Quadruple and octuple layer potentials in two dimensions I: Analytical apparatus

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Abstract

A detailed analysis is presented of all pseudo-differential operators of orders up to 2 encountered in classical potential theory in two dimensions. Each of the operators under investigation turns out to be a sum of one or more of standard operators (second derivative, derivative of the Hilbert transform, etc.), and an integral operator with smooth kernel. This classification leads to an extremely simple analysis of spectra of such operators, and simplifies the design of procedures for their numerical evaluation. In a sequel to this paper, the obtained apparatus will be used to construct stable discretizations of arbitrarily high order for a variety of boundary value problems for elliptic partial differential equations.

Keywords: Laplace equation; Potential theory; Pseudo-differential operators; Hypersingular integral equations

1. Introduction

Integral equations of classical potential theory are a tool for the solution of the Laplace equation; they have straightforward analogies to many other elliptic partial differential equations (PDEs). From the point of view of a modern mathematician, they are relatively simple objects. Indeed, a second kind integral equation (SKIE) is an equation involving the sum of the unity operator and a compact operator;
for most practical purposes, such an object behaves like a finite-dimensional system of linear algebraic
equations, with the Fredholm alternative replacing the theory of determinants. Integral equations of
the first kind (FKIEs) are a considerably more complicated object than those of the second kind. Since a first
kind integral operator is compact, solving a first kind integral equation involves the application of the
inverse of a compact operator to the right-hand side; depending on the right-hand side, the result might
or might not be a function. Since the classical boundary value problems (Dirichlet, Neumann, and Robin)
are easily reduced to SKIEs, the original creators of potential theory simply ignored FKIEs. Later, FKIEs
of classical potential theory have also been investigated, and are now a fairly well-understood object.

In a nutshell, when the solution of a Dirichlet problem is represented by the potential of a single layer,
the result is an FKIE; when the solution of a Dirichlet problem is represented by the potential of a double
layer, the result is an SKIE. When the solution of a Neumann problem is represented by a single layer
potential, the result is an SKIE; and when the solution of a Neumann problem is represented by a double
layer potential, the result is not a classical integral equation, but rather an integro-pseudo-differential one
(in computational electromagnetics, this particular object is known as a hypersingular integral equation). Once the integral equation is constructed, the question arises whether it has a solution, whether that
solution is unique, etc. Generally, questions of this type are easily answered for the Laplace and Yukawa
equations, and less so in other cases.

As a computational tool, SKIEs were popular before the advent of computers; between 1950 and
1970, they were almost completely replaced with Finite Differences and Finite Elements. The only areas
where integral equations survived as a numerical tool were those where discretizing the whole area of
definition of a PDE is impractical or very difficult, such as problems in radar scattering and certain
areas of aerodynamics. The reasons for this lack of favor have to do with the fact that discretization of
most integral equations of potential theory leads to dense systems of linear algebraic equations, while
the Finite Elements and Finite Differences result in sparse matrices. During the last 15 years or so, it
has been discovered that many integral operators of potential theory can be applied to arbitrary vectors
in a “fast” manner (for a cost proportional to $n$ for the Laplace and Yukawa equations, and for a cost
proportional to $n \cdot \log(n)$ for the Helmholtz equation, with $n$ the number of nodes in the discretization
of the integral operator). Detailed discussion of such numerical issues is outside the scope of this paper,
and we refer the reader to [5,6]. Here, we remark that the interest in integral formulations of problems
of mathematical physics has been increasing, and that classical tools of potential theory turned out to be
insufficient for dealing with many problems encountered in practice.

Specifically, many applications lead to integral formulations involving not only integral equations, but
also integro-pseudo-differential ones. More frequently, while it is possible to formulate a problem as an
FKIE or an SKIE, the numerical behavior (stability) of the resulting schemes leaves much to be desired. In
such cases, it is sometimes possible to reformulate the problem as an integro-pseudo-differential equation
with drastically improved stability properties (perhaps, after an appropriate preconditioning). A simple
example of such a situation is the exterior Neumann problem for the Helmholtz equation, where the
classical SKIE has so-called spurious resonances, coinciding with those for the interior Dirichlet problem
on the same surface, and having nothing to do with the behavior of the exterior Neumann problem being
solved. The so-called “combined field equation” solves the problem of spurious resonances at the expense
of replacing an integral equation with an integro-pseudo-differential one (see, for example, [1,13,15,18]).
Other examples of such situations include problems in scattering theory, computational elasticity, and
fluid dynamics.
In this paper, we investigate in detail the analytical structure of the integro-pseudo-differential equations obtained when Neumann problems are solved via double layer potentials, when Dirichlet problems are solved via quadruple layer potentials, and several other cases (see (11)–(29) in Section 2 for a detailed list). It turns out that the analytical structure of the obtained equations is quite simple, and involves several standard pseudo-differential operators (derivative, Hilbert transform, derivative of Hilbert transform, inverse of the derivative of the Hilbert transform, and the second derivative), composed (from the left or the right) with simple diagonal operators. We also show that the product of the derivative of the Hilbert transform (a standard hypersingular integral operator) with the standard first kind integral operator of classical potential theory is a second kind integral operator; in other words, these two operators are perfect preconditioners for each other, asymptotically speaking.

In short, the purpose of this paper is a detailed analytical investigation of integro-pseudo-differential operators converting the densities of charge, dipole, quadrupole, and octapole distributions on a smooth curve in two dimensions into the potential, normal derivative of the potential, second normal derivative of the potential, and third normal derivative of the potential on that curve. We will show that each of these operators is the sum of a standard singular or hypersingular operator (obtained by replacing the curve with a circle), an integral operator with a smooth kernel, and a diagonal operator. Once such expressions are obtained, it is quite easy to construct discretizations of the underlying integro-pseudo-differential operators that are adaptive, stable and of arbitrarily high order. Such discretizations (and resulting PDE solvers) have been constructed and will be reported in [10,11].

Remark. While the results reported here are easily generalized to three dimensions, it should be pointed out that there exist important classes of problems in three dimensions leading to integro-pseudo-differential equations that are outside the scope of this paper. Specifically, when frequency-domain equations of electromagnetic scattering are reduced to integral equations on the boundary of the scatterer (yielding the so-called Stratton–Chu equations), the resulting integro-pseudo-differential operators are of a type not investigated here (in addition to normal derivatives on the boundary, they involve tangential derivatives); similarly, integral equations of elastic (as opposed to acoustic) scattering lead to integral expressions whose analysis is not a straightforward extension of that presented in this paper. Needless to say, such operators are frequently encountered in applications; they are currently under investigation.

The structure of this paper is as follows. In Section 2, we list the identities that are the purpose of this paper, and discuss their computational consequences. The remainder of the paper is devoted to proving these identities. In Section 3 the necessary mathematical preliminaries are introduced. In Section 4 we present proofs of some of the results formulated in Section 2; when the proofs of several results are almost identical, we only prove one of them. Finally, in Section 5 we briefly discuss extensions of results of this paper for the Helmholtz equation and to three dimensions.

Remark. The principal purpose of this paper is to present the explicit formulae (31)–(49), (70)–(88), to be used in the design of numerical tools for the solution of partial differential equations. The proofs of these formulae in Section 4 below are a fairly standard exercise in classical analysis, provided here for the sake of completeness. The authors expect that many readers will find it unnecessary to read this paper beyond Section 2.
2. Statement of results

2.1. Notation

We will be considering Dirichlet and Neumann problems for Laplace’s equation in the interior or the exterior of an open region \( \Omega \) bounded by a Jordan curve \( \gamma(t) = (x_1(t), x_2(t)) \) in \( \mathbb{R}^2 \) where \( t \in [0, L] \). We will assume that \( \gamma \) is sufficiently smooth, and parametrized by its arclength. The image of \( \gamma \) will be denoted by \( \Gamma \), so that \( \partial \Omega = \Gamma \). For a vector \( y = (y_1, y_2) \in \mathbb{R}^2 \) we will denote its Euclidean norm by \( \|y\| \). Further, \( c(t) \) will denote the curvature, and \( N_{\gamma}(t) \) or simply \( N(t) \), the exterior unit normal to \( \Gamma \) at \( \gamma(t) \). Clearly,

\[
N(t) = \left( x_2'(t), -x_1'(t) \right);
\]  

the situation is illustrated in Fig. 1.

A charge of unit intensity located at the point \( x_0 \in \mathbb{R}^2 \) generates a potential, \( \Phi_{x_0} : \mathbb{R} \setminus \{x_0\} \to \mathbb{R} \), given by the expression

\[
\Phi_{x_0}(x) = -\log(\|x - x_0\|),
\]  

for all \( x \neq x_0 \). Further, the potential of a unit strength dipole located at \( x_0 \in \mathbb{R}^2 \), and oriented in the direction \( h \in \mathbb{R}^2 \), \( \|h\| = 1 \), is described by the formula

\[
\Phi_{x_0, h}(x) = \frac{\langle h, x - x_0 \rangle}{\|x - x_0\|^2}.
\]  

As is well known, the potential due to a point charge at \( x_0 \in \mathbb{R}^2 \), defined by formula (2), is harmonic in any region excluding the source point \( x_0 \).

**Definition 2.1.** Suppose that \( \sigma : [0, L] \to \mathbb{R} \) is an integrable function. Then we will refer to the functions \( p_{0, \gamma, \sigma} : \mathbb{R}^2 \to \mathbb{R} \) and \( p_{1, \gamma, \sigma} : \mathbb{R}^2 \setminus \Gamma \to \mathbb{R} \), given by the formulae

\[
p_{0, \gamma, \sigma}(x) = \int_0^L \Phi_{\gamma(t)}(x) \sigma(t) \, dt,
\]  

\[
p_{1, \gamma, \sigma}(x) = \int_0^L \frac{\partial \Phi_{\gamma(t)}(x)}{\partial N(t)} \sigma(t) \, dt,
\]

the situation is illustrated in Fig. 1.

\[
\gamma[0, L]
\]

\[\Omega\]

\[N\]

Fig. 1. Boundary value problem in \( \mathbb{R}^2 \).
\[ p^2_{\gamma,\sigma}(x) = \int_0^L \frac{\partial^2 \Phi_{\gamma(t)}(x)}{\partial N(t)^2} \sigma(t) \, dt, \]

\[ p^3_{\gamma,\sigma}(x) = \int_0^L \frac{\partial^3 \Phi_{\gamma(t)}(x)}{\partial N(t)^3} \sigma(t) \, dt, \]

as the single, double, quadruple and octuple layer potentials, respectively.

