Figures (9a) and (9b). Individual values of the “form factor” may be greater than or less than unity. Where \( g(\tau_i, F) \) is less than unity collaborative entropy costs are damping the capacity gain (assuming \( \tau_i > \tau_0 \)). Note that the above describes only capacity changes rather than changes in actual workload: the latter would require added information or assumptions.

The time scale for capacity growth is not specified but there is an implicit assumption: staff growth must be “slow enough” to maintain “local equilibrium” as the growth occurs, meaning that collaborative linkages and specialized roles must emerge and stabilize quickly relative to the time for adding substantial numbers of new actors. The organization is then never in a state far off the evolution path built into the models. A massive, impulsive injection of staff would alter the evolutionary path followed in response, perhaps change the final equilibrium state, and perhaps require non-equilibrium amendments to the theory.

Figures (10a) and (10b) show the “form factor” \( g(\tau_i, F) \) calculated using the baseline parameter sets introduced above. They continue to assume that the models in Equations (2.1) and (4.21) govern the evolution. Each curve in each family pertains to a single value of the population ratio \( F \) in the range 2 through 256. \( F = 2 \) might apply to a technical project that must double staff. \( F = 256 \) might describe long-term growth from a small to a large enterprise over several years. A point plotted at an initial population \( \tau_i \) on one of the curve represents the effect of evolving from that population to the endpoint population \( \tau_f = F \tau_i \). The figures show that:

- The “form factor” \( g(\tau_i, F) \) equals unity (the intuitively expected value) only at a few isolated points where it is changing rapidly. Capacity does not scale linearly with the staff in general, on account of productivity variation that is attributable to collaborative coordination costs.

- Where \( g(\tau_i, F) \) is less than unity much of the capacity gain that adding staff might potentially bring is not realized but is lost to coordination costs. This is so for all of the curves shown when the initial population \( \tau_i \) is at or above the saturation value \( \tau_s = 1 \). Above about \( \tau_s \approx 0.8 \), the larger population growth ratios \( F \) produce much severer productivity impairment than small values of \( F \). For the larger population ratios shown, the coordination costs are severe \( (g(\tau_i, F) << 1) \) even when the initial population is only about 10% of the saturation value. The highest coordination costs (at minima of \( g(\tau_i, F) \)) are found between \( \tau_s \approx 1 \) and \( \tau_s \approx 10 \) for all the parameters chosen.

- The form factor \( g(\tau_i, F) \) approaches unity as the initial population \( \tau_i \) approaches infinity. This is attributable to the flat slope of the curves in Figures (9a) & (9b) at very large initial populations \( (\tau_i \gg 1) \).

- The form factor \( g(\tau_i, F) \) is greater than unity only when the starting population \( \tau_i \) is small enough to ensure that the final population \( \tau_f \) is also below unity (saturation); i.e., only when both points are on the positive slope portions of the productivity curves in Figures (9a) & (9b). Such growth episodes are bringing unused capacity into service in addition to adding staff, thereby masking the coordination costs.

A saturated organization that increases staff to meet project needs will thus feel “Mythical Man-Month” type effects, in that adding staff brings adds much less capacity than the new staff size might suggest. The results above agree with intuition and experience, although they incorporate specific parameters and approximations. Actual organizations can be treated by building detailed individual models, and some may follow evolution paths that differ from those assumed by the two power laws (4.20) and (4.22).
Figure 8a: Productivity versus total population for an organization – constant collaborative/solo index $\xi$, high decision complexity

Figure 8b: Productivity versus total population for an organization – constant collaborative/solo index $\xi$, low decision complexity
Figure 9a: Productivity versus total population for an organization – power law model for \( \xi(\tau) \), high decision complexity

Figure 9b: Productivity versus total population for an organization – power law model for \( \xi(\tau) \), low decision complexity
\[ mcrit = 8, mcap = 16, \gamma = 0.3, \xi = 0.1, \sigma = 1.0 \]

\[ g(\tau_{\text{initial}}, F) = \frac{S_{\text{tot}}(\tau_{\text{final}})}{S_{\text{tot}}(\tau_{\text{initial}})} \]

Figure 10a: Capacity gain form factor – high decision complexity application

Figure 10b: Capacity gain form factor – low decision complexity application
5. Conclusions and discussion

Collaborative entropy can produce coordination costs having significant impact. The implied upper limit on \textit{per capita} productivity is \textit{fundamental}, not merely a consequence of imperfect management and control. Productivity trends that were discussed above agree qualitatively with long-standing experience and intuition and are quantitatively large enough to significantly affect organizations’ competitiveness and culture. The trends were observed phenomenologically for a long time; the new contribution here is a mathematical model that predicts them quantitatively from first principles.

The results summarized immediately below all depend on having included collaborative entropy in the model calculations; that is, they are absent if the logarithmic terms in the model representing collaborative information are replaced by unity.

5.1. Conclusions summary

For collaborative sub-nets – referred to loosely as workgroups:

- The “collaborative entropy” cost reduces \textit{per capita} productivity in a workgroup of any size, compared to what it would have been with the entropy mechanism excluded. The cost is a measure of additional information that must be created specifically to make the distributed functionality work.
- The collaboration cost remains significant even taking into account the compensatory “value” of using experts and hiding complexity.
- There is an optimum (i.e., peak productivity) workgroup size (span) that depends on the type of tasks done. If a workgroup starts small and grows the \textit{per capita} productivity grows until a saturation point is reached. With further growth the \textit{per capita} productivity falls off, essentially as the logarithm of the span.
- Coordination costs are dramatically amplified if the collaborative fraction of the workload grows at the expense of solo effort. The largest coordination cost (minimum productivity) is incurred when an effort is purely collaborative.
- Large productivity changes occur when the workload shifts from predominantly individual to predominantly collaborative, holding other parameters such as the span constant during the transition. The magnitude of the productivity changes may be large enough to decisively affect competitiveness, in markets where basic \textit{per capita} productivity is a driver.
- Productivity may be reduced significantly if even a modest fraction of individuals’ effort is collaborative.

For organizations as a whole (containing many sub-nets), two simple assumptions were incorporated into models for evolutionary growth in the total population of an organization:

- The average span tends to grow as organizations grow, owing to increasing specialization.
- The fraction of an average actors’ total effort spent in collaborative versus individual work tends to grow larger as well.

These models may suggest an entropic principle for coordination costs in organizations, analogous to the one for physical systems. Staff growth increases the number of collaborative states (pairings). Actors tend to populate the state space (unless checked) over time, as a result of which collaborative entropy (information that much be generated) \textit{generally} tends to increase. The validity of such a basic principle is not asserted or relied on now.

The evolution models were incorporated into a picture of productivity changes in (formal or informal) organizations as a function of total population:

- The coordination cost trends seen for sub-nets (workgroups) also apply to entire organizations.
- Productivity becomes peaked fairly sharply about an optimum organization size (Figures 9a, 9b). Above the peak size, productivity deterioration is greatly amplified, owing to the growing ratio of collaborative/solo effort with its implied growth of collaborative entropy. The average \textit{per capita} productivity bound deteriorates much faster than logarithmically when population grows past saturation unless the collaborative effort is essentially negligible.
- As Brooks noted [Br1995], an organization’s work capacity usually does not increase as fast as increases in staff, owing to growing collaborative costs (Figures 10a, 10b). The exceptions occur when groups are small and actors are lightly loaded, or if the two evolutionary assumptions above do not apply (perhaps via management intervention).

It is germane to ask how important collaborative entropy cost is in the scheme of things: how often is it the driving limit
on organizational performance? Coordination entropy takes a toll whether saturation has been reached or not and whether it is the largest transaction cost or not. There is potentially a high payoff if organizations can be successfully managed to match collaborative entropy costs to the scale of tasks. As a caveat, though, recall that the productivity metric adopted here is measured throughout; it does not reflect intangibles related to the quality of decision-making that often justifies paying a substantial coordination cost.

