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Abstract—**We propose relaying strategies for uncoded two-way relay channels, where two terminals transmit simultaneously to each other with the help of a relay. In particular, we consider a memoryless system, where the signal transmitted by the relay is obtained by applying an instantaneous relay function to the previously received signal. For binary antipodal signaling, a class of so called absolute (abs)-based schemes is proposed in which the processing at the relay is solely based on the absolute value of the received signal. We analyze and optimize the symbol-error performance of existing and new abs-based and non-abs-based strategies under an average power constraint, including abs-based and non-abs-based versions of amplify and forward (AF), detect and forward (DF), and estimate and forward (EF). Additionally, we optimize the relay function via functional analysis such that the average probability of error is minimized at the high signal-to-noise ratio (SNR) regime. The optimized relay function is shown to be a Lambert W function parameterized on the noise power and the transmission energy. The optimized function behaves like abs-AF at low SNR and like abs-DF at high SNR, respectively; EF behaves similarly to the optimized function over the whole SNR range. We find the conditions under which each class of strategies is preferred. Finally, we show that all these results can also be generalized to higher order constellations.**

Index Terms—**Two-way channel, wireless relay networks, functional analysis.**

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absolute value

fi fi

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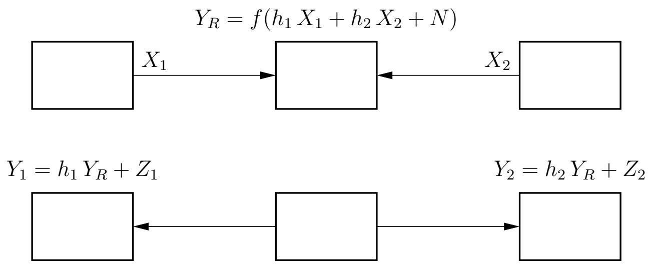


fi

Notation $p_X(x)$ X $\mathcal{G}(x, \sigma^2) \triangleq \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$ X 0 σ^2 $Q(\cdot)$

X_i

i $i = 1, 2$ Y_i Y_R

$Y_R =$

$f(h_1\, X_1 + h_2\, X_2 + N)$ $f(\cdot)$ h_1 h_2

$h_1 \geq h_2 \geq 0$ N

σ_r^2 Y_R Z_i i $\sigma_{s_i}^2$

$$Y_i = h_i\, f(h_1\, X_1 + h_2\, X_2 + N) + Z_i, \quad i = 1,\, 2.$$

fi $U = h_1\, X_1 + h_2\, X_2 + N$

Y_i *a priori* X_i X_i $E\{|X_i|^2\} \leq P_s, \quad i = 1,\, 2$ $E\{|f(h_1 X_1 + h_2 X_2 + N)|^2\} \leq P_r$

$\sigma_{s_1}^2 = \sigma_{s_2}^2 = \sigma_s^2$

h_1 h_2

fi h_1 h_2

$X_i = \pm\sqrt{P_s}$

$|U|$

U

fi

- $x_i = \sqrt{P_s}$ i $\sqrt{P_s}$ $y_i \geq v_i$ $-\sqrt{P_s}$ v_i y_i Y_i

$$P_e^{(1)} = \frac{1}{2} + \frac{1}{2}\int_0^{w} A(u)du \underbrace{\left[\int_{-\infty}^{v_1} \mathcal{G}\left(y - h_1 b, \sigma_s^2\right) dy\right]}_{C(v_1,b)} + \frac{1}{2}\int_w^{+\infty} A(u)du \underbrace{\left[\int_{-\infty}^{v_1} \mathcal{G}\left(y - h_1 a, \sigma_s^2\right) dy\right]}_{D(v_1,a)}$$
$$+ \frac{1}{2}\int_0^{w} B(u)du \underbrace{\left[\int_{-\infty}^{v_1} \mathcal{G}\left(y + h_1 b, \sigma_s^2\right) dy\right]}_{E(v_1,b)} + \frac{1}{2}\int_w^{+\infty} B(u)du \underbrace{\left[\int_{-\infty}^{v_1} \mathcal{G}\left(y + h_1 a, \sigma_s^2\right) dy\right]}_{F(v_1,a)}.$$