**Remark.** The functions \( \frac{\partial \Phi_{\gamma(t)}}{\partial N(t)} \), \( \frac{\partial^2 \Phi_{\gamma(t)}}{\partial N(t)^2} \), \( \frac{\partial^3 \Phi_{\gamma(t)}}{\partial N(t)^3} \): \( \mathbb{R}^2 \setminus \{ \gamma(t) \} \to \mathbb{R} \) are often referred to as the dipole, quadrupole and octapole potentials, respectively. Obviously, \( \frac{\partial \Phi_{\gamma(t)}(x)}{\partial N(t)} = \langle N(t), x - \gamma(t) \rangle \| x - \gamma(t) \|^2 \), \( \frac{\partial^2 \Phi_{\gamma(t)}(x)}{\partial N(t)^2} = 2 \langle N(t), x - \gamma(t) \rangle \| x - \gamma(t) \|^4 - \frac{1}{\| x - \gamma(t) \|^2} \), \( \frac{\partial^3 \Phi_{\gamma(t)}(x)}{\partial N(t)^3} = 8 \langle N(t), x - \gamma(t) \rangle \| x - \gamma(t) \|^6 - \frac{6}{\| x - \gamma(t) \|^4}. \)

Clearly, the potentials \( p^1_{\gamma,\sigma}, p^2_{\gamma,\sigma}, p^3_{\gamma,\sigma} \) are analytic in the interior of \( \Omega \) for any integrable \( \sigma \). However, for sufficiently smooth \( \sigma \) and \( \gamma \), they can be extended to \( \overline{\Omega} \) as smooth functions. Similarly, the potentials \( p^4_{\gamma,\sigma}, p^5_{\gamma,\sigma}, p^6_{\gamma,\sigma} \) are analytic functions in the exterior \( \mathbb{R}^2 \setminus \overline{\Omega} \) of \( \Omega \), and can be extended as smooth functions to \( \mathbb{R}^2 \setminus \Omega \). Furthermore, the normal derivatives of these potentials also can be extended up to the boundary as smooth functions. Needless to say, the interior and exterior extensions do not necessarily agree on the boundary \( \Gamma \) (with the obvious exception of \( p^0_{\gamma,\sigma} \)), and the following definition introduces several integral operators concerning such extensions.

**Definition 2.2.** Suppose that the function \( \sigma : [0, L] \to \mathbb{R} \) is twice continuously differentiable, and that \( \gamma \) is sufficiently smooth. Then we define the operators \( K^0_{\gamma,\sigma}, K^{1,0}_{\gamma,\sigma}, K^{1,0}_{\gamma,\sigma}, K^{2,0}_{\gamma,\sigma}, K^{2,0}_{\gamma,\sigma}, K^{3,0}_{\gamma,\sigma}, K^{3,0}_{\gamma,\sigma}, K^{0,1}_{\gamma,\sigma}, K^{0,1}_{\gamma,\sigma}, K^{1,1}_{\gamma,\sigma}, K^{1,1}_{\gamma,\sigma}, K^{2,1}_{\gamma,\sigma}, K^{2,1}_{\gamma,\sigma}, K^{0,2}_{\gamma,\sigma}, K^{0,2}_{\gamma,\sigma}, K^{1,2}_{\gamma,\sigma}, K^{1,2}_{\gamma,\sigma}, K^{0,3}_{\gamma,\sigma}, K^{0,3}_{\gamma,\sigma} : c^2[0, L] \to c[0, L] \) via the formulae

\[ K^0_{\gamma}(\sigma)(s) = \int_0^L \Phi_{\gamma(t)}(\gamma(s)) \sigma(t) \, dt, \]

\[ K^{1,0}_{\gamma,i}(\sigma)(s) = \lim_{h \to 0} \int_0^L \frac{\partial \Phi_{\gamma(t)}(\gamma(s) - h \cdot N(s))}{\partial N(t)} \sigma(t) \, dt, \]

\[ K^{1,0}_{\gamma,e}(\sigma)(s) = \lim_{h \to 0} \int_0^L \frac{\partial \Phi_{\gamma(t)}(\gamma(s) + h \cdot N(s))}{\partial N(t)} \sigma(t) \, dt, \]
\[ K_{\gamma,i}^{2,0}(\sigma)(s) = \lim_{h \to 0} \int_0^L \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s) - h \cdot N(s))}{\partial N(t)^2} \sigma(t) \, dt, \]  
(14)

\[ K_{\gamma,e}^{2,0}(\sigma)(s) = \lim_{h \to 0} \int_0^L \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s) + h \cdot N(s))}{\partial N(t)^2} \sigma(t) \, dt, \]  
(15)

\[ K_{\gamma,i}^{3,0}(\sigma)(s) = \lim_{h \to 0} \int_0^L \frac{\partial^3 \Phi_{\gamma(t)}(\gamma(s) - h \cdot N(s))}{\partial N(t)^3} \sigma(t) \, dt, \]  
(16)

\[ K_{\gamma,e}^{3,0}(\sigma)(s) = \lim_{h \to 0} \int_0^L \frac{\partial^3 \Phi_{\gamma(t)}(\gamma(s) + h \cdot N(s))}{\partial N(t)^3} \sigma(t) \, dt, \]  
(17)

\[ K_{\gamma,i}^{0,1}(\sigma)(s) = \lim_{h \to 0} \int_0^L \frac{\partial \Phi_{\gamma(t)}(\gamma(s) - h \cdot N(s))}{\partial N(s)} \sigma(t) \, dt, \]  
(18)

\[ K_{\gamma,e}^{0,1}(\sigma)(s) = \lim_{h \to 0} \int_0^L \frac{\partial \Phi_{\gamma(t)}(\gamma(s) + h \cdot N(s))}{\partial N(s)} \sigma(t) \, dt, \]  
(19)

\[ K_{\gamma,i}^{1,1}(\sigma)(s) = \lim_{h \to 0} \int_0^L \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s) - h \cdot N(s))}{\partial N(s) \partial N(t)} \sigma(t) \, dt, \]  
(20)

\[ K_{\gamma,e}^{1,1}(\sigma)(s) = \lim_{h \to 0} \int_0^L \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s) + h \cdot N(s))}{\partial N(s) \partial N(t)} \sigma(t) \, dt, \]  
(21)

\[ K_{\gamma,i}^{2,1}(\sigma)(s) = \lim_{h \to 0} \int_0^L \frac{\partial^3 \Phi_{\gamma(t)}(\gamma(s) - h \cdot N(s))}{\partial N(s) \partial N(t)^2} \sigma(t) \, dt, \]  
(22)

\[ K_{\gamma,e}^{2,1}(\sigma)(s) = \lim_{h \to 0} \int_0^L \frac{\partial^3 \Phi_{\gamma(t)}(\gamma(s) + h \cdot N(s))}{\partial N(s) \partial N(t)^2} \sigma(t) \, dt, \]  
(23)

\[ K_{\gamma,i}^{0,2}(\sigma)(s) = \lim_{h \to 0} \int_0^L \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s) - h \cdot N(s))}{\partial N(s)^2} \sigma(t) \, dt, \]  
(24)

\[ K_{\gamma,e}^{0,2}(\sigma)(s) = \lim_{h \to 0} \int_0^L \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s) + h \cdot N(s))}{\partial N(s)^2} \sigma(t) \, dt, \]  
(25)
\[ K_{\gamma,i}^{1,2}(\sigma)(s) = \lim_{h \to 0} \int_{0}^{L} \frac{\partial^3 \Phi_{\gamma}(\gamma(s) - h \cdot N(s))}{\partial N(s)^2 \partial N(t)} \sigma(t) \, dt, \]

(26)

\[ K_{\gamma,e}^{1,2}(\sigma)(s) = \lim_{h \to 0} \int_{0}^{L} \frac{\partial^3 \Phi_{\gamma}(\gamma(s) + h \cdot N(s))}{\partial N(s)^2 \partial N(t)} \sigma(t) \, dt, \]

(27)

\[ K_{\gamma,i}^{0,3}(\sigma)(s) = \lim_{h \to 0} \int_{0}^{L} \frac{\partial^3 \Phi_{\gamma}(\gamma(s) - h \cdot N(s))}{\partial N(s)^3} \sigma(t) \, dt, \]

(28)

\[ K_{\gamma,e}^{0,3}(\sigma)(s) = \lim_{h \to 0} \int_{0}^{L} \frac{\partial^3 \Phi_{\gamma}(\gamma(s) + h \cdot N(s))}{\partial N(s)^3} \sigma(t) \, dt. \]

(29)

**Remark.** Throughout the paper, the subscripts “i” and “e” will denote the limits from the interior and the exterior towards the boundary, respectively. Furthermore, the superscripts “i, j” (as, for example, in \( p_{i,j}^{\gamma,\sigma,e}(s) \)) refers to \( i \) times and \( j \) times differentiation with respect to \( N(t) \) and \( N(s) \), respectively.

**Remark.** Obviously, the operators \( K_{\gamma,i}^{0,1}, K_{\gamma,e}^{0,1}, K_{\gamma,i}^{0,2}, K_{\gamma,e}^{0,2}, K_{\gamma,i}^{0,3}, K_{\gamma,e}^{0,3}, K_{\gamma,i}^{1,2}, K_{\gamma,e}^{1,2} \) given by the formulae (18), (19), (24)–(29) are the adjoints of the operators \( K_{\gamma,e}^{1,0}, K_{\gamma,i}^{1,0}, K_{\gamma,e}^{2,0}, K_{\gamma,i}^{2,0}, K_{\gamma,e}^{3,0}, K_{\gamma,i}^{3,0}, K_{\gamma,e}^{2,1}, K_{\gamma,i}^{2,1} \) defined by (12)–(17), (22), (23), respectively. Furthermore, \( K_{\gamma,i}^{0}, K_{\gamma,i}^{1,1}, K_{\gamma,e}^{1,1} \) defined by (11), (20), (21) are self-adjoint.

### 2.2. Physical interpretation

Formulae (11)–(29) have simple physical interpretations. Specifically, \( K_{\gamma,i}^{0} \) is the linear operator converting a charge distribution on the curve \( \Gamma \) into the potential of that charge distribution on \( \Gamma \). The operator \( K_{\gamma,i}^{1,0} \) converts a dipole distribution on \( \Gamma \) into the potential created by that distribution on the inside of \( \Gamma \); the operator \( K_{\gamma,e}^{1,0} \) converts a dipole distribution on \( \Gamma \) into the potential created by that distribution on the outside of \( \Gamma \). The operator \( K_{\gamma,i}^{0,1} \) converts a charge distribution on \( \Gamma \) into the normal derivative of the potential created by that distribution on the outside of \( \Gamma \), etc.

Generally, the first superscript denotes the number of differentiations at the source (charges, dipoles, quadrupoles, or octapoles); the second superscript denotes the number of differentiations at the point where the potential is evaluated (potential, normal derivative of the potential, second normal derivative of the potential, third normal derivative of the potential). In agreement with standard practice in the theory of pseudo-differential operators, we will define the order \( k \) of either of the operators \( K_{\gamma,i}^{i,j} \) and \( K_{\gamma,e}^{i,j} \) by the formula

\[ k = i + j - 1, \]

(30)

and observe that in this paper, we describe in detail all operators of potential theory whose order does not exceed 2.