Competitiveness will tend to decline with size for organizations (e.g., business firms) that compete in productivity-sensitive, knowledge-intensive markets but do a poor job of controlling internal collaboration and its costs. The same effects apply in principle to collections of contractors working collaboratively, but the formal firm boundaries tend to monetarize the collaboration costs, thereby rationing the growth of collaboration. Current human-resource-limited conditions suggest that many organizations operate far from their most productive operating points. Experiments and further modeling can provide empirical assessments.

### 5.2. Impact of the productivity limit on knowledge-workers

The achievable information transfer rate (limiting decision rate) decreases when the span exceeds saturation or the work becomes heavily collaborative. Unless the per capita workload is decreased and knowledge transfer slows down the decision channels may be over-run; decision information is then “lost” in transmission much as symbols are lost when a data communication channel exceeds its maximum symbol transmission rate.

Knowledge workers feel this upper limit subjectively and often respond with diminished morale. Actors mis-interpret or entirely miss key information. Individuals who try to produce outputs faster than the collaborative net can absorb and respond to them have to slow down to be sure their information gets assimilated by colleagues.

The overloaded-channel condition may be mis-characterized as “bureaucracy”, “politics”, the presence of “dead wood”, etc. Individuals in large organizations often notice the correlation of group or firm size to sluggishness and sometimes fatalistically come to feel that organizations become stupid when they become large.

### 5.3. Potential scale dis-economies

Collaborative entropy cost is a fundamental (rather than merely an accidental) reason why large organizations may have dis-economies of scale in markets for knowledge-based goods and services that punish those who use intellectual labor inefficiently. Firms need to work at managing their collaborative architectures. Small firms with small collaborative costs may have a systematic edge over large organizations in much the same way that a small bias in gambling odds results in the “house” almost inevitably bankrupting gamblers who continue to play.

Large firms (meaning large in staff size) are in some sense general purpose tools that can execute very intricate and complex knowledge-based tasks by coordinating broad sets of people with highly specialized functions. Section 4 confirmed the empirical understanding that broad capability carries inherently high collaborative overhead and reduces productivity metrics. Truly complex efforts need the broad-capability support and there is no alternative but to pay for their high intrinsic coordination expense. Where broad-capability firms are already in place they must find complex tasks to defend their size and cost. They also tend to create overly complex and overly expensive products and solutions to relatively simple tasks that they may undertake. The infrastructure and culture needed for breadth is a detriment to competitiveness when applied to problems that don’t demand such broad capability. The models above portray resources as overly broad when the number of collaborating actors exceeds the “value” caps.

Large, established firms that have traditionally enjoyed scale economies and barriers to competition hope that their competitive advantages will continue as their products and services become more knowledge-based. But the relative importance of collaborative costs grows as economies become more knowledge-based and globally egalitarian, and the traditional advantages may be diluted. When firms consolidate – often expecting to find “synergies” due to shared technology and expertise - much of the potential benefit may be given up to increased collaboration costs that undermine the expected benefits.

Organizations that are not sensitive to knowledge-market price competition may nevertheless be severely impaired if the upper limit on their decision rates is allowed to throttle their decision cycle times; they must still form strategies and respond on time scales set by their market competitors or (for the military) by enemies. There are often restrictions on expanding staff to accommodate this.

For simple or highly specialized tasks, organizations with “appropriately” restricted functionality should be most productive. The results reinforce well-understood imperatives that instruct managers to keep the collaborative architecture of firms sparse and lean but still broad enough to handle the tasks to be done. Reorganization methods already in use have adopted this principle empirically for a long time: optimization of the collaborative architecture does not often happen spontaneously via self-organization, otherwise management would be largely superfluous.

Organizations may mitigate high coordination costs by adopting better knowledge-management and design tools. But
these tools may not be differentially more affordable to large versus small firms and thus may not change the large firm/small firm competitive balance.

Some people argue that organizations modeled on “open source” software development may approach unmanaged self-optimization [Ra1999]; i.e., that organizations having non-traditional structures, rewards, and goals can bypass or minimize the usual coordination costs when the choice of expert collaborators is not constrained by organizational boundaries, goals, schedule constraints, or explicit contracts. Problem complexity and the number of collaborators needed may become reduced, inasmuch as each collaboration can hide more information by accessing crucial expertise from contributors whose involvement is less intense but whose expertise is sharply tuned to the problem. This issue is unresolved, but in any case the open source model may not often be acceptable to business firms or institutional actors that have competitive agendas, well-defined architectures, and externally imposed timetables.

5.4. An approach to organizational learning

The modeling structure developed above may be able to incorporate organizational learning. Organizational learning depends on allowing collaborative structure to evolve and lower the decision complexity, which depends in turn on a stable environment. Over time, knowledge workers develop application-specific high level languages that hide information. If there are frequent reorganizations, down-sizings, or rapidly changing markets the process is disrupted. Small groups may have the fastest learning curves and be favored for survival.

The equations presented in Section 4 may accommodate learning if the decision complexity functions $A_{coll}$ and $A_{solo}$ were to be generalized as decreasing functions of “cumulative experience”. As more “experience” is gained the coordination costs should become smaller and the (admittedly narrow) productivity metric used should improve. The well-known forecasting method using “experience curves” (see for example [BCG1972] and [Mt1993] chapter 7) may be a useful approach to quantify learning processes tied to coordination costs. Experience curves depend on a simple empirical rule that describes a host of manufacturing and services markets: the fractional reduction in price or cost is proportional to the fractional increase in cumulative “experience” (power law dependence).

5.5. Application to business firms

The smallest firms with just enough intellectual breadth to accomplish particular tasks should become the low-cost providers (assuming equally talented individual contributors). The market leaders will be those whose collaboration and specialization levels just match the complexity of the jobs to be done.

The wave of large-firm restructuring during recent decades can be read in part as a move to rectify excessive coordination costs. Firms responded to competitive pressure by deconstructing vertically integrated and multi-product businesses into autonomous business units, spinning off new firms, and/or switching to third party sourcing. These tactics ration collaborative costs (as well as other costs) by directly interposing a market price for them.

On the other hand, defense and aerospace firms that take responsibility for enormously complex systems and design problems have tended to consolidate instead of breaking up, indicating that their breadth and high coordination costs are essential to the class of problems they engage.

When knowledge-related jobs are outsourced to low wage areas, extra collaboration costs are often incurred in high wage locales to retain control of the outsourced operations. These may offset much of the gain. Sometimes, operations may be outsourced without greatly increasing collaborative overheads; the workers in outsourced call centers for example seemingly interact mainly with the general public and with data processing systems as if no change in locale has occurred, thanks to ubiquitous, cheap communication. Manufacturing operations usually have fairly low sensitivity to collaborative costs.

Automated knowledge management may reduce decision complexity by allowing the re-use of work done previously. But it is not clear whether large firms will be able to afford better access to and quality of systems than are small competitors, and thus a differential advantage linked to organization size may not accrue from KM systems.

Examples of businesses highly sensitive to collaborative coordination costs may be: consulting and creative design operations of many kinds, product development (rather than long term research), architecture and engineering design services, marketing, publishing, administrative and legal services, supply chain management, customer service and sales support, and accounting.

5.6. Application to R&D organizations

Many large technology-intensive firms that depend on R&D have outsourced much of it [An2004]. Small, venture-style firms conduct much of the short-term, product-oriented R&D that used to be internal to microelectronics, pharmaceutical, and computer/networking hardware & software firms. Given the short-term perspectives dictated by competition, the high collaborative costs of basic R&D and forward-looking product realization have become an
argument for outsourcing them. Centralized R&D Lab cultures harbored broad capability and emphasized disruptive technology advances with long-focus payoffs. But short-term competitive pressure diminished firms’ abilities to support such high cost, long-term R&D. Attempts to build “small firm R&D environments” as isolated entities inside big firms have most often been disappointing: parent firm collaborative culture and costs seem to inexorably seep into the internal “startups” and impede their work or limit their impact on the rest of the firm.