$x_i = -\sqrt{P_s}$ i
$\sqrt{P_s}$ $y_i \geq -v_i$ $-\sqrt{P_s}$

- $X_1 = \sqrt{P_s}$ $X_2 = \sqrt{P_s}$ $X_1 = -\sqrt{P_s}$ $X_2 = -\sqrt{P_s}$

v_i

v_i $X_1 = \sqrt{P_s}$
$X_2 = -\sqrt{P_s}$ $X_1 = -\sqrt{P_s}$ $X_2 = \sqrt{P_s}$

Theorem 1: $\sqrt{P_s}$ $-\sqrt{P_s}$

$f(U)$ $f(|U|)$ f
U $|U|$

A. Non-Abs-Based Strategies

1

$$\begin{aligned}
P_e^{(1)} =& \frac{1}{4}\Bigg(\Pr(y_1 < v_1 | x_1 = x_2 = \sqrt{P_r}) \\
&+ \Pr(y_1 > v_1 | x_1 = \sqrt{P_r}, x_2 = -\sqrt{P_r}) \\
&+ \Pr(y_1 < -v_1 | x_1 = -\sqrt{P_r}, x_2 = \sqrt{P_r}) \\
&+ \Pr(y_1 > -v_1 | x_1 = x_2 = -\sqrt{P_r}) \Bigg) \\
=& \frac{1}{2} + \frac{1}{2}\int_{-\infty}^{+\infty} \left(\mathcal{G}\left(u - (h_1 + h_2)\sqrt{P_s}, \sigma_r^2\right) \right. \\
&\left. - \mathcal{G}\left(u - (h_1 - h_2)\sqrt{P_s}, \sigma_r^2\right) \right) \\
&\times \left[\int_{-\infty}^{v_1} \mathcal{G}\left(y - h_1 f(u), \sigma_s^2\right) dy \right] du.
\end{aligned}$$

2

1 2

1) Amplify-and-Forward:

$f(\cdot)$
$f(\cdot)$

$f(u) = \sqrt{\frac{P_r}{(h_1^2+h_2^2)P_s+\sigma_r^2}}u$
i

$$\begin{aligned}
Y_i =& h_i\sqrt{\frac{P_r}{(h_1^2+h_2^2)P_s+\sigma_r^2}}(X_1 + X_2) \\
&+ \left(h_i\sqrt{\frac{P_r}{(h_1^2+h_2^2)P_s+\sigma_r^2}}N + Z_i \right), \ i = 1, 2.
\end{aligned}$$

x_1 x_2
Y_i

$$p_{Y_i|X_1,X_2}(y_i|x_1,x_2) = \mathcal{G}\left(y_i - h_i\sqrt{\frac{P_r}{(h_1^2+h_2^2)P_s+\sigma_r^2}}(x_1+x_2), \frac{h_i^2 P_r \sigma_r^2}{(h_1^2+h_2^2)P_s+\sigma_r^2} + \sigma_s^2\right),$$

$\mathcal{G}(x, \sigma^2)$ fi x_i
i
$v_i = h_i\sqrt{\frac{P_r}{(h_1^2+h_2^2)P_s+\sigma_r^2}}x_i$
i $i = 1, 2$

$$P_e^{(i)} = Q\left(\sqrt{\frac{h_i^2 P_r P_s}{h_i^2 P_r \sigma_r^2 + (h_1^2+h_2^2)P_s\sigma_s^2 + \sigma_r^2\sigma_s^2}}\right).$$

2) Detect-and-Forward:

fi

$$f(u) = \begin{cases} a, & u \geq w, \\ b, & w > u \geq 0, \\ -f(-u), & , \end{cases}$$

w a b
v_1 v_2

$$\begin{aligned}
A(u) \triangleq& \mathcal{G}\left(u - (h_1+h_2)\sqrt{P_s}, \sigma_r^2\right) - \mathcal{G}\left(u - (h_1-h_2)\sqrt{P_s}, \sigma_r^2\right), \\
B(u) \triangleq& \mathcal{G}\left(u + (h_1+h_2)\sqrt{P_s}, \sigma_r^2\right) - \mathcal{G}\left(u + (h_1-h_2)\sqrt{P_s}, \sigma_r^2\right).
\end{aligned}$$