An examination of formulae (31)–(49) below shows that the complexity of the expressions describing the operators (11)–(29) on the circle hardly increases as the order of the operator grows. On the other
hand, the differences between the operators (11)–(29) on the circle and those on an arbitrary curve become more complicated with the growth of the order of the operator. For example, the operators $K^0_{γ}, K^{1,0}_{γ,i}, K^{0,1}_{γ,i}, K^{1,0}_{γ,e}, K^{0,1}_{γ,e}$ on an arbitrary smooth curve always differ from these operators on the circle by a compact operator (see formulae (70)–(74)). Similar differences for the operators $K^{2,0}_{γ,i}, K^{2,0}_{γ,e}, K^{1,1}_{γ,i}, K^{2,0}_{γ,e}, K^{0,2}_{γ,i}, K^{0,2}_{γ,e}, K^{1,2}_{γ,i}, K^{1,2}_{γ,e}, K^{0,3}_{γ,i}, K^{0,3}_{γ,e}$ involve the curvature of $γ$ (see (75)–(80)). For the operators $K^{3,0}_{γ,i}, K^{3,0}_{γ,e}, K^{2,1}_{γ,i}, K^{2,1}_{γ,e}, K^{1,2}_{γ,i}, K^{1,2}_{γ,e}, K^{0,3}_{γ,i}, K^{0,3}_{γ,e}$, the corresponding formulae (81)–(88) already involve the square and the derivative of the curvature, as well as the Hilbert transform of the derivative of the function.

**Remark.** While it is certainly possible to derive explicit expressions for boundary integral operators of orders higher than 2, the complexity of the resulting formulae grows, while their numerical utility decreases. The authors have chosen to draw the line at the order 2, mostly because in the applications they anticipate, order 1 is sufficient.

**Remark.** While many of the facts presented in this paper can be obtained “automatically” from the standard theory of pseudo-differential operators, the purpose of this paper is to provide the explicit expressions (31)–(49), (70)–(88) to be used in numerical calculations. Thus, we are ignoring the connections between the formulae (31)–(49), (70)–(88), and the more general theory of pseudo-differential operators.

### 2.3. Results

The limits (12), (13), (18), (19) have been studied in detail in the literature (see, for example, [12,14]). In Section 4, we conduct a similar investigation of (14)–(17), (20)–(29); first for a circle, and then for a sufficiently smooth Jordan curve. In this section we summarize the results of these findings.

The following theorem provides explicit expressions for the action of the operators (11)–(29) on the circle for functions of the form $e^{iks/r}$, with $k = 0, \pm 1, \pm 2, \ldots$; it is proved by direct evaluation of the relevant integrals via the theory of residues (see Section 4 for details).

**Theorem 2.1.** Suppose that $γ$ is a circle of radius $r$ parametrized by its arclength, $k$ is an arbitrary integer, and $s \in [−πr, πr]$. Then,

(a) $K^0_{γ,i}(e^{iks/r})(s) = \begin{cases} \frac{\pi|k|}{2πr} \frac{1}{r} e^{iks/r}, & \text{for } k \neq 0, \\ -\frac{\pi}{2πr} \log(r), & \text{for } k = 0, \end{cases}$

(b) $K^{1,0}_{γ,i}(e^{iks/r})(s) = \begin{cases} -\pi e^{iks/r}, & \text{for } k \neq 0, \\ -2\pi, & \text{for } k = 0, \end{cases}$

(c) $K^{1,0}_{γ,e}(e^{iks/r})(s) = \begin{cases} \pi e^{iks/r}, & \text{for } k \neq 0, \\ 0, & \text{for } k = 0, \end{cases}$

(d) $K^{2,0}_{γ,i}(e^{iks/r})(s) = \begin{cases} \pi(|k| + 1)r^{-1} e^{iks/r}, & \text{for } k \neq 0, \\ 2\pi r^{-1}, & \text{for } k = 0, \end{cases}$

(e) $K^{2,0}_{γ,e}(e^{iks/r})(s) = \begin{cases} \pi(|k| - 1)r^{-1} e^{iks/r}, & \text{for } k \neq 0, \\ 0, & \text{for } k = 0, \end{cases}$

(f) $K^{3,0}_{γ,i}(e^{iks/r})(s) = \begin{cases} -\pi(|k| + 1)(|k| + 2)r^{-2} e^{iks/r}, & \text{for } k \neq 0, \\ -4πr^{-2}, & \text{for } k = 0, \end{cases}$
for the operators (11)–(29) acting on any sufficiently smooth function when $\gamma$ is a circle.

**Corollary 2.1.** Suppose that $\gamma$ is a circle of radius $r$ parametrized by its arclength, and that the function $
abla : [-\pi r, \pi r] \to \mathbb{C}$ is given by its Fourier series

$$
\sigma(t) = \sum_{k=-\infty}^{\infty} \hat{\sigma}_k e^{ikt/r},
$$

with $\hat{\sigma}_k$ denoting the $k$th Fourier coefficient of $\sigma$. Then,

(a) $K^0_\gamma(\sigma)(s) = -2\pi r \log(r) \hat{\sigma}_0 + \pi r \sum_{k=-\infty}^{\infty} \frac{1}{|k|} \hat{\sigma}_k e^{iks/r}$. 

The following corollary is an immediate consequence of Theorem 2.1; it provides explicit expressions for the operators (11)–(29) acting on any sufficiently smooth function when $\gamma$ is a circle.
(b) \[ K^{1,0}_{\gamma,i}(\sigma)(s) = -\pi \sigma(s) - \pi \hat{\sigma}_0, \]
\[ K^{1,0}_{\gamma,e}(\sigma)(s) = \pi \sigma(s) - \pi \hat{\sigma}_0, \]  
\[ K^{1,0}_{\gamma,e}(\sigma)(s) = \pi \sigma(s) - \pi \hat{\sigma}_0, \]  
\[ K^{2,0}_{\gamma,i}(\sigma)(s) = \frac{\pi}{r} \sigma(s) + \pi \hat{\sigma}(\sigma')(s) + \frac{\pi}{r} \hat{\sigma}_0, \]  
\[ K^{2,0}_{\gamma,e}(\sigma)(s) = -\frac{\pi}{r} \sigma(s) + \pi \hat{\sigma}(\sigma')(s) + \frac{\pi}{r} \hat{\sigma}_0, \]  
\[ K^{3,0}_{\gamma,i}(\sigma)(s) = -\frac{2\pi}{r^2} \sigma(s) + \pi \sigma''(s) - \frac{3\pi}{r} H(\sigma')(s) - \frac{2\pi}{r^2} \hat{\sigma}_0, \]  
\[ K^{3,0}_{\gamma,e}(\sigma)(s) = \frac{2\pi}{r^2} \sigma(s) - \pi \sigma''(s) - \frac{3\pi}{r} H(\sigma')(s) - \frac{2\pi}{r^2} \hat{\sigma}_0, \]  
\[ \text{with } H \text{ denoting the Hilbert transform (see (113) in Section 3.3).} \]

The following theorem follows directly from well-known results (see, for example, [14,20]); here stated in a slightly different form.

**Theorem 2.2.** Suppose that \( \gamma : [0, L] \rightarrow \mathbb{R}^2 \) is a \( k \) times continuously differentiable Jordan curve parametrized by its arclength, and that \( \eta : [0, L] \rightarrow \mathbb{R}^2 \) denotes the circle of radius \( r \). Then, there exist such integral operators \( M_0, M_1, N_1 : c[0, L] \rightarrow c[0, L] \) with kernels \( m_0(s,t) \in c^{k-1}([0, L] \times [0, L]), \) \( m_1(s,t), n_1(s,t) \in c^{k-2}([0, L] \times [0, L]) \) that for any sufficiently smooth function \( \sigma : [0, L] \rightarrow \mathbb{R}, \)

(a) \[ K^0_{\gamma,i}(\sigma)(s) = K^0_{\eta,i}(\sigma)(s) + M_0(\sigma)(s), \]  
(b) \[ K^{1,0}_{\gamma,i}(\sigma)(s) = K^{1,0}_{\eta,i}(\sigma)(s) + M_1(\sigma)(s) = -\pi \sigma(s) + N_1(\sigma)(s), \]
(c) \( K^{0.1}_{\gamma, i}(\sigma)(s) = K^{0.1}_{\eta, i}(\sigma)(s) + M^*_i(\sigma)(s) = \pi\sigma(s) + N^*_i(\sigma)(s), \)
(73)
\[ K^{0.1}_{\gamma, e}(\sigma)(s) = K^{0.1}_{\eta, e}(\sigma)(s) + M^*_e(\sigma)(s) = -\pi\sigma(s) + N^*_e(\sigma)(s). \]  
(74)

Furthermore, \( M^*_1, N^*_1 \) are the adjoints of \( M_1, N_1 \), respectively, and the operator \( M_0 \) is self-adjoint.

Theorem 2.2 approximates the operators \( K^0_{\gamma}, K^{1,0}_{\gamma}, K^{1,0}_{\gamma, i}, K^{0.1}_{\gamma, i}, K^{0.1}_{\gamma, e} \) for an arbitrary smooth Jordan curve by the same operators on the circle; Theorem 2.3 below extends these results to the operators \( (14), (15), (20), (21), (24), (25) \). While Theorem 2.2 is well known, the authors failed to find Theorem 2.3 in the literature.

**Theorem 2.3.** Suppose that \( \gamma : [0, L] \to \mathbb{R}^2 \) is a \( k \) times continuously differentiable Jordan curve parametrized by its arclength, and that \( \eta : [0, L] \to \mathbb{R}^2 \) denotes the circle of radius \( \frac{r}{2} \), also parametrized by its arclength. Then, there exist such integral operators \( M_2, N_2, G_2 : c[0, L] \to c[0, L] \) with kernels \( m_2(s, t), n_2(s, t), g_2(s, t) \in C^{k-2}([0, L] \times [0, L]) \) that for any sufficiently smooth function \( \sigma : [0, L] \to \mathbb{R} \),

(a) \( K^{2,0}_{\gamma, i}(\sigma)(s) = \left( \pi c(s) - \frac{2\pi^2}{L} \right) \sigma(s) + K^{2,0}_{\eta, i}(\sigma)(s) + M_2(\sigma)(s) \)
= \( \pi c(s)\sigma(s) + \pi H(\sigma')(s) + N_2(\sigma)(s), \)  
(75)
\[ K^{2,0}_{\gamma, e}(\sigma)(s) = -\left( \pi c(s) - \frac{2\pi^2}{L} \right) \sigma(s) + K^{2,0}_{\eta, e}(\sigma)(s) + M_2(\sigma)(s) \]
= \( -\pi c(s)\sigma(s) + \pi H(\sigma')(s) + N_2(\sigma)(s), \)  
(76)
(b) \( K^{1,1}_{\gamma, i}(\sigma)(s) = K^{1,1}_{\eta, i}(\sigma)(s) + G_2(\sigma)(s) = -\pi H(\sigma')(s) + G_2(\sigma)(s), \)
(77)
\[ K^{1,1}_{\gamma, e}(\sigma)(s) = K^{1,1}_{\eta, e}(\sigma)(s) + G_2(\sigma)(s) = -\pi H(\sigma')(s) + G_2(\sigma)(s), \]  
(78)
(c) \( K^{0,2}_{\gamma, i}(\sigma)(s) = -\left( \pi c(s) - \frac{2\pi^2}{L} \right) \sigma(s) + K^{0,2}_{\eta, i}(\sigma)(s) + M^*_i(\sigma)(s) \)
= \( -\pi c(s)\sigma(s) + \pi H(\sigma')(s) + N^*_i(\sigma)(s), \)  
(79)
\[ K^{0,2}_{\gamma, e}(\sigma)(s) = \left( \pi c(s) - \frac{2\pi^2}{L} \right) \sigma(s) + K^{0,2}_{\eta, e}(\sigma)(s) + M^*_e(\sigma)(s) \]
= \( \pi c(s)\sigma(s) + \pi H(\sigma')(s) + N^*_e(\sigma)(s), \)  
(80)

where \( c(s) \) denotes the curvature of \( \gamma \) at \( \gamma(s) \). Furthermore, \( M^*_2, N^*_2 \) are the adjoints of \( M_2, N_2 \), the operator \( G_2 \) is self-adjoint, and \( H \) denotes the Hilbert transform (see (113) in Section 3.3).