5.7. Application to military organizations

New weapons systems and warfare concepts have become more sensitive to timeline advantages while basing each decision on ever larger amounts of current battlefield information. U.S. military doctrine emphasizes real-time command, control, and battle management timelines that are shorter than those of enemies. Critical operations include fusing and processing intelligence information, formulating strategy and tactics, assessing the success of operations, and providing logistical control of assets in the field.

These time-sensitive functions depend on fast collaborative networks of skilled human decision-makers - a limited resource. Military managers need to minimize coordination overhead while being able to dynamically restructure C³I organizations as combats evolve.

Combat and anti-terror tactics might introduce “entropy-based warfare” [He1999] techniques in order to bring about precisely the opposite of the ideal situation cited above within enemies’ own combat C³I organizations; that is, to oversaturate enemies’ C³I decision channels by forcing their analysts away from analysis and into grossly excessive collaboration. This happened spontaneously to the Warsaw Pact’s over-centralized C³ doctrine. Forcing added collaboration on terrorists and supporters also exposes them to surveillance and detection.

5.8. An approach to managing collaborative coordination costs

Quantitative collaboration management tools based on the ideas developed in this paper would have significant practical value; they would supplement rather than replace traditional methods that incorporate strategic and other considerations not measured by our metrics. The key notion is “sparsification”: limiting collaborations and expert staff, redefining tasks and processes, specifying where to use information hiding, etc. As remarked, managers have already been using these heuristics intuitively for a long time but generally without quantitative guidance.

The tools would quantitatively model and minimize entropy-related coordination costs. One approach might combine classical optimization techniques with the equations developed above, used as components of linear programming cost functions. Analysts would have to measure structure and work-related parameters needed to run the model, and so the analysis of a large firm would very likely require a significant effort. Detailed, validated models and simulations are needed to venture beyond the simplified but analytically solvable solutions above, meaning that much experience and experimentation is needed.
References

[Ah1974]: Aho, Alfred V., John E. Hopcroft, & Jeffrey D. Ullman, “The design and Analysis of Computer Algorithms”, Addison Wesley, 1974, page.77. The sorting literature most often uses base 2 logarithms, which differ from natural logarithms by a factor of \( \log_2(2) \sim 0.6921 \).


6. Appendices

6.1. Entropy and choice

The connection between entropy, information, and choice has been realized since the 19th century. As early as 1894, physicist Ludwig Boltzmann [To1938, chapter 6] remarked that the entropy of a physical system is related to "missing information": it measures how many of a physical system’s alternative ("degenerate") microscopic states in phase space can produce a single macroscopic (observable) state. The entropy grows with the size of the phase space volume that a system occupies. It is a maximum when all the phase space cells (microscopic system states) are equally probable. The system is then highly disordered, meaning that it takes a lot of information to specify which microscopic state it is in. By contrast, the state of a highly ordered system (say, a solid at absolute zero temperature) has low entropy and its state can be described using comparatively little information.

Claude Shannon’s quantitative definition of information in symbol transmission also has choice as its key concept. The freedom associated with choosing a symbol to be transmitted from among those in the whole symbol set led to a definition of information that incorporates a function identical to that of physical entropy. Whenever a symbol sequence is highly predictable, the range of choices associated with it is small and the amount of information conveyed is also small.

Collaboration also involves choice – the choice of partners in producing and using information in the course of an organization’s work. Collaborative decision-making is thus associated with an entropy-like information measure that grows with the size of the decision sub-networks (called “collaborative spans” below and in the main text). This collaborative entropy (like physical entropy) should be a so-called “extensive” quantity, meaning that it is proportional to the system size, i.e. the number of collaborators. The collaborative entropy of an organization measures the choice of alternative pairings between actors inside the organizational structure: a subset of these states is used to perform each particular knowledge-intensive task.

When the decision network in an organization has a large number of nodes it has a large number of states and its collaborative entropy is large. The organization then is something of a multi-purpose tool: it can reconfigure to handle diverse tasks. Conversely, when its entropy is small an organization will probably accomplish a narrowly prescribed set of specialized tasks efficiently and others not at all. The high collaborative entropy level is the price of having the capability to execute a broad range of complicated, multi-person decision tasks. As is well-known, general-purpose capability tends to impair efficiency when doing simple tasks and should thus be applied sparingly if costs have much importance.

6.2. Some essentials of information theory applied to symbol transmission

Claude Shannon [Sh1998] was the first to define information and apply the concept to communications systems transmitting symbols. Choice was a key insight: whenever a symbol sequence is highly predictable there is very little choice and so very little information is conveyed.

Shannon quantified the average information content of each symbol to be transmitted by an information source to a receiver as the number of bits per symbol needed when using the optimal compression coding scheme. The amount of information conveyed depends on the amount of choice, as does the entropy of physical systems [To1938, chapter 6]. The information carried by symbol transmission has the same functional form as physical entropy, namely:

\[ H = -k \sum_{j=1}^{n} p_j \log_2(p_j) \]

In the above, \( p_j \) is the probability for a particular symbol to be chosen. The Shannon entropy \( H \) has dimensions of bits/symbol and is essentially the expected value of the information content of each symbol, measured by the logarithmic factor.

A communication channel's capacity \( C \) is the maximum rate at which it can transmit raw bits/second without making errors. It is determined by physical characteristics of the channel, such as its bandwidth and the modulation scheme used. It also depends on signal strength and the amount of noise present.

The Fundamental Theorem for a Noiseless Channel (see [Sh1998], page 58ff. for proof) showed that there is a maximum rate \( S \) of symbol transmission that is simply the channel capacity \( C \) - taken as a given - divided by the Shannon entropy \( H \):
\[ S = \frac{C}{H} \]

If a symbol source tries to transmit faster than \( S \) symbols/unit time it will overrun the channel and lose information. The recipient will receive garbled data and the sender may not realize that unless the data contains error detection features (which require transmitting redundant information).

As implied by its name, the Fundamental Theorem does not explicitly consider noise. The famous “Shannon Capacity Expression” [Sh1949] relates the maximum bit rate \( C \) for a channel to the bandwidth \( W \) and signal-to-noise ratio \( R \):

\[ C = W \log_2(1 + R) \]

Although this latter expression is famous among communication engineers, this work does not apply noise concepts to describe collaboration. For more detail see Mischa Schwartz’s book [Sw1980, pp300 and 517] or any other standard work on digital communication theory.

6.3. Mathematical model for collaborative decision networks

This section provides a somewhat more general and mathematically detailed development of key equations than was presented in Sections 2 and 4 of the main text.

An organization is abstractly viewed as a network of \( n \) collaborative human decision makers [Be1972]. A particular individual actor - node \( i \) – is one of \( n \) that populate an organization.

Each individual actor connects to the rest of the organization via a “collaborative sub-network” – a set of inbound and outbound collaboration links that define his functional position in the organization. Each sub-net in a larger organization has a maximum rate at which it can carry decisions – limited ultimately by the cognitive processing rates of its (human) nodes. If that rate is exceeded the collaborative information may be lost or arrive too late to have an impact. Actors may use a portion of their capacity for performing solitary tasks.

Complex knowledge work can be theoretically broken into a sequence of many basic binary choices - quanta of decision information to be called “dits”. In principle, each high level decision can be represented by constructing a binary decision tree and can thus be reduced to a sequence of independent binary choices [Be1972]. A string of “dits” would represent an “application-level” decision pertaining to some practical task. The average number of dits per decision (the number of nodes in the decision tree) is proportional to the decision complexity, which measures the amount of choice information in an application-level decision.

Each decision-maker (actor) is the terminus of collaborative links with \( m \) other actors in a sub-net. An individual actor’s sub-net behaves like a single collaborative channel since the total “dit” throughput is limited by the cognitive capacity of the node (person) where all the inbound and outbound links connect together.

The maximum decision rate (decisions/unit time) for a single node \( M_{i\text{tot}} \) is the maximum channel capacity (dits/unit time) divided by the decision complexity (dits per decision). The decision complexity measures choice information (dits/decision) just as entropy for a communication channel measures the choice content (bits/symbol). The decision complexity is assumed proportional in turn to the amount of cognitive labor involved with a task. In that sense it measures the cost of dividing up labor and functions, paid for by requiring that extra information that must be created to make the functional division work smoothly.