$P_e^{(1)} + P_e^{(2)}$
w

$$\begin{aligned}
\frac{\partial(P_e^{(1)} + P_e^{(2)})}{\partial w} =& A(w)\left(C(v_1, b) - D(v_1, a)\right) \\
&+ B(w)\left(E(v_1, b) - F(v_1, a)\right) + \frac{\partial P_e^{(2)}}{\partial w} = 0.
\end{aligned}$$

w a, b, v_1, v_2

k $w^{(k)}$
$w^{(0)} = h_1\sqrt{P_s}$ $a^{(k)}, b^{(k)}, v_1^{(k)}, v_2^{(k)}$
$w^{(k)}, a^{(k)}, b^{(k)}$ $v_1^{(k)}, v_2^{(k)}$
$a^{(k)}, b^{(k)}$

$$
\begin{aligned}
g(u) &= E\{h_1x_1+h_2x_2|u\} \\
&= \frac{\sinh\left(\frac{(h_1+h_2)\sqrt{P_s}u}{\sigma_r^2}\right)e^{-\frac{(h_1+h_2)^2P_s}{2\sigma_r^2}}(h_1+h_2)+\sinh\left(\frac{(h_1-h_2)\sqrt{P_s}u}{\sigma_r^2}\right)e^{-\frac{(h_1-h_2)^2P_s}{2\sigma_r^2}}(h_1-h_2)}{\cosh\left(\frac{(h_1+h_2)\sqrt{P_s}u}{\sigma_r^2}\right)e^{-\frac{(h_1+h_2)^2P_s}{2\sigma_r^2}}+\cosh\left(\frac{(h_1-h_2)\sqrt{P_s}u}{\sigma_r^2}\right)e^{-\frac{(h_1-h_2)^2P_s}{2\sigma_r^2}}}\sqrt{P_s}
\end{aligned}
$$

$$
\begin{aligned}
&G(f)=P_e^{(1)}+P_e^{(2)} \\
&=1+\frac{1}{2}\int_{-\infty}^{+\infty}\left(\mathcal{G}\left(u-(h_1+h_2)\sqrt{P_s},\sigma_r^2\right)-\mathcal{G}\left(u-(h_1-h_2)\sqrt{P_s},\sigma_r^2\right)\right)\left[\int_{-\infty}^{v_1}\mathcal{G}\left(y-h_1f(u),\sigma_s^2\right)dy\right]du \\
&+\frac{1}{2}\int_{-\infty}^{+\infty}\left(\mathcal{G}\left(u-(h_2+h_1)\sqrt{P_s},\sigma_r^2\right)-\mathcal{G}\left(u-(h_2-h_1)\sqrt{P_s},\sigma_r^2\right)\right)\left[\int_{-\infty}^{v_2}\mathcal{G}\left(y-h_2f(u),\sigma_s^2\right)dy\right]du.
\end{aligned}
$$

$a^{(k)}, b^{(k)}$ $w^{(k+1)}$

$a^{(k)}, b^{(k)}, v_1^{(k)}, v_2^{(k)}$

fi fi

fi

$w = h_1\sqrt{P_s}$

$A(w)=0$

w $w = h_1\sqrt{P_s}$

v_1

$$v_1 = \frac{h_1(a+b)}{2}, \quad v_2 = \frac{h_2(a-b)}{2}.$$

a b

$P_e^{(1)}+P_e^{(2)}$

$$
\begin{aligned}
P_e^{(1)}+P_e^{(2)} &\approx Q\left(\frac{h_1(a-b)}{2\sigma_s}\right)+Q\left(\frac{h_2(a+b)}{2\sigma_s}\right) \\
&+Q\left(\frac{h_2\sqrt{P_s}}{\sigma_r}\right)\left(1+\frac{1}{2}Q\left(\frac{h_2(3b-a)}{2\sigma_s}\right)\right).
\end{aligned}
$$

fi a, b

$a^2+b^2=2P_r$ fi

$P_r, P_s, \sigma_r, \sigma_s, h_1$

h_2 fi fi

$$\frac{b}{a} = \frac{h_1-h_2}{h_1+h_2}, \quad a^2+b^2=2P_r.$$

$$
\begin{aligned}
P_e^{(1)}+P_e^{(2)} &\approx 2Q\left(\sqrt{\frac{P_r}{h_1^2+h_2^2}}\frac{h_1h_2}{\sigma_s}\right)+Q\left(\frac{h_2\sqrt{P_s}}{\sigma_r}\right) \\
&\times\left(1+\frac{1}{2}Q\left(\sqrt{\frac{P_r}{h_1^2+h_2^2}}\frac{h_2(h_1-2h_2)}{\sigma_s}\right)\right).
\end{aligned}
$$