**Remark.** The formulae (71)–(74) above are somewhat misleading, in that they state very simple facts in a relatively complicated manner. Specifically, each of the operators \( K^{1,0}_{\gamma, i}, K^{1,0}_{\gamma, e}, K^{0.1}_{\gamma, i}, K^{0.1}_{\gamma, e} \) is a second kind integral operator with smooth \( (C^{k-2}) \) kernel (see, for example, [14]). In the case of the circle, the kernels of the operators \( K^{1,0}_{\eta, i}, K^{1,0}_{\eta, e}, K^{0.1}_{\eta, i}, K^{0.1}_{\eta, e} \) are identically equal to \( -\frac{1}{r^2} \). Thus, (71)–(74) state the trivial fact that the difference of two smooth kernels is smooth. We list (71)–(74) for compatibility with the formulae (70), (75)–(80).
Observation 2.4. Formulae (70)–(80) have a straightforward interpretation. Specifically, each of the operators $K_{0}^{0}, K_{1}^{0}, K_{1}^{1}, K_{2}^{0}, K_{2}^{1}, K_{3}^{0}, K_{3}^{1}, K_{4}^{1}, K_{5}^{0}, K_{5}^{1}, K_{6}^{0}, K_{6}^{1}, K_{7}^{0}, K_{7}^{1}$, is a sum of a standard operator (the corresponding operator on the circle) and an integral operator with a smooth kernel.

In Section 4, a proof of formulae (75) and (76) is given; the proofs of the formulae (77)–(80) in Theorem 2.3 are similar and are omitted. Theorem 2.5 below extends the results of Theorem 2.3 above by its arclength. Then, there exist such integral operators $M_{2}^{2}$, also parametrized by its arclength, and that $\eta : [0, L] \rightarrow \mathbb{R}^{2}$ denotes the circle of radius $\frac{L}{2\pi}$, also parametrized by its arclength. Then, there exist such integral operators $M_{3}, N_{3}, F_{3}, G_{3} : c[0, L] \rightarrow c[0, L]$ with kernels $m_{3}(s, t), n_{3}(s, t), f_{3}(s, t), g_{3}(s, t) \in c^{k-4}([0, L] \times [0, L])$ that for any sufficiently smooth function $\sigma : [0, L] \rightarrow \mathbb{R}$,

(a) $K_{0}^{3,0}(\sigma)(s) = -\left(2\pi \left(\frac{c(s)}{L} \right)^{2} - \frac{4\pi^{2}}{L} c(s) \right) \sigma(s) - \frac{2\pi c'(s)}{L} H(\sigma)(s) + L \frac{c''(s)}{2\pi} \sigma''(s)$

(b) $K_{1}^{3,0}(\sigma)(s) = \left(\pi - \frac{L}{2} c(s) \right) \sigma''(s) + \frac{L}{2\pi} c(s) K_{0}^{3,0}(\sigma)(s) + F_{3}(\sigma)(s)$

(c) $K_{1}^{3,0}(\sigma)(s) = \left(\pi - \frac{L}{2} c(s) \right) \sigma''(s) + \frac{L}{2\pi} c(s) K_{0}^{3,0}(\sigma)(s) + F_{3}(\sigma)(s)$

Theorem 2.5. Suppose that $\gamma : [0, L] \rightarrow \mathbb{R}^{2}$ is a $k$ times continuously differentiable Jordan curve parametrized by its arclength, and that $\eta : [0, L] \rightarrow \mathbb{R}^{2}$ denotes the circle of radius $\frac{L}{2\pi}$, also parametrized by its arclength. Then, there exist such integral operators $M_{3}, N_{3}, F_{3}, G_{3} : c[0, L] \rightarrow c[0, L]$ with kernels $m_{3}(s, t), n_{3}(s, t), f_{3}(s, t), g_{3}(s, t) \in c^{k-4}([0, L] \times [0, L])$ that for any sufficiently smooth function $\sigma : [0, L] \rightarrow \mathbb{R}$,
(d) \[ K_{\gamma,i}^{0,3}(\sigma)(s) = \left( 2\pi c(s) \right)^2 - \frac{4\pi^2}{L} c(s) \sigma(s) - \left( \pi - \frac{L}{2} c(s) \right) \sigma''(s) \]

\[ - \pi c'(s) H(\sigma)(s) + \frac{L}{2\pi} c(s) K_{n,i}^{0,3}(\sigma)(s) + M_i^*(\sigma)(s) \]

\[ = 2\pi c(s)^2 \sigma(s) - \pi \sigma''(s) - \pi c'(s) H(\sigma)(s) \]

\[ - 3\pi c(s) H(\sigma)'(s) + N_i^*(\sigma)(s), \]

\[ K_{\gamma,e}^{0,3}(\sigma)(s) = - \left( 2\pi c(s) \right)^2 - \frac{4\pi^2}{L} c(s) \sigma(s) + \left( \pi - \frac{L}{2} c(s) \right) \sigma''(s) \]

\[ - \pi c'(s) H(\sigma)(s) + \frac{L}{2\pi} c(s) K_{n,e}^{0,3}(\sigma)(s) + M_i^*(\sigma)(s) \]

\[ = -2\pi c(s)^2 \sigma(s) + \pi \sigma''(s) - \pi c'(s) H(\sigma)(s) \]

\[ - 3\pi c(s) H(\sigma)'(s) + N_i^*(\sigma)(s), \]

where \( c(s) \) denotes the curvature of \( \gamma \) at \( \gamma(s) \). Furthermore, \( M_i^*, N_i^*, F_i^*, G_i^* \) are the adjoints of \( M_3, N_3, F_3, G_3 \), and \( H \) denotes the Hilbert transform (see (113) in Section 3.3).

2.4. Computational observations

In the numerical solution of elliptic PDEs, one is often confronted with the task of evaluating some (or all) of the operators (11)–(29) numerically. While this class of issues will be discussed in detail in a sequel to this work (see [10,11]), here we observe that an inspection of the formulae (31)–(49), indicates that each of the operators \( K_{\gamma,i}^{0,0}, K_{\gamma,i}^{1,0}, K_{\gamma,e}^{0,1}, K_{\gamma,e}^{1,0}, K_{\gamma,i}^{0,1}, K_{\gamma,i}^{1,0}, K_{\gamma,e}^{0,0}, K_{\gamma,e}^{1,0}, K_{\gamma,i}^{1,1}, K_{\gamma,e}^{1,1}, K_{\gamma,i}^{0,2}, K_{\gamma,e}^{0,2}, K_{\gamma,i}^{3,0}, K_{\gamma,e}^{3,0}, K_{\gamma,i}^{2,1}, K_{\gamma,e}^{1,2}, K_{\gamma,i}^{0,2}, K_{\gamma,e}^{0,3}, K_{\gamma,i}^{0,3} \) (see (70)–(88)) is a sum of some of the following: integral operators with smooth kernels, integral operators with logarithmic singularities, the Hilbert transform, the derivative of the Hilbert transform, and the second derivative. The techniques for the accurate integration of smooth functions have been available for hundreds of years, and numerical evaluation of the second derivative presents no serious problems. Effective techniques for the numerical evaluation of the Hilbert transform are less well-known, but have also been available for many years (see, for example, [17]). Efficient integration of logarithmically singular functions is also not very difficult (see [2,8,16]). The only possible source of problems is the derivative of the Hilbert transform; quadrature rules for the evaluation of the latter have been constructed, and will be published in [10]. Thus, there exist rapidly convergent schemes for the numerical evaluation of all of the operators (11)–(29), and, therefore, for the discretization of any problem of mathematical physics that has been reduced to a set of integro-pseudo-differential equations involving any (or all) of the operators (11)–(29).

Of course, when a problem of mathematical physics is discretized, one of principal issues is the condition number of the obtained system of equations. An examination of the formulae (32), (38), (39) immediately shows that the operators \( K_{\gamma,i}^{0,0}, K_{\gamma,i}^{1,0}, K_{\gamma,e}^{0,1}, K_{\gamma,e}^{1,0}, K_{\gamma,i}^{0,1}, K_{\gamma,i}^{1,0}, K_{\gamma,e}^{0,0}, K_{\gamma,e}^{1,0}, K_{\gamma,i}^{1,1}, K_{\gamma,e}^{1,1}, K_{\gamma,i}^{0,2}, K_{\gamma,e}^{0,2}, K_{\gamma,i}^{3,0}, K_{\gamma,e}^{3,0}, K_{\gamma,i}^{2,1}, K_{\gamma,e}^{1,2}, K_{\gamma,i}^{0,2}, K_{\gamma,e}^{0,3}, K_{\gamma,i}^{0,3} \) (see (70)–(88)) have a spectrum that grows linearly, and the \( n \)-point discretization of each of them will also have condition number \( \sim n \). Finally, each of the operators \( K_{\gamma,i}^{3,0}, K_{\gamma,i}^{2,1}, K_{\gamma,e}^{3,0}, K_{\gamma,e}^{2,1}, K_{\gamma,i}^{0,3}, K_{\gamma,e}^{0,3} \) has a spectrum
that grows as $k^2$; an $n$-point discretization of any of them will have condition number $\sim n^2$. Thus, whenever the problem to be solved results in the discretization of any one of the operators $K^0_y$, $K^2_0$, $K^1_{y,i}$, $K^1_{y,e}$, $K^2_{y,i}$, $K^2_{y,e}$, $K^3_{y,i}$, $K^3_{y,e}$, $K^1_{y,1}$, $K^2_{y,1}$, $K^3_{y,1}$, $K^0_{y,1}$, $K^0_{y,2}$, $K^1_{y,2}$, $K^2_{y,2}$, $K^3_{y,2}$, $K^0_{y,3}$, $K^1_{y,3}$, $K^2_{y,3}$, $K^3_{y,3}$, there is a potential for condition number problems, similar to those encountered with discretization of differential equations.