The decision complexity for one collaborative link has an overall amplitude factor relating to the intrinsic complexity of the collaborative tasks. This is multiplied by a collaborative entropy factor that measures the amount of choice within the set of collaborators. The total decision complexity also includes a component representing individual work (see discussion below).

The key metric is the per capita maximum decision rate: the fundamental upper limit on productivity in using intellectual capital.

6.3.1 Measures of “dit” capacity for collaborative sub-nets and individual contributors

Each person’s maximum rate for making elementary binary decisions (“dits”/unit time) is assumed to have an upper limit \( D_{i\text{tot}} \) that is fixed for that individual. It depends on intrinsic cognitive speed limits and environmental factors. The “dit” capacity might be applied to making many simple or a few relatively complex application-level decisions. It can be split between collaborative and individual activity, with the proportion of each depending on the actor’s tasks and functions. Some capacity may also be reserved for other purposes, but we can assume without loss of generality that it is zero.
A portion $D_i^{\text{solo}}$ of the total decision capacity is allocated (as a time average, perhaps) to be the upper limit on individual tasks. Another portion $D_i^{\text{coll}}$ is reserved for the aggregate capacity of inbound and outbound collaborative channels, and is usually further subdivided among a number of collaborators. Hence:

$$D_i^{\text{coll}} + D_i^{\text{solo}} = D_i^{\text{tot}} \quad \text{and} \quad D_i^{\text{other}} = 0 \quad (6.1)$$

A dimensionless parameter $\eta_i$ is defined to keep track of the cognitive capacity allocation between collaborative and individual work:

$$\eta_i = \frac{D_i^{\text{coll}}}{D_i^{\text{solo}}} \quad \text{which implies} \quad D_i^{\text{solo}} = D_i^{\text{tot}} \left( \frac{1}{1 + \eta_i} \right) \quad \text{and} \quad D_i^{\text{coll}} = D_i^{\text{tot}} \left( \frac{\eta_i}{1 + \eta_i} \right) \quad (6.2)$$

When $\eta_i >> 1$, $D_i^{\text{coll}} \to D_i^{\text{tot}}$ and $D_i^{\text{solo}} \to 0$. When $\eta_i << 1$, $D_i^{\text{solo}} \to D_i^{\text{tot}}$ and $D_i^{\text{coll}} \to 0$.

The preceding parameters describe “dit” capacity that is potentially assignable.

The portion of “dit” capacity that is actually assigned is symbolized by:

$$R_i^{\text{tot}} = R_i^{\text{solo}} + R_i^{\text{coll}}$$

where $R_i^{\text{coll}}$ and $R_i^{\text{solo}}$ are respectively portions of $D_i^{\text{coll}}$ and $D_i^{\text{solo}}$ that are actually assigned. Actors become “saturated” as noted in Sections 2 and 4 when all of the potentially assignable capacity is actually allocated.

$R_i^{\text{coll}}$ and $R_i^{\text{solo}}$ are proportional respectively to the number of collaborations and solitary tasks assigned to an actor. For collaborations, assuming equal numbers $m_i$ (on the average) of inbound and outbound linkages:

$$R_i^{\text{coll}}(m) = 2R_0 \min[m_i, m_i^{\text{crit}}] \leq D_i^{\text{coll}} \quad (6.3a)$$

as in Sections 2 and 4 above. $R_0$ is the average “dit” capacity needed per uni-directional collaboration link, assumed to be a constant for all linkages. The notation $\min[x,y]$ is an instruction to select the smaller of $x$ or $y$. The condition at saturation is:

$$2m_i^{\text{crit}}R_0 = D_i^{\text{coll}} \quad (6.3b)$$

The individual contributor “dit” capacity $R_i^{\text{solo}}$ is similarly proportional to the number $k_i$ of solo tasks assigned to actor $i$. Assuming that each task requires an average decision capacity of $T_0$ “dits”/second, the $k_i$ identical solo tasks in progress require “dit” capacity $R_i^{\text{solo}}$ given by:

$$R_i^{\text{solo}}(k_i) = T_0 \min[k_i, k_i^{\text{crit}}] \leq D_i^{\text{solo}} \quad (6.4a)$$

The implied upper limit $k_i^{\text{crit}}$ coincides with the allocated individual workload reaching saturation at $D_i^{\text{solo}}$, i.e.:

$$k_i^{\text{crit}}T_0 = D_i^{\text{solo}} \quad (6.4b)$$

In the general case $m_i$ and $k_i$ are independent:

- $k_i$ represents the number of independent tasks per actor while...

- $m_i$ represents the total partitioning of functions across distinct actors; i.e., the number of other individual contributors participating in the $k_i$ tasks.

- We assumed for simplicity that the “spans” for solo task partitioning and for collaboration are the same, which should often be the case but is not necessarily so.

### 6.3.2 Definitions of collaboration and solo dominance

An actor is “collaboration-dominated” when his work is primarily to engage other members of the organization; that is, when $R_i^{\text{coll}} >> R_i^{\text{solo}}$ meaning that most of his allocated decision capacity is applied to collaborations. An actor is “solo-dominated” when the reverse condition applies; i.e., when $R_i^{\text{solo}} >> R_i^{\text{coll}}$, meaning that he works primarily as an individual contributor without much collaborative exchange. A solo contributor’s own individual tasks may be part of larger efforts that are partitioned among a sizable number of expert contributors who work (almost) autonomously.
The limiting cases thus defined are developed in Sections 2 and 4. The assignment of capacity to collaborations versus individual activities depends on the nature of the tasks, and in general is intermediate between the extremes.

Another dimensionless parameter $\xi_i$ will be called the collaborative/solo index. It concisely keeps track of these dominance relations:

$$
\xi_i(m,k) = \frac{R_i^{coll}(m)}{R_i^{solo}(k)} \quad \text{which implies}
$$

$$
R_i^{solo} = R_i^{tot}(\frac{1}{1+\xi_i}) \quad \text{and} \quad R_i^{coll} = R_i^{tot}(\frac{\xi_i}{1+\xi_i}).
$$

(6.5)

Collaboration dominance is conveyed by $\xi_i >> 1$ for which case $R_i^{coll} \rightarrow R_i^{tot}$ and $R_i^{solo} \rightarrow 0$ in the above. An actor is “solo-dominated” in the opposite limit when $\xi_i << 1$, for which case $R_i^{solo} \rightarrow R_i^{tot}$ and $R_i^{coll} \rightarrow 0$.

Note that $\xi_i$ depends on $m$ and $k$, in general. The definition of the limiting cases above is more meaningful than would be one based on Equations (6.1) and (6.2); i.e., comparing the cognitive capacity quantity $D_i^{coll}$ to $D_i^{solo}$. The saturation fractions (see below) contain actual capacity allocation information that is included in the current definition. Setting $\eta_i >> 1$ would not, for example, ensure collaboration dominance as the collaborative saturation fraction $\beta_i^{coll}$ may nonetheless be small (see below).

### 6.3.3 Saturation fractions

The saturation fractions $\beta_i^{coll}$, $\beta_i^{solo}$, and $\beta_i^{tot}$ are ratios that measure how close actors are to their saturation limits for collaboration and individual work. They are defined follows:

$$
\beta_i^{coll} = \frac{R_i^{coll}}{D_i^{coll}} = \frac{\min[m_i,m_i^{crit}]}{m_i^{crit}} \leq 1
$$

$$
\beta_i^{solo} = \frac{R_i^{solo}}{D_i^{solo}} = \frac{\min[k_i,k_i^{crit}]}{k_i^{crit}} \leq 1
$$

$$
\beta_i^{tot} = \frac{R_i^{tot}}{D_i^{tot}} \leq 1
$$

(6.6)

where $R_i^{tot} = R_i^{coll} + R_i^{solo} \leq D_i^{tot}$

When $\beta_i^{coll}$ and/or $\beta_i^{solo}$ equal unity an actor is “saturated”; that is, an actor is using all of his “dit” capacity for collaboration or individual work. If additional collaboration links or tasks are assigned the “dit” capacity available on the average for each one must be diminished.