$\max(P_e^1, P_e^2)$

$P_e^{(1)}+P_e^{(2)}$

$\max(P_e^1, P_e^2)$

fi fi

$\{-h_1-h_2, -h_1+h_2, h_1-h_2, h_1+h_2\}$

fi fi

$$a = \sqrt{\frac{9P_r}{5}}, \quad b = \sqrt{\frac{P_r}{5}}.$$

$$P_e^{(1)}+P_e^{(2)} \approx Q\left(\sqrt{\frac{P_r}{5}}\frac{h_1}{\sigma_s}\right)+Q\left(\sqrt{\frac{4P_r}{5}}\frac{h_2}{\sigma_s}\right)+\frac{5}{4}Q\left(\frac{h_2\sqrt{P_s}}{\sigma_r}\right).$$

fi

$\sqrt{\frac{5P_s\sigma_s^2}{P_r\sigma_r^2}} < \frac{h_1}{h_2} < 2$

fi

$h_1 = h_2$

$$a = \sqrt{2P_r}, \quad b = 0,$$

$h_1 - h_2 = 0$

h_1/h_2

h_1/h_2

3) Estimate-and-Forward:

$h_1X_1+h_2X_2$

u $g(u)$

$f(u)$

$g(u)$

fi $g(u)$

4) Optimized Relay Function:

$$\min_{f,v_1,v_2} G(f)$$

$$\int_0^{+\infty} \mathcal{G}\left(u-(h_1+h_2)\sqrt{P_s},\sigma_r^2\right) f^2(u)du + \int_0^{+\infty} \mathcal{G}\left(u-(h_1-h_2)\sqrt{P_s},\sigma_r^2\right) f^2(u)du$$
$$+ \int_0^{+\infty} \mathcal{G}\left(u+(h_1+h_2)\sqrt{P_s},\sigma_r^2\right) f^2(u)du + \int_0^{+\infty} \mathcal{G}\left(u+(h_1-h_2)\sqrt{P_s},\sigma_r^2\right) f^2(u)du = 2P_r.$$

$$P_e = \frac{1}{2} + \frac{1}{2}\int_0^{+\infty}\left(\mathcal{G}\left(u-2\sqrt{P_s},\sigma_r^2\right)+\mathcal{G}\left(u+2\sqrt{P_s},\sigma_r^2\right)-2\mathcal{G}\left(u,\sigma_r^2\right)\right)\left[\int_{-\infty}^{v}\mathcal{G}\left(y-\beta\left(u-C\right),\sigma_s^2\right)dy\right]du.$$

$$P_e = \frac{1}{2} + \frac{1}{4}\int_{-\infty}^{+\infty}\left(\mathcal{G}\left(u-2\sqrt{P_s},\sigma_r^2\right)+\mathcal{G}\left(u+2\sqrt{P_s},\sigma_r^2\right)-2\mathcal{G}\left(u,\sigma_r^2\right)\right)\int_{-\infty}^{v}\mathcal{G}\left(y-f(u),\sigma_s^2\right)du\,dy$$
$$= \frac{1}{2} + \frac{1}{2}\int_0^{w}\left(\mathcal{G}\left(u-2\sqrt{P_s},\sigma_r^2\right)+\mathcal{G}\left(u+2\sqrt{P_s},\sigma_r^2\right)-2\mathcal{G}\left(u,\sigma_r^2\right)\right)du$$
$$\times\int_{-\infty}^{v}\left(\mathcal{G}\left(y+\sqrt{P_r},\sigma_s^2\right)-\mathcal{G}\left(y-\sqrt{P_r},\sigma_s^2\right)\right)dy.$$

fi v_1 v_2
v_1 v_2

v_1 v_2
v_1 v_2

h_1 h_2

B. Abs-Based Strategies

$h_1 = h_2 = 1$
$h_1 > h_2$ fl

i

$$P_e = \frac{1}{2}\Pr(y > v_i|x_1 \neq x_2) + \frac{1}{2}\Pr(y < v_i|x_1 = x_2).$$