Fortunately, formulae (31)–(49) suggest a solution. Specifically, an examination of the formulae (31), (34), (70), (75) immediately indicates that each of the operators $K^0_y \circ K^2_{y,i}$, $K^2_0 \circ K^0_y$ is a sum of multiplication by a constant with a compact operator, i.e.,

\begin{align*}
K^0_y \circ K^2_{y,i} &= \pi^2 \cdot I + M_i^{00,20}, \\
K^2_0 \circ K^0_y &= \pi^2 \cdot I + M_i^{20,00},
\end{align*}

with $M_i^{00,20}$, $M_i^{20,00}$ compact operators $L^2[0, L] \to L^2[0, L]$ defined by the formulae

\begin{align*}
M_i^{00,20}(\sigma)(s) &= -\pi^2 \cdot \left( 4 \log \left( \frac{L}{2\pi} \right) + 1 \right) \cdot \hat{d}_0 + \pi^2 \cdot \sum_{k=\pm \infty \setminus k \neq 0} \frac{1}{|k|} \hat{\phi}_k e^{2\pi i k s/L} \\
&\quad + \frac{2\pi^2}{L} \cdot M_0(\sigma)(s) + \frac{2\pi^2}{L} \cdot M_0(\hat{d}_0)(s) + \pi \cdot M_0(H(\sigma'))(s) \\
&\quad + \pi \cdot K^0_y(\hat{c}(\sigma))(s) - \frac{2\pi^2}{L} \cdot K^0_y(\sigma)(s) + K^0_y(M_2(\sigma))(s), \quad (91)
\end{align*}

\begin{align*}
M_i^{20,00}(\sigma)(s) &= -\pi^2 \cdot \left( 4 \log \left( \frac{L}{2\pi} \right) + 1 \right) \cdot \hat{d}_0 + \pi^2 \cdot \sum_{k=\pm \infty \setminus k \neq 0} \frac{1}{|k|} \hat{\phi}_k e^{2\pi i k s/L} \\
&\quad + \frac{2\pi^2}{L} \cdot M_0(\sigma)(s) + \frac{2\pi^2}{L} \cdot (\hat{M}_0(\sigma))_0 + \pi \cdot H((M_0(\sigma))')(s) \\
&\quad + \pi \cdot c(s) \cdot K^0_y(\sigma)(s) - \frac{2\pi^2}{L} \cdot K^0_y(\sigma)(s) + M_2(K^0_y(\sigma))(s), \quad (92)
\end{align*}

respectively. Similarly,

\begin{align*}
K^0_y \circ K^2_{y,e} &= \pi^2 \cdot I + M_e^{00,20}, \\
K^2_0 \circ K^0_y &= \pi^2 \cdot I + M_e^{20,00},
\end{align*}

and

\begin{align*}
K^0_y \circ K^1_{y,1} &= -\pi^2 \cdot I + M_i^{00,11}, \\
K^1_{y,i} \circ K^0_y &= -\pi^2 \cdot I + M_i^{11,00}, \\
K^0_y \circ K^1_{y,e} &= -\pi^2 \cdot I + M_e^{00,11}, \\
K^1_{y,e} \circ K^0_y &= -\pi^2 \cdot I + M_e^{11,00},
\end{align*}

and
all of the operators \( M_{11,00}^1, M_{11,00}^e, M_{00,11}^1, M_{00,11}^e, M_{02,00}^1, M_{02,00}^e, M_{00,02}^1, M_{00,02}^e \) are compact; explicit expressions for these are analogous to (91), (92) and are omitted. In other words, the operator \( K_\gamma^0 \) is a perfect preconditioner (asymptotically speaking) for each of the second order pseudo-differential operators of potential theory in two dimensions; in turn, \( K_\gamma^0 \) is preconditioned by each of the operators (75)–(80).

Expressions (81)–(88) contain the second derivative, and are, clearly, preconditioned by the operator of repeated integration \( I_2 : L^2[0, L] \to L^2[0, L] \), defined by its action on the functions \( e^{i \cdot m \cdot x/L} \) via the formula

\[
I_2(e^{i \cdot m \cdot x/L}) = \frac{1}{m^2} \cdot e^{i \cdot m \cdot x/L}.
\]  

In other words, for each of the operators (11)–(29), there is available a straightforward preconditioner. Numerical implications of these (and related) observations will be discussed in [11].

3. Analytical preliminaries

In this section we summarize several results from classical analysis to be used in the remainder of the paper. The principal goal of this section are Theorems 3.1–3.3 that are well-known, and can be found, for example, in [7,9].

3.1. Principal value integrals

Integrals of the form

\[
\int_a^b \frac{\varphi(t)}{t-s} \, dt,
\]

where \( s \in (a, b) \), do not exist in the classical sense, and are often referred to as singular integrals.

**Definition 3.1.** Suppose that \( \varphi \) is a function \( [a, b] \to \mathbb{R}, s \in (a, b) \), and the limit

\[
\lim_{\varepsilon \to 0} \left( \int_a^{s-\varepsilon} \frac{\varphi(t)}{t-s} \, dt + \int_{s+\varepsilon}^b \frac{\varphi(t)}{t-s} \, dt \right)
\]

exists and is finite. Then we will denote the limit (105) by

\[
p.v. \int_a^b \frac{\varphi(t)}{t-s} \, dt,
\]
and refer to it as a principal value integral.

**Theorem 3.1.** Suppose that the function \( \varphi : [a, b] \to \mathbb{R} \) is continuously differentiable in a neighborhood of \( s \in (a, b) \). Then the principal value integral (106) exists.

### 3.2. Finite part integrals

In this paper, we will be dealing with integrals of the form
\[
\int_a^b \frac{\varphi(t)}{(t - s)^2} dt,
\]
where \( s \in (a, b) \), which are divergent in the classical sense. This type of integrals are often referred to as hypersingular or strongly singular.

**Definition 3.2.** Suppose that \( \varphi \) is a function \([a, b] \to \mathbb{R}, s \in (a, b)\), and the limit
\[
\lim_{\varepsilon \to 0} \left( \int_a^s \frac{\varphi(t)}{(t - s)^2} dt + \int_s^{s+\varepsilon} \frac{\varphi(t)}{(t - s)^2} dt - \frac{2\varphi(s)}{\varepsilon} \right)
\]
exists and is finite. Then we will denote the limit (108) by
\[
f.p. \int_a^b \frac{\varphi(t)}{(t - s)^2} dt,
\]
and refer to it as a finite part integral (see, for example, [7]).

The following obvious theorem provides sufficient conditions for the existence of the finite part integral (108), and establishes a connection between finite part and principal value integrals.

**Theorem 3.2.** Suppose that the function \( \varphi : [a, b] \to \mathbb{R} \) is twice continuously differentiable in a neighborhood of \( s \in (a, b) \). Then the finite part integral (109) exists, and
\[
f.p. \int_a^b \frac{\varphi(t)}{(t - s)^2} dt = \frac{d}{ds} \text{p.v.} \int_a^b \frac{\varphi(t)}{t - s} dt.
\]

### 3.3. The Hilbert transform

For an arbitrary periodic function \( \varphi \in L^2[-\pi, \pi] \) and any integer \( k \), we will denote by \( \hat{\varphi}_k \) the \( k \)th Fourier coefficient of \( \varphi \), defined by the formula,
\[
\hat{\varphi}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(s) e^{-iks} ds,
\]
(111)
so that

\[ \varphi(t) = \sum_{k=-\infty}^{\infty} \hat{\varphi}_k e^{ikt}, \]  

(112)

for all \( t \in [-\pi, \pi] \).

**Definition 3.3.** The Hilbert transform is the mapping \( H : L^2[-\pi, \pi] \to L^2[-\pi, \pi] \), given by the formula

\[ H(\varphi)(s) = \sum_{k=-\infty}^{\infty} -i \text{sgn}(k) \hat{\varphi}_k e^{iks}, \]  

(113)

with \( \varphi \in L^2[-\pi, \pi] \) an arbitrary function. The function \( H(\varphi) : [-\pi, \pi] \to \mathbb{C} \) is often referred to as the conjugate function of \( \varphi \).

The following theorem summarizes several well-known properties of the Hilbert transform (see, for example, [9]).

**Theorem 3.3.**

(a) The mapping \( H : L^2[-\pi, \pi] \to L^2[-\pi, \pi] \) is bounded.

(b) For any integrable \( \varphi \), the identity

\[ H(\varphi)(s) = \text{p.v.} \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t) \frac{1}{\tan((s-t)/2)} \, dt, \]  

(114)

holds almost everywhere.

(c) For any function \( \varphi \in c^1[-\pi, \pi] \),

\[ H(\varphi')(s) = \left( (H(\varphi))' \right)(s) = \sum_{k=-\infty}^{\infty} |k| \hat{\varphi}_k e^{iks}. \]  

(115)

In other words,

\[ HD = DH, \]  

(116)

where \( D = \frac{d}{ds} \) is the differentiation operator.

### 3.4. Boundary integral operators

In this section, we define the boundary integral operators \( K_{y,0}^{1,0}, K_{y,0}^{2,0}, K_{y,1}^{3,0}, K_{y,1}^{0,1}, K_{y,2}^{1,1}, K_{y,2}^{1,2}, K_{y,3}^{1,2}, K_{y,3}^{0,3} \), that are closely related to the operators (12)–(29) defined in Section 2.