The saturation fractions are related to the parameters defined earlier by the following:

$$
\frac{\beta_i^{coll}}{\beta_i^{solo}} = \frac{\xi_i}{\eta_i}
$$

(6.7)

from which one can easily find the following alternative forms for the total saturation fraction:

$$
\beta_i^{tot} = \frac{1}{2} \left\{ \frac{1+\xi_i}{1+\eta_i} \left[ \beta_i^{solo} + \frac{\eta_i}{\xi_i} \beta_i^{coll} \right] \right\} = \frac{1+\xi_i}{1+\eta_i} \beta_i^{solo} = \frac{1+\xi_i}{1+\eta_i} \frac{\eta_i}{\xi_i} \beta_i^{coll}
$$

(6.8)

The collaborative/solo index $\xi_i$ depends implicitly on the spans $m_i$ and $k_i$ or equivalently on the saturation fractions $\beta_i^{coll}$ and $\beta_i^{solo}$. But if the ratio of the saturation fractions is held constant (i.e. if $m_i$ and $k_i$ are in constant ratio), then the left hand side of Equation (6.7) is constant and $\xi_i$ is independent of $m_i$ and $k_i$. It is then reasonable to use either $\xi_i$, $\eta_i$, or the “attention” coefficients defined in the next section in power series expansions that apply for arbitrary values of the spans.
6.3.4 Parameters for limiting cases and expansion

A pair of “attention” coefficients $\alpha_i^{\text{coll}}$ and $\alpha_i^{\text{solo}}$ - related to the quantities defined above – is the most convenient set of expansion parameters for use in Section 4; they are dominance parameters equivalent to $\xi_i$. Their sum is normalized to unity. The “attention coefficients” are defined in terms of the quantities introduced above by:

$$\alpha_i^{\text{coll}} = \frac{R_i^{\text{coll}}}{R_i^{\text{tot}}} = \frac{\xi_i}{1 + \xi_i}$$

$$\alpha_i^{\text{solo}} = \frac{R_i^{\text{solo}}}{R_i^{\text{tot}}} = \frac{1}{1 + \xi_i}$$

(6.9) (see 4.2)

$$\alpha_i^{\text{coll}} + \alpha_i^{\text{solo}} = 1$$

When $\xi_i$ approaches infinity (collaboration dominance) the expression above for $\alpha_i^{\text{coll}}$ approaches unity and $\alpha_i^{\text{solo}}$ approaches zero. In the opposite limit when $\xi_i$ approaches zero (solo dominance) $\alpha_i^{\text{coll}}$ approaches zero and $\alpha_i^{\text{solo}}$ approaches unity. When $\xi_i$ or $1/\xi_i$ are small, Taylor series expansions using them converge rapidly.

6.3.5 General method for evaluating the collaborative “dit” capacity

It is a great simplification (and hopefully often a good approximation) to assume that all actors in an organization have the same collaborative parameters: in effect justifying the omission of the indices above.

When parameters vary for different actors and different collaborations, $R_i^{\text{coll}}$ is evaluated for a particular actor by summing over his own collaborative sub-net, limited by $D_i^{\text{coll}}$ as in Equation (6.10b) below. As long as that condition is met (generally meaning that the number of links is small enough so that saturation is not approached) specific collaboration links are assumed to have fixed, individualized “dit” capacity allocations: $r_{i,j}^+$ for outbound or $r_{i,j}^-$ for inbound. The factors $\rho_{i,j}^+$ and $\rho_{i,j}^-$ are the assigned capacities (not observed values) and may vary for each collaborative link (named by it’s endpoints).

The total collaborative capacity $R_i^{\text{coll}}$ is given by the sum below, incorporating a specific set of collaboration partners at node $i$:

$$R_i^{\text{coll}} = \sum_{j=1,|\neq i|}^{n} \rho_{i,j}^+ r_{i,j}^+ + \sum_{j=1,|\neq i|}^{n} \rho_{i,j}^- r_{i,j}^-$$

(6.10a)

$$R_i^{\text{coll}} \leq D_i^{\text{coll}}$$

(6.10b) (2.4)

The (+/-) sums are carried out respectively over the outbound and inbound linkages. Each sum can theoretically select up to $n - 1$ collaborators from the organization’s total population ($n$). The switch functions $\rho_{i,j}^+$ and $\rho_{i,j}^-$ equal unity for collaboration linkages that are actually chosen and they are zero otherwise. Each one defines one edge of the specific directed graph whose edges map out the collaborative network topology for the organization.

The sets $\{\rho_{i,j}^+\}$ and $\{\rho_{i,j}^-\}$ include all $i$ and $j$ values in the interval from 1 to $n$. They specify the outbound and inbound collaboration graphs and are identical if every link that begins within the organization also ends somewhere within it. This is assumed to be approximately the case, consistent with the assumption in Section 2 that the organization is not seriously perturbed by external linkages.

The elements of $\{\rho_{i,j}^-\}$ are then the same as those of $\{\rho_{i,j}^+\}$ but with the indices swapped. The cardinality of these sets grows with the product $2n<m>$, where $<m>$ is an average (inbound or outbound) number of collaborators per node. It would scale as $n(n-1) \sim n^2$ if everyone were to collaborate with everyone else, but in most sizable
organizations there is a much sparser collaborative structure that excludes most of the \( n - 1 \) choices available to one individual.

The outbound and inbound links at a particular node need not connect to the same partners (e.g., some or all of the decision processes may be unidirectional flows). If there are exactly \( m^* \) open outbound paths from node \( i \), then \( p^+_{i,j} = 1 \) for exactly \( m^* \) values of \( j \). The same logic applies to the number of inbound paths at that node, which equals \( m^- \) and may not equal \( m^* \).

Equation (6.10b) defines the saturation limit - an upper bound on the total elementary binary decision rate an individual can support for collaborations. The individual channel rates \( r^+_{i,j} \) and \( r^-_{i,j} \) remain constant (by assumption for simplicity) as the number of collaborators grows until a saturation threshold is reached when \( R^\text{coll}_i = D^\text{coll}_i \). Above that, further growth in the number of collaborators implies throttling back the individual channel rates. There is thus an impediment to collaborative throughput when knowledge workers assume too broad a collaborative span or an excessive workload. In the absence of coordination costs, productivity would reach and sustain a plateau when the span is above saturation; the inclusion of collaborative entropy costs (see below) leads us to forecast an actual per capita productivity reduction in that case.

### 6.3.6 Total decision complexity and collaborative entropy

The total decision complexity for node \( i \) measures the average number of dits per decision. The function \( A^\text{tot}_i \) has contributions for both collaborative work and for individual contributions (solo), weighted by the “attention” coefficients \( \alpha^\text{coll}_i \) and \( \alpha^\text{solo}_i \) that were defined in Equation (6.9):

\[
A^\text{tot}_i = \alpha^\text{coll}_i A^\text{coll}_i + \alpha^\text{solo}_i A^\text{solo}_i \quad (6.11)
\]

In the “collaboration-dominated” limiting case \( D^\text{coll} \rightarrow 0 \), hence \( R^\text{coll}_i \rightarrow 0 \) and \( \alpha^\text{coll}_i \rightarrow 1 \). In the opposite, “solo-dominated” limit \( D^\text{coll} \rightarrow 0 \), hence \( R^\text{coll}_i \rightarrow 0 \) and \( \alpha^\text{solo}_i \rightarrow 1 \). \( A^\text{solo}_i \) contains only a single term that pertains to the one particular node while \( A^\text{coll}_i \) includes summed contributions from all collaborators.

The decision complexities on the right side of Equation (6.11) couple to each other when tasks are large and multifacetted. Groups can “divide and conquer” complex problems by parceling out subtasks to specialized experts who execute them efficiently while collaborating and/or working independently. Task “partitioning” matches subtasks to appropriate subject matter experts who split their effort between individual work (sub-divided say among several tasks) and collaboration with several others on each task; in the latter case collaboration costs increase as the number of collaborating experts grows.