$h_1 = h_2$ $v_1 = v_2 = v$

1) Abs-Based Amplify-and-Forward:
fi

C

$$f(u) = \beta\left(|u| - C\right),$$

β fi

$h_1 = h_2 = 1$

v C

fi $v = 0$ $C = h_1\sqrt{P_s}$
$C = h_1\sqrt{P_s} + \sigma_r/\sqrt{2}$

2) Abs-Based Detect-and-Forward:

$2\sqrt{P_s}$ 0 $-2\sqrt{P_s}$
x_1 x_2
$h_1x_1 + h_2x_2$
$\sqrt{P_r}$ $-\sqrt{P_r}$

$$f(u) = \begin{cases} \sqrt{P_r}, & |u| \geq w, \\ -\sqrt{P_r}, & , \end{cases}$$

w

$h_1 = h_2 = 1$

w v w
v w

$$w = \sqrt{P_s}\left(1 + \frac{\sigma_r^2}{2P_s}\log\left(1+\sqrt{1-e^{-4P_s/\sigma_r^2}}\right)\right),$$

v $v = 0$

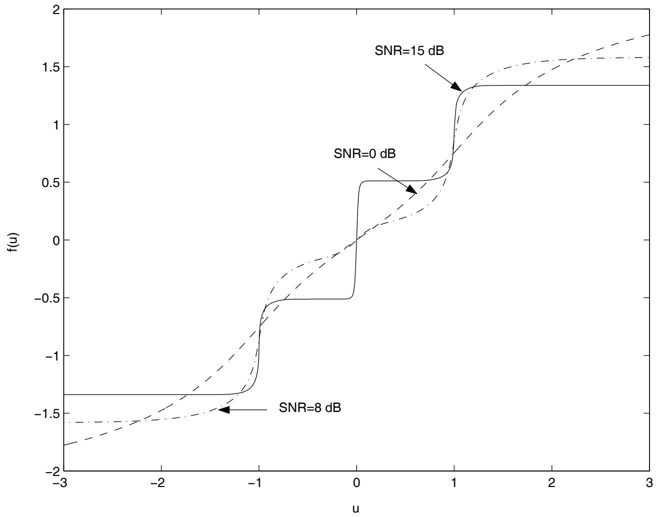
$$P_e = \frac{1}{2} + \frac{1}{2}\left(Q\left(\frac{2\sqrt{P_s}-w}{\sigma_r}\right) + 2Q\left(\frac{w}{\sigma_r}\right)\right.$$
$$\left. - Q\left(\frac{2\sqrt{P_s}+w}{\sigma_r}\right) - 1\right)\left(1 - 2Q\left(\frac{\sqrt{P_r}}{\sigma_s}\right)\right).$$

$\sigma_r^2 \to 0$ w $\sqrt{P_s}$
w v w

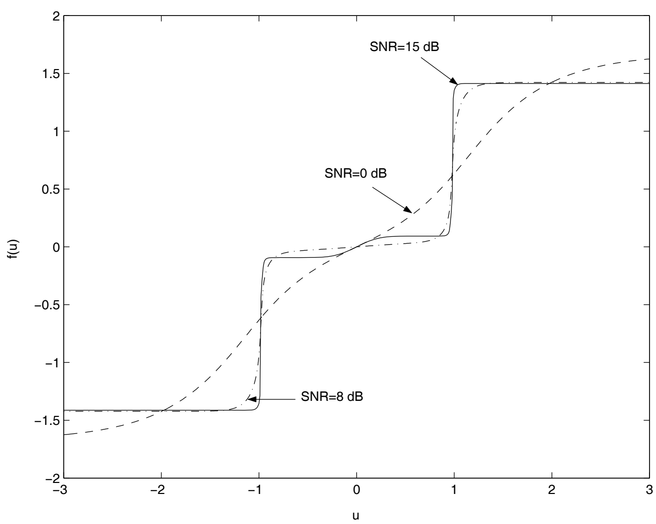
$h_1 > h_2$ $w = h_1\sqrt{P_s}$

3) Abs-Based Estimate-and-Forward:

$|h_1x_1 + h_2x_2|$ fi $h_1 = h_2 = 1$



$h_1 = 1$ $h_2 = 0.5$



$h_1 = 1$ $h_2 = 0.8$

h_1 h_2

$$g(u) = E\left\{|x_1 + x_2|\big|u\right\} = \frac{2\sqrt{P_s}\cosh\left(\frac{2\sqrt{P_s}u}{\sigma_r^2}\right)}{e^{2P_s/\sigma_r^2} + \cosh\left(\frac{2\sqrt{P_s}u}{\sigma_r^2}\right)}.$$

$f(u)$ $g(u) - C$

$$f(u) = \begin{cases} \beta\left(g(u) - C\right), & u \geq 0, \\ f(-u), & , \end{cases}$$

C $\beta \ \geq \ 0$ $E\{f^2(u)\} = P_r$

$h_1 > h_2$

$g(u)$

4) Optimized Relay Strategy:

$f(\cdot)$

$h_1 \ = \ h_2$

$B(u)$ u

$$D(u) \triangleq \mathcal{G}\left(u + 2\sqrt{P_s}, \sigma_r^2\right) + \mathcal{G}\left(u - 2\sqrt{P_s}, \sigma_r^2\right) + 2\mathcal{G}\left(u, \sigma_r^2\right).$$

$$\min_{f,v} \ H(f) = \int_0^{+\infty} B(u)A(f)du, \qquad \frac{1}{2}\int_0^{+\infty} D(u)f^2(u)du \leq P_r,$$

$$\phi(\lambda, f) = H(f) + \frac{\lambda}{2}\left(\int_0^{+\infty} D(u)f^2(u)du - 2P_r\right),$$

$\lambda \ \geq \ 0$

$\phi(\lambda, f)$ $f(u)$

u

$$\frac{\mathcal{G}\left(f(u) - v, \sigma_s^2\right)}{f(u)} = \lambda\frac{D(u)}{B(u)}.$$

$\lambda > 0, D(u) > 0$ $|u| \geq w$ $B(u) \geq 0$

$B(u) < 0$

$$\begin{cases} f(u) \geq 0, & |u| \geq w, \\ f(u) < 0, & , \end{cases}$$

w fi

Lemma 1: $f(u)$

$$\begin{cases} f(u) \geq v, & |u| \geq w, \\ f(u) < v, & , \end{cases}$$

$P_e(f)$ f

Proof: f g

$\lambda \in [0, 1]$ $\gamma = 1 - \lambda$ $\lambda f + \gamma g$ fi

$$\frac{\partial^2 A(f)}{\partial f^2} = \frac{1}{2\sigma_s^2}\left(f(u) - v\right)\mathcal{G}\left(v - f(u), \sigma_s^2\right),$$

$f(u) \ \geq \ v$

$B(u)\frac{\partial^2 A(f)}{\partial f^2}$ $|u| \geq w$

$$\begin{aligned} P_e(\lambda f + \gamma g) =& \frac{1}{2} + \frac{1}{2}\int_0^{+\infty} B(u)A(\lambda f + \gamma g)du \\ \leq& \lambda P_e(f) + \gamma P_e(g). \end{aligned}$$

■

$v = 0$ fi

$$\frac{e^{-\left(f(u)/\sqrt{2\sigma_s^2}\right)^2}}{f(u)/\sqrt{2\sigma_s^2}} = \lambda 2\sqrt{\pi}\sigma_s^2\frac{\cosh\left(\frac{2\sqrt{P_s}u}{\sigma_r^2}\right) + e^{2P_s/\sigma_r^2}}{\cosh\left(\frac{2\sqrt{P_s}u}{\sigma_r^2}\right) - e^{2P_s/\sigma_r^2}},$$