**Definition 3.4.** Suppose that the function \( \sigma : [0, L] \to \mathbb{R} \) is sufficiently smooth. Then we denote by \( K_{y,0}^{1,0}, K_{y,0}^{0,1} : c[0, L] \to c[0, L] \) and \( K_{y,1}^{2,0}, K_{y,1}^{3,0}, K_{y,1}^{1,1}, K_{y,1}^{2,1}, K_{y,2}^{0,2}, K_{y,2}^{1,2}, K_{y,3}^{1,2}, K_{y,3}^{0,3} : c^2[0, L] \to c[0, L] \) the operators defined by the formulae...
\[ K_{1,0}^0(\sigma)(s) = \int_0^L \frac{\partial \Phi_{y(t)}(y(s))}{\partial N(t)} \sigma(t) dt, \]  
(117)

\[ K_{2,0}^0(\sigma)(s) = \text{f.p.} \int_0^L \frac{\partial^2 \Phi_{y(t)}(y(s))}{\partial N(t)^2} \sigma(t) dt, \]  
(118)

\[ K_{3,0}^0(\sigma)(s) = \text{f.p.} \int_0^L \frac{\partial^3 \Phi_{y(t)}(y(s))}{\partial N(t)^3} \sigma(t) dt, \]  
(119)

\[ K_{0,1}^1(\sigma)(s) = \int_0^L \frac{\partial \Phi_{y(t)}(y(s))}{\partial N(s)} \sigma(t) dt, \]  
(120)

\[ K_{1,1}^1(\sigma)(s) = \text{f.p.} \int_0^L \frac{\partial^2 \Phi_{y(t)}(y(s))}{\partial N(s) \partial N(t)} \sigma(t) dt, \]  
(121)

\[ K_{2,1}^1(\sigma)(s) = \text{f.p.} \int_0^L \frac{\partial^3 \Phi_{y(t)}(y(s))}{\partial N(s) \partial N(t)^2} \sigma(t) dt, \]  
(122)

\[ K_{0,2}^2(\sigma)(s) = \text{f.p.} \int_0^L \frac{\partial^2 \Phi_{y(t)}(y(s))}{\partial N(s)^2} \sigma(t) dt, \]  
(123)

\[ K_{1,2}^2(\sigma)(s) = \text{f.p.} \int_0^L \frac{\partial^3 \Phi_{y(t)}(y(s))}{\partial N(s)^2 \partial N(t)} \sigma(t) dt, \]  
(124)

\[ K_{0,3}^3(\sigma)(s) = \text{f.p.} \int_0^L \frac{\partial^3 \Phi_{y(t)}(y(s))}{\partial N(s)^3} \sigma(t) dt, \]  
(125)

respectively.

**Remark.** Obviously, the operators \( K_{1,0}^0, K_{2,0}^0, K_{3,0}^0, K_{1,2}^1 \) given by the formulae (120), (123)–(125) are the adjoints of the operators \( K_{1,0}^0, K_{2,0}^0, K_{3,0}^0, K_{2,1}^1 \) defined by (117)–(119), (122). Furthermore, \( K_{1,1}^1 \), defined by (121), is self-adjoint.

### 4. Proof of results

In this section we prove some of the results in Section 2. The outline of this section is as follows: First, we consider the case when \( y \) is a circle. We start with proving Theorem 2.1 in this special case, and follow with Lemma 4.1, providing explicit formulae for the boundary integral operators (117)–(125), also
on the circle. Then, combining Theorem 2.1 with Lemma 4.1, we obtain the so-called jump conditions for the operators (12)–(29) on the circle, summarized in Theorem 4.1.

Next, we consider the case when $\gamma$ is an arbitrary sufficiently smooth Jordan curve. Since the proofs of all of the identities (75)–(80) in Theorem 2.3 are similar to each other, we only prove (75) and (76). Specifically, the identities (75) and (76) in Theorem 2.3 follow immediately by combining Theorem 4.2 and Lemma 4.4 below.

Proof of Theorem 2.1. Since the proofs for the identities (31)–(49) are nearly identical, we only provide the proof for the interior limit of the quadruple layer potential (34). Further, it is sufficient to prove (34) for the case $r = 1$; the general case follows by a simple transformation of variables.

We choose the parametrization
$$\gamma(t) = (\cos(t), \sin(t)), \quad (126)$$
where $t \in [-\pi, \pi]$. It immediately follows from (126) that
$$\int_{-\pi}^{\pi} \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s) - h \cdot N(s))}{\partial N(t)^2} e^{ikt} dt$$
$$= \int_{-\pi}^{\pi} \frac{1 - 2 \cdot (1 - h) \cdot \cos(t - s) + (1 - h)^2 \cdot \cos(2(t - s))}{(1 + (1 - h)^2 - 2 \cdot (1 - h) \cdot \cos(t - s))^2} e^{ikt} dt$$
$$= e^{iks} \cdot \int_{-\pi}^{\pi} \frac{1 - 2 \cdot (1 - h) \cdot \cos(t) + (1 - h)^2 \cdot \cos(2t)}{(1 + (1 - h)^2 - 2 \cdot (1 - h) \cdot \cos(t))^2} e^{ikt} dt, \quad (127)$$
for any $s \in [-\pi, \pi]$. We will use calculus of residues to evaluate the integral (127). To this effect, the substitution
$$z = e^{it}, \quad (128)$$
and simple algebraic manipulation convert the right-hand side of (127) into the integral
$$e^{iks} \cdot \int_{\{z \mid |z| = 1\}} \frac{1}{2} \left( -\frac{i z^{k+1}}{(1 - h) - z^2} - \frac{i z^{k-1}}{(z(1 - h) - 1)^2} \right) dz. \quad (129)$$
Now, formula (34) for $r = 1$ follows by applying a standard residue calculation to (129). \qed

Remark. Formulae (31)–(33), (38)–(39) follow immediately from well-known results (see, for example, [3,12]). While the derivation of (34)–(37), (40)–(49) is quite similar, the authors failed to find them in the literature.

The operators $K^{1,0}_\gamma, K^{2,0}_\gamma, K^{3,0}_\gamma, K^{1,1}_\gamma, K^{2,1}_\gamma, K^{0,1}_\gamma, K^{0,2}_\gamma, K^{0,3}_\gamma, K^{1,2}_\gamma$ defined by (117)–(125), assume a particularly simple form on the circle. The following lemma follows immediately from an elementary computation.

Lemma 4.1. Suppose that $\gamma$ is a circle of radius $r$ parametrized by its arclength, with exterior unit normal denoted by $N$. Then, for any sufficiently smooth function $\sigma : [-\pi r, \pi r] \to \mathbb{C}$,
\[
(a) \quad K^{1,0}_y (\sigma)(s) = \int_{-\pi r}^{\pi r} -\frac{\sigma(t)}{2r} \, dt = -\pi \hat{\sigma}_0, \quad (130)
\]

\[
(b) \quad K^{2,0}_y (\sigma)(s) = \text{f.p.} \int_{-\pi r}^{\pi r} \left( \frac{1}{2r^2} + \frac{1}{2r^2 \cos((t-s)/r) - 2r^2} \right) \sigma(t) \, d t
\]
\[
= \pi r^{-1} \hat{\sigma}_0 + \pi H(\sigma')(s), \quad (131)
\]

\[
(c) \quad K^{3,0}_y (\sigma)(s) = \text{f.p.} \int_{-\pi r}^{\pi r} \left( -\frac{1}{r^3} - \frac{3}{2r^3 \cos((t-s)/r) - 2r^3} \right) \sigma(t) \, d t
\]
\[
= -2\pi r^{-2} \hat{\sigma}_0 - 3\pi r^{-1} H(\sigma')(s), \quad (132)
\]

\[
(d) \quad K^{0,1}_y (\sigma)(s) = \int_{-\pi r}^{\pi r} -\frac{\sigma(t)}{2r} \, dt = -\pi \hat{\sigma}_0, \quad (133)
\]

\[
(e) \quad K^{1,1}_y (\sigma)(s) = \text{f.p.} \int_{-\pi r}^{\pi r} \frac{\sigma(t)}{2r^2 - 2r^2 \cos((t-s)/r)} \, d t = -\pi H(\sigma')(s), \quad (134)
\]

\[
(f) \quad K^{2,1}_y (\sigma)(s) = \text{f.p.} \int_{-\pi r}^{\pi r} \frac{\sigma(t)}{2r^3 \cos((t-s)/r) - 2r^3} \, d t = \pi r^{-1} H(\sigma')(s), \quad (135)
\]

\[
(g) \quad K^{0,2}_y (\sigma)(s) = \text{f.p.} \int_{-\pi r}^{\pi r} \left( \frac{1}{2r^2} + \frac{1}{2r^2 \cos((t-s)/r) - 2r^2} \right) \sigma(t) \, d t
\]
\[
= \pi r^{-1} \hat{\sigma}_0 + \pi H(\sigma')(s), \quad (136)
\]

\[
(h) \quad K^{1,2}_y (\sigma)(s) = \text{f.p.} \int_{-\pi r}^{\pi r} \frac{\sigma(t)}{2r^3 \cos((t-s)/r) - 2r^3} \, d t = \pi r^{-1} H(\sigma')(s), \quad (137)
\]

\[
(i) \quad K^{0,3}_y (\sigma)(s) = \text{f.p.} \int_{-\pi r}^{\pi r} \left( -\frac{1}{r^3} - \frac{3}{2r^3 \cos((t-s)/r) - 2r^3} \right) \sigma(t) \, d t
\]
\[
= -2\pi r^{-2} \hat{\sigma}_0 - 3\pi r^{-1} H(\sigma')(s), \quad (138)
\]

where \( H \) denotes the Hilbert transform (see (113) in Section 3.3).
Lemma 4.2. Suppose that $\gamma : [0, L] \rightarrow \mathbb{R}^3$ is a sufficiently smooth Jordan curve parametrized by its arclength, with the exterior unit normal and the unit tangent vectors at $\gamma(s)$ denoted by $N(s)$ and $T(s)$, respectively. Then, there exist a positive real number $a$ (dependent on $\gamma$), and two continuously differentiable functions $f, g : (-a, a) \rightarrow \mathbb{R}$ (dependent on $\gamma$), such that for any $s \in [0, L]$,\

$$\gamma(s + t) - \gamma(s) = \left( t + t^3 \cdot f(t) \right) \cdot T(s) - \left( \frac{ct^2}{2} + t^3 \cdot g(t) \right) \cdot N(s),$$  \hspace{1cm} (156)\

for all $t \in (-a, a)$, where the coefficient $c$ in (156) is the curvature of $\gamma$ at the point $\gamma(s)$. Furthermore, for all $t \in (-a, a)$,

$$\|f(t)\| \leq \|\gamma''(t)\|,$$  \hspace{1cm} (157)

$$\|g(t)\| \leq \|\gamma''(t)\|.$$  \hspace{1cm} (158)

We now proceed to the case where $\gamma$ is an arbitrary sufficiently smooth Jordan curve. The following obvious lemma can be found in most elementary textbooks on differential geometry (see, for example, [4]).

Lemma 4.2. Suppose that $\gamma : [0, L] \rightarrow \mathbb{R}^3$ is a sufficiently smooth Jordan curve parametrized by its arclength, with the exterior unit normal and the unit tangent vectors at $\gamma(s)$ denoted by $N(s)$ and $T(s)$, respectively. Then, there exist a positive real number $a$ (dependent on $\gamma$), and two continuously differentiable functions $f, g : (-a, a) \rightarrow \mathbb{R}$ (dependent on $\gamma$), such that for any $s \in [0, L]$,\

$$\gamma(s + t) - \gamma(s) = \left( t + t^3 \cdot f(t) \right) \cdot T(s) - \left( \frac{ct^2}{2} + t^3 \cdot g(t) \right) \cdot N(s),$$  \hspace{1cm} (156)\

for all $t \in (-a, a)$, where the coefficient $c$ in (156) is the curvature of $\gamma$ at the point $\gamma(s)$. Furthermore, for all $t \in (-a, a)$,

$$\|f(t)\| \leq \|\gamma''(t)\|,$$  \hspace{1cm} (157)

$$\|g(t)\| \leq \|\gamma''(t)\|.$$  \hspace{1cm} (158)

In the local parametrization (156), the potential of a quadrupole located at $\gamma(s)$ and oriented in the direction $N(s)$ assumes a particularly simple form, given by the following lemma.
Lemma 4.3. Suppose that \( \gamma : [0, L] \to \mathbb{R}^2 \) is a sufficiently smooth Jordan curve parametrized by its arclength. Then, there exist real positive numbers \( A, a \) and \( h_0 \) such that for any \( s \in [0, L] \)

\[
\left| \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s) - h \cdot N(s))}{\partial N(s + t)^2} - \frac{h^2 - t^2}{(h^2 + t^2)^2} \cdot \frac{cht}{(h^2 + t^2)^3} \right| \leq A,
\]

(159)

for all \( t \in (-a, a) \), \( 0 \leq h < h_0 \), where the coefficient \( c \) in (159) is the curvature of \( \gamma \) at the point \( \gamma(s) \).