The groups involved in collaborative and individual tasks are the same fairly often, and the same value of the collaborative span \( m \) can often pertain to both the number of collaborators and the size of the expert group that the subtasks are partitioned onto. There is a direct tradeoff between working individually and collaborating, which is favorable when it buys a reduction in the total decision complexity \( A^\text{tot}_i \).

As a result, the total decision complexity \( A^\text{tot}_i \) is more meaningful than either component (collaborative or individual) alone inasmuch as the productivity of an actor is sensitive to interactions between the collaborative and individual components. This assumption is incorporated into the productivity metrics defined in the following section.

When an application has been mapped onto a group with the appropriate skills and structure, \( A^\text{solo}_i \) and \( A^\text{coll}_i \) can both be represented as “value functions” (see Section 4 of the main text) that decrease the total decision complexity \( A^\text{tot}_i \), thereby improving productivity up to a point where no further benefit can be gained by expanding the expert group.

The collaborative complexity \( A^\text{coll}_i \) incorporates a specific set of linkages terminating on node \( i \). The range of choices is analogous to the set of communication symbol choices or to the set of available system states in statistical physics [To1938, chapter 6]. \( A^\text{coll}_i \) measures information associated with the intrinsic task complexity and with the range of available collaborators, from which it takes on the functional form of entropy.

For a single outbound link, the collaborative choice information is assumed to be proportional to the amount of choice information \( h^+_{i,j} \) multiplied by an amplitude \( a^+_{i,j} \) that measures the complexity of jointly performed tasks. Similar quantities apply to inbound links so that products having the form \( a^+_{i,j} h^+_{i,j} \) and \( a^-_{i,j} h^-_{i,j} \) measure decision complexity.
An increase in specialization (increased span, greater collaboration cost) without compensating reductions in the complexity amplitudes in $A^\text{solo}_i$ or $A^\text{coll}_i$ would cause overall productivity to deteriorate. As long as these factors diminish productivity may rise.

The choice information factors $h^+_i,j$ and $h^-_i,j$ have the same form as Shannon entropy; that is, they depend on the base 2 logarithms of $p^+_i(j)$ and $p^-_i(j)$, where these are the conditional probabilities for choosing collaborator $j$ when at node $i$ (see Appendix Section 6.6 for a added discussion of this). The use of the logarithm guarantees correct additive properties. The expected values of the entropy for single outbound and inbound links are the products:

$$h^+_i,j = -p^+_i(j) \log_2[p^+_i(j)]$$

$$h^-_i,j = -p^-_i(j) \log_2[p^-_i(j)]$$

Inasmuch as probabilities are always less than 1, each of the above quantities is always positive.

The collaborative decision complexity for node $i$ is just the sum over all the outbound and inbound links:

$$A^\text{coll}_i = -\sum_{j=1, j \neq i}^n a^+_i,j p^+_i(j) \log_2[p^+_i(j)] - \sum_{j=1, j \neq i}^n a^-_i,j p^-_i(j) \log_2[p^-_i(j)]$$  \hspace{1cm} (6.12)  \hspace{1cm} (4)

The amplitudes $a^+_i,j$ and $a^-_i,j$ are decision complexities for individual links, possibly including “value” contributions as discussed in Section 4.4. The left hand side of Equation (6.12) is just the choice-weighted average (expected value) of the link decision complexity. The complexity amplitudes are proportional to the amount of cognition needed to translate spoken, written, or visual material at both ends of a conversation. They can be compared functionally to codecs (coder/decoders) in data communication, but here they refer to high level interchanges without regard to a physical medium associated with the links.

When there is a high degree of professional and cultural compatibility between parties to an information exchange the amplitudes may be comparatively small.

If the amplitudes $a^+_i,j$ and $a^-_i,j$ are constants for all $i, j$ they factor out of the sums and become just overall multiplicative coefficients. Equation (6.12) then has the mathematical form of Shannon's information function [Sh1948] and also of physical entropy. The probabilities are zero when individuals do not collaborate with each other, so they have a switching effect inasmuch as the limit of $p \log_2(p)$ is zero as $p$ approaches zero.

If collaborative entropy were ignored, Equation (6.12) would contain the conditional probabilities but with the logarithmic factor replaced by unity. The theory would then still choose a statistically weighted mix of partners and predict saturation effects, but it would fail to predict per capita productivity deterioration as a function of collaborative span and all of the effects discussed in Section 4.

The probabilities cited above might be determined empirically by field surveys and activity reporting - a major undertaking that involves mapping out and observing the collaboration network over time. A family of standard modeling coefficients will probably be developed by experience and reused.

6.3.7 The maximum decision rate $M^\text{tot}_i$

The principal metric is the maximum decision rate (per capita) at node $i$; it is the quantity analogous to the maximum symbol transmission rate on a shared noiseless channel. The function defining it is the quotient of the maximum “total dit capacity” (including both collaborative and individual contributions terms) to the “total” decision complexity (also including collaborative and individual terms). The following is the proposed defining expression:
In the above, the collaborative and individual decision rates interact with each other through the total decision complexity and are not cleanly separable; an actor's productivity is sensitive to the weighted decision complexity $A_i^{\text{tot}}$ of all his tasks, forming the denominator above, as discussed in the preceding section. This definition of $M_i^{\text{tot}}$ appears to be more meaningful than a sum of separate terms purporting to describe collaborations and individual contributions that omit their competition. The following is an example of such a (rejected) candidate definition incorporating a sum of separable contributions:

$$\begin{align} N_i^{\text{tot}} &= \frac{R_i^{\text{solo}}}{A_i^{\text{solo}}} + \frac{R_i^{\text{coll}}}{A_i^{\text{coll}}} \end{align}$$

To understand the differences in these definitions, compare the one above to Equation (6.13) by forming the ratio $N_i^{\text{tot}}/M_i^{\text{tot}}$. After a bit of algebra the ratio takes the form:

$$\begin{align} \frac{N_i^{\text{tot}}}{M_i^{\text{tot}}} &= 1 + \alpha_i^{\text{solo}}\alpha_i^{\text{coll}} \left( \frac{A_i^{\text{coll}} - A_i^{\text{solo}}}{A_i^{\text{solo}}A_i^{\text{coll}}} \right)^2 \end{align}$$

The rightmost term measures the interference cost between collaborative and individual work. In the collaboration-dominated limit ($\alpha_i^{\text{solo}} \to 0$) or the individual-contribution-dominated limit ($\alpha_i^{\text{coll}} \to 0$) the rightmost term is always zero, the ratio above becomes unity, and the metrics $M_i^{\text{tot}}$ and $N_i^{\text{tot}}$ are equivalent. If the decision complexities in the numerator are exactly equal the rightmost term can vanish away from those limits as well, but one should expect this to be a rare occurrence inasmuch as the decision complexities ordinarily have different functional forms. The rightmost term is always positive, indicating that rejected candidate definition $N_i^{\text{tot}}$ would overestimate the productivity.

The decision rate for the entire organization is simply the sum over all $n$ actors (nodes, sub-nets):

$$\begin{align} M^{\text{tot}}(n) &= \sum_{i=1}^{n} M_i^{\text{tot}} \quad (6.14) \quad (2.1) \end{align}$$

The average per capita decision rate is simply $M^{\text{tot}}(n)$ divided by $n$, viz.:

$$\begin{align} \mu^{\text{tot}}(n) &= \frac{M^{\text{tot}}(n)}{n} \quad (6.15) \quad (2.1) \end{align}$$

If all the nodes are identical, Equations (6.13) and (6.15) will yield the same result.