$f(u)$ $W(\cdot)$

fi $W(x)e^{W(x)} = x$ λ

fi

$f(u)$

fi

$$g(u) = \frac{|h_1+h_2|\sqrt{P_s} e^{-\frac{(h_1+h_2)^2 P_s}{2\sigma_r^2}} \cosh\left(\frac{(h_1+h_2)\sqrt{P_s}u}{\sigma_r^2}\right)}{e^{-\frac{(h_1+h_2)^2 P_s}{2\sigma_r^2}} \cosh\left(\frac{(h_1+h_2)\sqrt{P_s}u}{\sigma_r^2}\right) + e^{-\frac{(h_1-h_2)^2 P_s}{2\sigma_r^2}} \cosh\left(\frac{(h_1-h_2)\sqrt{P_s}u}{\sigma_r^2}\right)} + \frac{|h_1-h_2|\sqrt{P_s} e^{-\frac{(h_1-h_2)^2 P_s}{2\sigma_r^2}} \cosh\left(\frac{(h_1-h_2)\sqrt{P_s}u}{\sigma_r^2}\right)}{e^{-\frac{(h_1+h_2)^2 P_s}{2\sigma_r^2}} \cosh\left(\frac{(h_1+h_2)\sqrt{P_s}u}{\sigma_r^2}\right) + e^{-\frac{(h_1-h_2)^2 P_s}{2\sigma_r^2}} \cosh\left(\frac{(h_1-h_2)\sqrt{P_s}u}{\sigma_r^2}\right)}.$$

$$P_e(f) = \frac{1}{2} + \frac{1}{2}\int_0^{+\infty} \underbrace{\left(\mathcal{G}\left(u+2\sqrt{P_s},\sigma_r^2\right) + \mathcal{G}\left(u-2\sqrt{P_s},\sigma_r^2\right) - 2\mathcal{G}\left(u,\sigma_r^2\right)\right)}_{\triangleq B(u)} \underbrace{\left[\int_{-\infty}^{v} \mathcal{G}\left(y-f(u),\sigma_s^2\right) dy\right]}_{\triangleq A(f)} du,$$

$$f(u) = \begin{cases} \sqrt{\sigma_s^2 W\left(\frac{1}{2\pi\lambda^2\sigma_s^4}\left[\frac{\cosh\left(\frac{2\sqrt{P_s}u}{\sigma_r^2}\right) - e^{2P_s/\sigma_r^2}}{\cosh\left(\frac{2\sqrt{P_s}u}{\sigma_r^2}\right) + e^{2P_s/\sigma_r^2}}\right]^2\right)}, & u \geq w, \\ -\sqrt{\sigma_s^2 W\left(\frac{1}{2\pi\lambda^2\sigma_s^4}\left[\frac{\cosh\left(\frac{2\sqrt{P_s}u}{\sigma_r^2}\right) - e^{2P_s/\sigma_r^2}}{\cosh\left(\frac{2\sqrt{P_s}u}{\sigma_r^2}\right) + e^{2P_s/\sigma_r^2}}\right]^2\right)}, & w > u \geq 0, \\ f(-u), & u < 0. \end{cases}$$

$f^2(u)$

$f(u)$

$v = 0$

$$\lim_{\sigma_r^2 \to 0} \frac{\mathcal{G}\left(u+2\sqrt{P_s},\sigma_r^2\right) + \mathcal{G}\left(u-2\sqrt{P_s},\sigma_r^2\right) - 2\mathcal{G}\left(u,\sigma_r^2\right)}{\mathcal{G}\left(u+2\sqrt{P_s},\sigma_r^2\right) + \mathcal{G}\left(u-2\sqrt{P_s},\sigma_r^2\right) + 2\mathcal{G}\left(u,\sigma_r^2\right)} = \begin{cases} 1, & |u| > w, \\ -1, & w > |u|, \end{cases}$$

$$f(u) = \begin{cases} C_1, & |u| > w, \\ -C_2, & w > |u|, \end{cases}$$

$C_1, C_2 > 0$

fi

$$\frac{\mathcal{G}\left(C_1 - v, \sigma_s^2\right)}{C_1} = \lambda = \frac{\mathcal{G}\left(C_2 + v, \sigma_s^2\right)}{C_2},$$

$$v = \frac{\log C_1 - \log C_2}{C_1 + C_2}\sigma_s^2 + \frac{C_1 - C_2}{2} \xrightarrow[\sigma_s^2 \to 0]{} \frac{C_1 - C_2}{2}.$$

$C_1 = C_2 = C$

λ

$$\lambda = \frac{\mathcal{G}\left(C, \sigma_s^2\right)}{C}.$$

$$\min_{C,v} \quad Q\left(\frac{C}{\sigma_s}\right) + \frac{\mathcal{G}\left(C,\sigma_s^2\right)}{C}\left(C^2 - P_r\right).$$