Proof. Without loss of generality, it is sufficient to prove the lemma for the case where \( s = 0 \), \( \gamma(0) = 0 \), and \( \gamma'(0) = (1, 0) \). Substituting (156) into (9) and evaluating the result at \( x = (0, h) \), we obtain

\[
\frac{\partial^2 \Phi_{\gamma(t)}(x)}{\partial N(t)^2} = \frac{p_0(h, t)}{(h^2 + t^2 + r(h, t))^2},
\]

(160)

where \( p_0, r : \mathbb{R}^2 \to \mathbb{R} \) are functions given by the formulae

\[
p_0(h, t) = \left[ h - t + cht + \frac{ct^3}{2} \right. 
= \left. \frac{cht}{2} \left( f(t) + g(t) \right) + \frac{cht^3}{2} \left( f'(t) + g'(t) \right) - t^4 \left( f''(t) - f'(t) \right) \right]
- \frac{cht^5}{2} \left( f''(t) - f'(t) \right) + t^6 \left( f''(t) - f'(t) \right)
+ \left[ h + t - cht + \frac{ct^3}{2} \right. 
= \left. \frac{cht}{2} \left( f(t) + g(t) \right) + \frac{cht^3}{2} \left( f'(t) + g'(t) \right) - t^4 \left( f''(t) - f'(t) \right) \right]
- \frac{cht^5}{2} \left( f''(t) - f'(t) \right) + t^6 \left( f''(t) - f'(t) \right)
+ \left[ h^2 + r(h, t) + \frac{ct^4}{4} \right. 
= \left. \frac{cht}{2} \left( f(t) + g(t) \right) + \frac{cht^3}{2} \left( f'(t) + g'(t) \right)
+ \left( h^2 + r(h, t) + \frac{ct^4}{4} \right) \right].
\]

(161)

(162)

We also introduce the notation

\[
p_1(h, t) = (h^2 + t^2 + r(h, t))^2 - (h^2 + r(h, t))^2 = 2(h^2 + r(h, t))^2.
\]

(163)

Utilizing the fact that the functions \( f, g, f', g' \) are bounded for sufficiently small \( t \) (see Lemma 4.2 above) and the trivial inequality that for any \( m + n \geq 2k \),

\[
h^m \cdot t^n \leq (h^2 + t^2)^k,
\]

(164)

for sufficiently small \( h \) and \( t \), we observe that there exist positive real numbers \( a, h_0, \) and \( C \) (dependent on \( \gamma \)) such that

\[
| p_0(h, t) - h^2 + t^2 - 3cht^2 | \leq C(h^2 + t^2)^2,
\]

(165)

\[
| p_0(h, t) \cdot p_1(h, t) - 2cht^2(h^2 + t^2)(h^2 - t^2) | \leq C(h^2 + t^2)^4,
\]

(166)

\[
| p_0(h, t) \cdot p_1(h, t)^2 | \leq C(h^2 + t^2)^6,
\]

(167)
where the convergence of the series follows from (168). Combining (165)–(167), we obtain

**Lemma 4.4.** Suppose that $\sigma$ is a circle. The following lemma shows that the operator $K_0$ is a compact operator, and that

$$\int_0^L \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s))}{\partial N(t)^2} \sigma(t) \, dt = f \cdot p_1(h, t) < 1,$$

(168)

for all $h < h_0$, $t \in (-a, a)$. Substituting (163) into (160), we have

$$\frac{\partial^2 \Phi_{\gamma(t)}(x)}{\partial N(t)^2} = \frac{p_0(h, t)}{(h^2 + t^2)^2 (1 + \frac{p_0(h, t)}{(h^2 + t^2)^2})} = \frac{p_0(h, t)}{(h^2 + t^2)^2} \sum_{k=0}^{\infty} (-1)^k \frac{p_1(h, t)^k}{(h^2 + t^2)^{2k}},$$

(169)

where the convergence of the series follows from (168). Combining (165)–(167), we obtain

$$\frac{\partial^2 \Phi_{\gamma(t)}(x)}{\partial N(t)^2} \leq \frac{p_0(h, t)}{(h^2 + t^2)^2} \frac{h^2 - t^2}{(h^2 + t^2)^2} + \frac{cht^2 (5h^2 + t^2)}{(h^2 + t^2)^3} + \sum_{k=0}^{\infty} \frac{p_0(h, t) \cdot p_1(h, t)^k}{(h^2 + t^2)^{2k+2}},$$

(170)

with $\alpha$ defined by the formula

$$\alpha = \sup_{h < h_0, t \in (-a, a)} \left| \frac{p_1(h, t)}{(h^2 + t^2)^2} \right|.$$

(171)

Now, introducing the notation

$$A = 2C + C \cdot \frac{\alpha^2}{1 - \alpha},$$

(172)

we obtain (159). □

**Lemma 4.4.** Suppose that $\gamma : [0, L] \rightarrow \mathbb{R}^2$ is a sufficiently smooth Jordan curve parametrized by its arclength, and that $\eta : [0, L] \rightarrow \mathbb{R}^2$ denotes the circle of radius $\frac{L}{2\pi}$, also parametrized by its arclength. In addition, suppose that $\sigma : [0, L] \rightarrow \mathbb{R}$ is a twice continuously differentiable function. Then,

$$\text{f.p.} \int_0^L \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s))}{\partial N(t)^2} \sigma(t) \, dt = f \cdot p_1(h, t) \sigma(t) \, dt + M_2(\sigma)(t),$$

(173)

where $M_2 : c[0, L] \rightarrow c[0, L]$ is a compact operator defined by the formula

$$M_2(\sigma)(s) = \int_0^L \left( \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s))}{\partial N(t)^2} - \frac{\partial^2 \Phi_{\gamma(t)}(\eta(s))}{\partial N(t)^2} \right) \sigma(t) \, dt.$$

(174)
Furthermore, for any \( t \neq s \),
\[
    m_2(s, t) = \frac{2(N(t), \gamma(s) - \gamma(t))^2}{\|\gamma(s) - \gamma(t)\|^4} - \frac{1}{2} \left( \frac{2\pi}{L} \right)^2 \frac{\|\gamma(s) - \gamma(t)\|^2 - 2 \left( \frac{L}{2\pi} \right)^2 (1 - \cos \left( \frac{2\pi}{L} (s - t) \right))}{\|\gamma(s) - \gamma(t)\|^2 \left( 1 - \cos \left( \frac{2\pi}{L} (s - t) \right) \right)}.
\]
(175)

and for \( t = s \),
\[
    m_2(s, s) = \frac{5}{12} (c(s))^2 - \frac{5}{12} \left( \frac{2\pi}{L} \right)^2,
\]
(176)
where \( c(s) \) is the curvature of \( \gamma \) at the point \( \gamma(s) \), and \( m_2 : [0, L] \times [0, L] \to \mathbb{R} \) is the kernel of the operator \( M_2 \).

The following theorem provides the so-called jump conditions for the operators (14) and (15) on the boundary \( \Gamma \), when \( \Gamma \) is sufficiently smooth.

**Theorem 4.2.** Suppose that \( \gamma : [0, L] \to \mathbb{R}^2 \) is a sufficiently smooth Jordan curve parametrized by its arclength. Then, for any sufficiently smooth function \( \sigma : [0, L] \to \mathbb{R} \),
\[
    K_{y, c}^{2, 0}(\sigma) - K_{y, i}^{2, 0}(\sigma) = \lim_{h \to 0} \int_0^L \left( \frac{\partial^2 \Phi_y(t)(\gamma(s) + h \cdot N(s))}{\partial N(t)^2} - \frac{\partial^2 \Phi_y(t)(\gamma(s) - h \cdot N(s))}{\partial N(t)^2} \right) \sigma(t) \, dt
\]
\[
    = -2\pi c(s) \sigma(s),
\]
(177)

and
\[
    K_{y, c}^{2, 0}(\sigma) + K_{y, i}^{2, 0}(\sigma) = \lim_{h \to 0} \int_0^L \left( \frac{\partial^2 \Phi_y(t)(\gamma(s) + h \cdot N(s))}{\partial N(t)^2} + \frac{\partial^2 \Phi_y(t)(\gamma(s) - h \cdot N(s))}{\partial N(t)^2} \right) \sigma(t) \, dt
\]
\[
    = 2 \cdot \text{f.p.} \int_0^L \frac{\partial^2 \Phi_y(t)(\gamma(s))}{\partial N(t)^2} \sigma(t) \, dt,
\]
(178)

where \( c(s) \) denotes the curvature of \( \gamma \) at \( \gamma(s) \). In other words, the quadruple layer potential with density \( \sigma \) (see (6)), can be continuously extended from \( \Omega \) to \( \overline{\Omega} \) and from \( \mathbb{R}^2 \setminus \overline{\Omega} \) to \( \mathbb{R}^2 \setminus \Omega \), with the limiting values given by the formulae
\[
    p_{y, c}^{2, 0}(\sigma)(s) = K_{y, c}^{2, 0}(\sigma)(s) = \pi c(s) \sigma(s) + \text{f.p.} \int_0^L \frac{\partial^2 \Phi_y(t)(\gamma(s))}{\partial N(t)^2} \sigma(t) \, dt,
\]
(179)
\[
    p_{y, c}^{2, 0}(\sigma)(s) = K_{y, c}^{2, 0}(\sigma)(s) = -\pi c(s) \sigma(s) + \text{f.p.} \int_0^L \frac{\partial^2 \Phi_y(t)(\gamma(s))}{\partial N(t)^2} \sigma(t) \, dt.
\]
(180)
Proof. Without loss of generality, we can assume that \( s \neq 0 \) and \( s \neq L \). We begin by proving (178). Suppose that \( \eta: [0, L] \rightarrow \mathbb{R}^2 \) is the circle of radius \( \frac{L}{\pi} \) parametrized by its arclength. We define the functions \( \Sigma^h_\gamma, \Sigma^h_\eta: [0, L] \times [0, L] \rightarrow \mathbb{R} \) via the formulae

\[
\Sigma^h_\gamma(s, t) = \frac{\partial^2 \Phi_\gamma(t)(\gamma(s) + h \cdot N(s))}{\partial N(t)^2} + \frac{\partial^2 \Phi_\gamma(t)(\gamma(s) - h \cdot N(s))}{\partial N(t)^2}, \tag{181}
\]

\[
\Sigma^h_\eta(s, t) = \frac{\partial^2 \Phi_\eta(t)(\eta(s) + h \cdot N(s))}{\partial N(t)^2} + \frac{\partial^2 \Phi_\eta(t)(\eta(s) - h \cdot N(s))}{\partial N(t)^2}, \tag{182}
\]

and, substituting (181), (182) into (178), obtain the identity

\[
K_{\gamma, e}^2 (\sigma)(s) + K_{\eta, e}^2 (\sigma)(s) = \lim_{h \to 0} \int_0^L \Sigma^h_\gamma(s, t) \sigma(t) \, dt + \lim_{h \to 0} \int_0^L (\Sigma^h_\gamma(s, t) - \Sigma^h_\eta(s, t)) \sigma(t) \, dt. \tag{183}
\]