Note that Equations (6.13), (6.14), and (6.15) depend on a particular snapshot $\{S\}$ of the state of an organization, which is of course changing dynamically with time. The collaborative topology, population, capacities, task assignments, and task complexities are likely to all be varying. To incorporate such changes one would evaluate the time average of $M^{\text{tot}}(\{S\})$, which is equivalent to an ensemble averaged maximum decision rate $<M^{\text{tot}}>\quad (\text{distinct from the average portion of the capacity that is actually used})$. Letting $\{S\}$ specify a possible state of the organization, the ensemble average is the sum of $M^{\text{tot}}(\{S\})$ weighted by a density of states function $\rho(\{S\})$ in an expression $[\text{To1938, chapter 3}]$ of the form:

$$<M^{\text{tot}}> = \sum_{\text{all states } \{S\}} \rho(\{S\}) M^{\text{tot}}(\{S\})$$
The density of states represents the occurrence probability for each possible state (including changes in the collaborative topology) in the ensemble.

This statistical approach may be pursued in the future but it merely mentioned for now, inasmuch as it requires much more information about organizational evolution than the current approach.

6.4. An exactly solvable model

The equations in the sections above could be used directly for numerical modeling if the constants and collaborative network connectivity were determined in detail. That would be a sizable undertaking, especially for a large organization. In the interim, it is useful to develop an approximate, analytically tractable model that provides qualitative insight and allows us to sidestep the need to find values of the many parameters that are not yet known. The path below uses the formalism above to find the same results presented in Section 2 by more intuitive arguments. The collaborative “value” as defined in Section 4.4.2 and 4.5 is neglected.

6.4.1 The collaboration-dominated approximation

The collaboration-dominated limit assumes that \( \alpha_i^{\text{solo}} \to 0 \) for all actors, thus \( \alpha_i^{\text{coll}} \to 1 \) (see Equation (6.9)). The collaborative terms in Equation (6.13) are dominant in both the numerator and denominator with the other terms neglected to lowest order, resulting in:

\[
M_i^{\text{coll}} = M_i^{\text{solo}} \leq \frac{R_i^{\text{coll}}}{A_i^{\text{coll}}} \leq \frac{D_i^{\text{coll}}}{A_i^{\text{coll}}} \quad (6.16a)
\]

By contrast, \( \alpha_i^{\text{coll}} \to 0 \) and \( \alpha_i^{\text{solo}} \to 1 \) when knowledge workers are all principally individual contributors. They have almost no collaborative activities but may have divided up tasks according to expertise areas to reduce decision complexities. Equation (6.13) would then become:

\[
M_i^{\text{coll}} = M_i^{\text{solo}} \leq \frac{R_i^{\text{solo}}}{A_i^{\text{solo}}} \leq \frac{D_i^{\text{tot}}}{A_i^{\text{tot}}} \quad (6.16b)
\]

Real knowledge workers operate between these extremes and real organizations are populated by a mix of workers in each category.

All workers are assumed to be collaboration-dominated below to find the lowest order influence of collaborative entropy. If there are purely solitary workers, the lowest order correction to the collaboration-dominated version of Equation (6.14) is an additive constant. Mixed cases require additional assumptions and are considered in Section 4.

6.4.2 Structure factor for collaboration-dominated work

Equations (6.10), (6.12), and (6.16a) simplify when all of a decision-maker’s collaborators are equivalent; that is, when the capacities and decision complexities are identical for all links and when the inbound and outbound values are the same. The dit capacities \( r_{ij}^{+} \) and \( r_{ij}^{-} \) are replaced by an average called \( R_{0,j} \), which factors out of the summation. The complexity amplitudes \( a_{ij}^{+} \) and \( a_{ij}^{-} \) are likewise replaced by a single average value called \( A_{0,i} \), which also factors out.

The collaborative decision rate (Equation (6.16a)) becomes a constant \( M_{0,i} \) multiplied by a dimensionless structure factor \( S_i^{\text{coll}} \) defined by:

\[
M_i^{\text{coll}} = M_{0,i} S_i^{\text{coll}} \quad (6.17)
\]

where \( M_{0,i} = 2R_{0,i} / A_{0,i} \)

The coefficient \( M_{0,i} \) as defined is the maximum per capita collaborative decision rate for the smallest group that can be meaningfully treated: one with three members each of whom has exactly 2 inbound and 2 outbound links at each node (see discussion below). The structure factor \( S_i^{\text{coll}} \) therefore measures per capita collaborative productivity relative to
the benchmark workgroup of 3 members with 2 collaborators per person. Below the saturation limit (associated with $D_{i}^{coll}$) the structure factor $S_{i}^{coll}$ is:

$$S_{i}^{coll} \approx \frac{-\frac{1}{2} \sum_{j=1,|j|}^{n} \left[p_{ij}^{+} + p_{ij}^{-}\right]}{\sum_{j=1,|j|}^{n} \left[p_{ij}^{+}\log_{2}(p_{ij}^{+}) + p_{ij}^{-}\log_{2}(p_{ij}^{-})\right]}$$  \hspace{1cm} (6.18a)

Above saturation, the inequality on the right in Equation (6.10b) limits the numerator so that $S_{i}^{coll}$ is:

$$S_{i}^{coll} \approx \frac{-m_{i}^{crit}}{\sum_{j=1,|j|}^{n} \left[p_{ij}^{+}\log_{2}(p_{ij}^{+}) + p_{ij}^{-}\log_{2}(p_{ij}^{-})\right]}$$  \hspace{1cm} (6.18b)

where $m_{i}^{crit} = D_{i}^{coll}/(2R_{0,i})$  \hspace{1cm} (6.3b) (2.7)

In the above, $m_{i}^{crit}$ is simply the number of outbound or inbound collaboration links that triggers saturation.

The structure factor is dimensionless and provides comparisons that depend on organization structure alone, even in the absence of accurate independent numerical values for the coefficients $A_{0,i}$ and $R_{0,i}$. The structure factor depends purely on the linkages and entropy associated with a particular worker $i$.

6.4.3 An exactly solvable model for the structure factor as a function of collaborative span

The summation over the switch functions $p_{ij}^{+}$ and $p_{ij}^{-}$ in Equation (6.18a) simply counts the numbers of outbound ($m_{i}^{+}$) and inbound ($m_{i}^{-}$) collaborations for node $i$. In general, the collaborative spans $m_{i}^{+}$ and $m_{i}^{-}$ can vary independently between 0 and $n - 1$ and need not be the same at all nodes, making possible a huge variety of organizational topologies.

As an additional approximation assume that the conditional probabilities are all equal for all $j$ and are simply:

$$p_{ij}^{+} = \frac{1}{m_{i}^{+}} ; \quad p_{ij}^{-} = \frac{1}{m_{i}^{-}}$$

The sums in the denominator of Equation (6.18a) contribute $m_{i}^{+}$ and $m_{i}^{-}$ identical terms respectively with the result:

$$S_{i}^{coll} \approx \frac{1}{2} \left(m_{i}^{+} + m_{i}^{-}\right)$$

and

$$S_{i}^{coll} \approx \frac{m_{i}^{crit}}{\log_{2}(m_{i}^{+}) + \log_{2}(m_{i}^{-})}$$

The outbound and inbound collaborators can in general be different individuals and unequal in number for each node. So the collaborative spans $m_{i}^{+}$ and $m_{i}^{-}$ can differ. But when all nodes are equivalent – meaning the collaborative spans are independent of $i$ at all nodes – one can show that:

$$m_{i}^{+} = m_{i}^{-} = m \quad \text{for all } i$$

This last assumption significantly restricts the range of organization structures allowed but still does not demand that all collaborations be bilateral. For example, when $n = 4$ and $m^{+} = 2$ (outbound) for all nodes, there are still 81
organizational topologies if we let $m^\text{-(incoming)}$ have any value from 0 to 3. If $m^\text{-( incoming)}$ is limited to 2 only, there are only 9 structures of which only 3 are composed solely of bilateral links. The others have some one-way collaboration links. These last modeling assumptions will cause us to miss some interesting behavior. Real organizations evolve so that some individuals do have many more linkages than others. These decision-makers may become choke points for this kind of collaboration cost (as they can for sequential collaboration) when their connectivity exceeds $m^\text{crit}$. Other individuals may be under-loaded and waiting for decision information as a result of the queuing congestion. These situations will have to be addressed by future models, possibly using numerical methods.

Now assume also that all the nodes are identical to each other; i.e., that there is no dependence on $i$ in any of the coefficients. The per capita structure factor is the same for all nodes and is simply:

$$S^{\text{coll}}(m) = \frac{1}{2} \min[m,m^\text{crit}] \log(m) \quad m \geq 2 \quad (6.19) \quad (2.11)$$

The expression $\min[a,b]$ simply chooses the smaller of $a$ or $b$. The per capita maximum decision rate $\mu^{\text{coll}}(m)$ equals $M_0S^{\text{coll}}(m)$, using Equations (6.15) and (6.17).

The collaboration-dominated limit is meaningless for structures having $m < 2$ (i.e., with less than 3 persons). There is no choice of collaborators. For the excluded case $m = 1$ the logarithmic denominator would vanish making $S^{\text{coll}}$ infinite. The graphs of the collaboration sub-nets for $m = 1$ are unidirectional ring structures having zero collaborative entropy (since there is no choice of available collaborators). Graphs of $m = 0$ are disconnected “bubble” diagrams and are therefore not collaborative nodes.

The maximum decision rate for the whole organization is now simply $n$ times $\mu^{\text{coll}}$, viz.:

$$M^{\text{coll}}(n,m) = M_0 n S^{\text{coll}}(m) \quad (6.20)$$

6.5. Three-actor groups as benchmarking tools

Modeling techniques based on this work will require values for parameters such as $M_0$ above. One way to fill this need is by measuring relative values of $M_0$ covering a range of representative knowledge-intensive tasks done by trios of collaborators used as benchmarks. The tabulated results can then be applied to models of workgroups having many more than three nodes.

Three actor groups are important because they are the simplest organizations that have non-zero collaborative entropy. Each of the three collaborators has bilateral links to the other two. The only structure possible is a ring-like topology with $m = 2$. For simplicity suppose any of the 3 actors can select either of the two others with equal probability $p_i(j) = \frac{1}{2}$, for which case $\log_2(p_i(j)) = -1$.

The structure factor is calculated using Equation (6.19) or (2.11), assuming $m^\text{crit} > m$ (below saturation) and $m = 2$. The result is:

$$S^{\text{coll}}(m = 2) = 1$$

The per capita maximum decision rate $\mu^{\text{coll}}(m) = M_0$ as defined in Equation (6.17). For the whole organization $M^{\text{coll}}(n=3,m=2) = 3M_0$. The same result can be obtained by directly evaluating Equation (6.16a); there are just $2A_0$ dits/decision in the entropy denominator (Equation (6.12)), counting inbound and outbound links. The maximum dit rate (from Equation (6.10a) or (2.6) is just $4R_0$.

Three-actor groups may not be efficient as actual competitive units, since the actors are not necessarily loaded near their saturation levels.

6.6. Expression for the collaborative entropy of a link

The entropy associated with a collaborative link is evaluated by re-tracing Hamming’s discussion of Shannon entropy for a communication channel [Ha1980, page 101 ff.]. The $i$th person chooses one of the remaining $n-1$ people in the organization whenever he inserts a decision into the network. Let $p_i(j)$ be the conditional probability that person $i$ chooses person $j$ as his target.

If the choice is known in advance there is no surprise and no information is needed to know whom person $i$ collaborates with; one of the $p_i(j)$ is then equal to 1 and all the other probabilities are zero. On the other hand, if the probabilities are all equal (and small if $n$ is large) then the “surprise” when one recipient is actually picked is at its
maximum and that choice carries a significant amount of information. The information needed to pick a recipient is thus related to the inverse of the probability \( p_i(j) \) for making that choice.

When two recipients are chosen independently the information associated with the joint event should simply be the sum of the information associated with for each event separately. For example, if person \( i \) selects recipients \( j \) and \( k \) independently, the associated information is:

\[
\text{Info}[E(i,j) \text{ and } E(i,k)] = \text{Info}[E(i,j)] + \text{Info}[E(i,k)]
\]

In the above, the notation \( E(i,j) \) represents the event in which source \( i \) chooses recipient \( j \). The function that satisfies all these requirements is a logarithm, viz.:

\[
\text{Info}[E(i,j)] = \log_2 (1/p_i(j)) = -\log_2 (p_i(j))
\]

When there is no surprise, \( \log_2(1) = 0 \) and zero information accompanies the event.

The expected value of the entropy \( c_i(j) \) (information in "dits") associated with the event \( E(i,j) \) is just the conditional probability of choosing \( j \) multiplied by the information associated with the choice, i.e.,

\[
h_i(j) = -p_i(j) \log_2(p_i(j)) \quad (6.21)
\]

The preceding discussion applied to information outbound from a node. A similar argument and result applies to inbound information.

Note that the information in \( h_i(j) \) reflects the frequencies of choice but not the complexity of the shared decision information. A multiplicative amplitude \( a_{i,j} \) is introduced in the main text to convert the raw entropy to a task complexity.

Like physical entropy, collaborative entropy grows with the size of the decision network.

6.7. Measuring the “value” added by expert collaborations

This section expands on the discussion of “value” functions (see sections 4.4.2 and 4.5) applied to "expert" collaborations. “Value” functions are modified solo and collaborative decision complexities - the latter including the collaborative entropy. For non-expert collaborations (such as those that merely increase the labor pool or share information and responsibility), the logarithmic terms in the denominators of Equations (4.4) and (4.5) would be replaced by unity.

Odlyzo and Tilly [OT2005] considered the problem of assigning “value” to communication networks that deliver information and entertainment. They were motivated by media popularity surrounding “Metcalfe’s Law” - a proposed rule for assigning economic “value” to the networks. Bob Metcalfe [Me1995/6] proposed that the value of networks is proportional to \( N^2 \) (approximately \( N(N-1) \)), where \( N \) is the number of users. This is equivalent to proposing that the value a single user gets is directly proportional to the number of other users in the network. Metcalfe ignored the likelihood that the value of unbounded access has “diminishing returns” as \( N \) becomes very large. The 1990’s “telecomm bubble” implosion was in part the marketplace refutation of this value rule.

O & T proposed an alternative model. They presented three theoretical arguments for proposing \( N \log_2(N) \) as a better rule-of-thumb valuation. This does incorporate a “diminishing returns” principle inasmuch as value per user grows only as \( \log_2(N) \).

6.8. Discussion of a previous definition of the maximum decision rate \( M(n) \)

The definition of the maximum decision rate used above is an improved version of the one used in previous papers on this subject [Ja2003, 2004]. The earlier work approximated an entire organization as a single fully connected common decision channel for simplicity. Each actor collaborated with every other actor. The maximum decision rate was simply taken to be the total maximum dit capacity \( R(n) \) divided by the total collaborative entropy \( A(n) \), where these functions are evaluated by summing Equations (6.3a) and (6.11) respectively over all \( n \) nodes, including only the collaborative terms.

The defining equation for \( M_{\text{full}}(n) \) in the fully connected approximation was:

\[
M_{\text{full}}^{\text{tot}}(n) = \frac{R_{\text{tot}}(n)}{A_{\text{tot}}(n)} = \sum_{i=1}^{n} \frac{R(i)}{A_{\text{tot}}(n)}
\]
Each term has the total decision complexity for the organization $A^{\text{tot}}(n)$ as its denominator. The collaborative entropy of the entire organization was assumed to limit the decision rate at each node, not just the contribution due to decisions at node $i$ alone which is smaller by roughly a factor of $n$. The per capita maximum decision rate deteriorated as $1/n \log_2(n)$ for organizations above saturation, which overestimates the falloff. At large values of $n$, the total decision rate for the whole organization falls off as $1/\log_2(n)$, meaning that total output diminishes. Brooks’ 1975 estimate in “The Mythical Man-month” [Br1995] suggested an even more severe output falloff proportional to $1/n$.

By comparison, the current definition (Equation (6.13)) allows only a node’s own decision complexity and collaboration costs in $A_i(n)$ to limit its decision rate, with each individual node seen as a common decision channel. The per capita maximum decision rate falls off more modestly as $1/\log_2(n)$ for large workgroups.