Substituting (141), (142) in Theorem 4.1 into (183), we have

\[
K_{\gamma, e}^2 (\sigma)(s) + K_{\eta, e}^2 (\sigma)(s) = 2 \cdot f.p. \int_0^L \frac{\partial^2 \Phi_{\eta(t)}(\eta(s))}{\partial N(t)^2} \sigma(t) \, dt + \lim_{h \to 0} \int_0^L (\Sigma^h_\gamma(s, t) - \Sigma^h_\eta(s, t)) \sigma(t) \, dt. \tag{184}
\]

Due to Lemma 4.3, there exist positive real constants \( C_0, a, h_0 \) such that for any \( s \in [0, L] \)

\[
| \Sigma^h_\gamma(s, t) - \Sigma^h_\eta(s, t) | \leq C_0, \tag{185}
\]

for all \( |t - s| < a, 0 \leq h < h_0 \). For any \( t \neq s \) and sufficiently small \( h \), both \( \Sigma^h_\gamma(s, t) \) and \( \Sigma^h_\eta(s, t) \) are \( C^\infty \)-functions. Therefore, there also exist positive real constants \( h_1, C_1 \) such that for any \( s \in [0, L] \)

\[
| \Sigma^h_\gamma(s, t) - \Sigma^h_\eta(s, t) | \leq C_1, \tag{186}
\]

for all \( |t - s| > a, 0 \leq h < h_1 \). Now, applying Lebesgue’s dominated convergence theorem (see, for example, [19]) to the second integral of the right-hand side of (184), we obtain

\[
\lim_{h \to 0} \int_0^L (\Sigma^h_\gamma(s, t) - \Sigma^h_\eta(s, t)) \sigma(t) \, dt = \int_0^L \lim_{h \to 0} (\Sigma^h_\gamma(s, t) - \Sigma^h_\eta(s, t)) \sigma(t) \, dt \tag{187}
\]

Finally, formula (178) immediately follows from the combination of (184), (187) with (173), (174) in Lemma 4.4.

We now turn our attention to the proof of (177). We define the functions \( \Delta^h_\gamma, \Delta^h_\eta: [0, L] \times [0, L] \rightarrow \mathbb{R} \) via the formulae

\[
\Delta^h_\gamma(s, t) = \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s) + h \cdot N(s))}{\partial N(t)^2} - \frac{\partial^2 \Phi_{\gamma(t)}(\gamma(s) - h \cdot N(s))}{\partial N(t)^2}, \tag{188}
\]

\[
\Delta^h_\eta(s, t) = \frac{\partial^2 \Phi_{\eta(t)}(\eta(s) + h \cdot N(s))}{\partial N(t)^2} - \frac{\partial^2 \Phi_{\eta(t)}(\eta(s) - h \cdot N(s))}{\partial N(t)^2}. \tag{189}
\]
and, by substituting (188), (189) into (177), obtain the identity

\[
K_{2,0}^2(\gamma) - K_{2,1}^2(\gamma)(s) = \frac{c(s)L}{2\pi} \cdot \lim_{h \to 0} \int_0^L \Delta_h^b(s, t)\sigma(t) dt
\]

\[
+ \lim_{h \to 0} \left( \Delta_h^b(s, t) - \frac{c(s)L}{2\pi} \cdot \Delta_h^b(s, t) \right)\sigma(t) dt.
\]  (190)

Substituting (141), (142) in Theorem 4.1 into (190), we get

\[
K_{2,0}^2(\gamma) - K_{2,1}^2(\gamma)(s) = -2\pi c(s)\sigma(s) + \lim_{h \to 0} \int_0^L \left( \Delta_h^b(s, t) - \frac{c(s)L}{2\pi} \cdot \Delta_h^b(s, t) \right)\sigma(t) dt.
\]  (191)

Due to Lemma 4.3, there exist positive real constants \(C_0, a, h_0\) such that for any \(s \in [0, L]\)

\[
\left| \Delta_h^b(s, t) - \frac{c(s)L}{2\pi} \cdot \Delta_h^b(s, t) \right| \leq C_0,
\]  (192)

for all \(|t - s| < a, 0 \leq h < h_0\). For any \(t \neq s\) and sufficiently small \(h\), both \(\Delta_h^b(s, t)\) and \(\Delta_h^b(s, t)\) are \(c^\infty\)-functions. Therefore, there also exist positive real constants \(h_1, C_1\) such that for any \(s \in [0, L]\)

\[
\left| \Delta_h^b(s, t) - \frac{c(s)L}{2\pi} \cdot \Delta_h^b(s, t) \right| \leq C_1,
\]  (193)

for all \(|t - s| > a, 0 \leq h < h_1\). Applying Lebesgue’s dominated convergence theorem (see, for example, [19]) to the second integral of the right-hand side of (191), we have

\[
\lim_{h \to 0} \int_0^L \left( \Delta_h^b(s, t) - \frac{c(s)L}{2\pi} \cdot \Delta_h^b(s, t) \right)\sigma(t) dt = \int_0^L \lim_{h \to 0} \left( \Delta_h^b(s, t) - \frac{c(s)L}{2\pi} \cdot \Delta_h^b(s, t) \right)\sigma(t) dt.
\]  (194)

Examining (188), (189), we obviously have

\[
\lim_{h \to 0} \left( \Delta_h^b(s, t) - \frac{c(s)L}{2\pi} \cdot \Delta_h^b(s, t) \right) = 0.
\]  (195)

Therefore, the integral on the right-hand side of (194) is zero, from which (177) follows immediately. \(\Box\)

5. Generalizations

We have presented explicit (modulo an integral operator with a smooth kernel) formulae for integro-pseudo-differential operators of potential theory in two dimensions (up to order 2). The work presented here admits several obvious extensions.
(a) Formulae (70)–(88) have their counterparts for elliptic PDEs other than the Laplace equation. Indeed, for any elliptic PDE in two dimensions, the Green’s function has the form

\[ G(x, y) = \phi(x, y) \cdot \log(\|x - y\|) + \psi(x, y), \tag{196} \]

with \( \phi, \psi \) a pair of smooth functions; derivations of Section 4 are almost unchanged when \( \log(\|x - y\|) \) is replaced with (196). In particular, the counterparts of the formulae (70)–(80) for the Helmholtz equation (with either real or complex Helmholtz coefficient) are identical to (70)–(80); the counterparts of the formulae (81)–(88) for the Helmholtz equation do not coincide with (81)–(88) exactly; instead, they assume the form

\[ (a) \quad K_{3,0}^{0.3,0}(\sigma)(s) = -2\pi (c(s))^2 \sigma(s) + 4\pi k^2 \sigma(s) + \pi \sigma''(s) - 2\pi c'(s)H(\sigma)(s) \]

\[ - 3\pi c(s)H(\sigma')(s) + N_3(\sigma)(s), \tag{197} \]

\[ K_{3,0}^{0.3,0}(\sigma)(s) = 2\pi (c(s))^2 \sigma(s) - 4\pi k^2 \sigma(s) - \pi \sigma''(s) - 2\pi c'(s)H(\sigma)(s) \]

\[ - 3\pi c(s)H(\sigma')(s) + N_3(\sigma)(s), \tag{198} \]

\[ (b) \quad K_{2,1}^{2.1,0}(\sigma)(s) = -4\pi k^2 \sigma(s) - \pi \sigma''(s) + \pi c'(s)H(\sigma)(s) + \pi c(s)H(\sigma')(s) \]

\[ + G_3(\sigma)(s), \tag{199} \]

\[ K_{2,1}^{2.1,0}(\sigma)(s) = 4\pi k^2 \sigma(s) + \pi \sigma''(s) + \pi c'(s)H(\sigma)(s) + \pi c(s)H(\sigma')(s) \]

\[ + G_3(\sigma)(s), \tag{200} \]

\[ (c) \quad K_{1,2}^{1.2,0}(\sigma)(s) = 4\pi k^2 \sigma(s) + \pi \sigma''(s) + \pi c(s)H(\sigma')(s) + \tilde{G}_3(\sigma)(s), \tag{201} \]

\[ K_{1,2}^{1.2,0}(\sigma)(s) = -4\pi k^2 \sigma(s) - \pi \sigma''(s) + \pi c(s)H(\sigma')(s) + \tilde{G}_3(\sigma)(s), \tag{202} \]

\[ (d) \quad K_{1,1}^{0.3,0}(\sigma)(s) = 2\pi (c(s))^2 \sigma(s) - 4\pi k^2 \sigma(s) - \pi \sigma''(s) - \pi c'(s)H(\sigma)(s) \]

\[ - 3\pi c(s)H(\sigma')(s) + \tilde{N}_3(\sigma)(s), \tag{203} \]

\[ K_{1,1}^{0.3,0}(\sigma)(s) = -2\pi (c(s))^2 \sigma(s) + 4\pi k^2 \sigma(s) + \pi \sigma''(s) - \pi c'(s)H(\sigma)(s) \]

\[ - 3\pi c(s)H(\sigma')(s) + \tilde{N}_3(\sigma)(s), \tag{204} \]

where \( k \in \mathbb{C} \) is the Helmholtz coefficient, and the operators \( N_3, G_3, \tilde{N}_3, \tilde{G}_3 : L^2[0, L] \to L^2[0, L] \) are compact.

(b) The derivation of the three-dimensional counterparts of formulae (70)–(88) is completely straightforward; such expressions have been obtained, and the paper reporting them is in preparation.

(c) In certain areas of mathematical physics, one encounters pseudo-differential equations whose analysis is outside the scope of this paper. An important example is the Stratton–Chu equations, to which Maxwell’s equations are frequently reduced in computational electromagnetics. Another source of such problems is the scattering of elastic waves in solids. Problems of this type are currently under investigation.

References