$Q\left(\frac{C}{\sigma_s}\right)$

$\frac{\sigma_s}{\sqrt{2\pi}C} e^{-\frac{C^2}{2\sigma_s^2}}$ $\mathcal{G}\left(C,\sigma_s^2\right) = \frac{1}{\sqrt{2\pi}\sigma_s} e^{-\frac{C^2}{2\sigma_s^2}}$

$v = 0$ $C_1 = C_2 = C = \sqrt{P_r}$ $\sigma_s^2 \to 0$ $\sigma_r^2 \to 0$

$v = 0$ $C_1 = C_2 = C = \sqrt{P_r}$

f^* λ^* $\min_f \phi(\lambda^*, f) = G(f^*)$

v

$h_1 > h_2$

Remarks:

- $f(u)$

$f(u)$ v

$f(u)$ v λ

v fi λ

v

$H(f)$ v

fi

$$f(u)=\begin{cases}\sqrt{\sigma_s^2 W\left(\frac{1}{2\pi\lambda^2 h_1^2\sigma_s^4}\left[\frac{e^{-\frac{(h_1+h_2)^2P_s}{2\sigma_r^2}}\cosh\left(\frac{(h_1+h_2)\sqrt{P_s}u}{\sigma_r^2}\right)-e^{-\frac{(h_1-h_2)^2P_s}{2\sigma_r^2}}\cosh\left(\frac{(h_1-h_2)\sqrt{P_s}u}{\sigma_r^2}\right)}{e^{-\frac{(h_1+h_2)^2P_s}{2\sigma_r^2}}\cosh\left(\frac{(h_1+h_2)\sqrt{P_s}u}{\sigma_r^2}\right)+e^{-\frac{(h_1-h_2)^2P_s}{2\sigma_r^2}}\cosh\left(\frac{(h_1-h_2)\sqrt{P_s}u}{\sigma_r^2}\right)}\right]^2\right)}, & u\ge w,\\ -\sqrt{\sigma_s^2 W\left(\frac{1}{2\pi\lambda^2 h_1^2\sigma_s^4}\left[\frac{e^{-\frac{(h_1+h_2)^2P_s}{2\sigma_r^2}}\cosh\left(\frac{(h_1+h_2)\sqrt{P_s}u}{\sigma_r^2}\right)-e^{-\frac{(h_1-h_2)^2P_s}{2\sigma_r^2}}\cosh\left(\frac{(h_1-h_2)\sqrt{P_s}u}{\sigma_r^2}\right)}{e^{-\frac{(h_1+h_2)^2P_s}{2\sigma_r^2}}\cosh\left(\frac{(h_1+h_2)\sqrt{P_s}u}{\sigma_r^2}\right)+e^{-\frac{(h_1-h_2)^2P_s}{2\sigma_r^2}}\cosh\left(\frac{(h_1-h_2)\sqrt{P_s}u}{\sigma_r^2}\right)}\right]^2\right)}, & w>u\ge 0,\\ f(-u), & u<0.\end{cases}$$

- fi fi
- fi h_1 h_2 h_1 h_2

h_1 h_2

$$P_e^{(1)}+P_e^{(2)}\approx\begin{cases}\frac{5}{8}e^{-\frac{h_2^2P_s}{2\sigma_r^2}}, & 2>\frac{h_1}{h_2}>\sqrt{\frac{5P_s\sigma_s^2}{P_r\sigma_r^2}},\\ e^{-\frac{h_1^2h_2^2P_r}{2(h_1^2+h_2^2)\sigma_s^2}}+\frac{1}{2}e^{-\frac{h_2^2P_s}{2\sigma_r^2}}, & .\end{cases}$$

$h_1>h_2$

$$P_e^{(1)}+P_e^{(2)}\approx\frac{1}{2}\left(e^{-\frac{h_1^2P_r}{2\sigma_s^2}}+e^{-\frac{h_2^2P_r}{2\sigma_s^2}}\right)+e^{-\frac{h_2^2P_s}{2\sigma_r^2}}.$$

$_r\sim\frac{P_s}{\sigma_r^2}$ $_s\sim\frac{P_r}{\sigma_s^2}$

- $_s<$ $_r$ $\frac{1}{2}e^{-\frac{h_2^2P_r}{2\sigma_s^2}}$ $e^{-\frac{h_1^2h_2^2P_r}{2(h_1^2+h_2^2)\sigma_r^2}}$ 1/2
- $_s>$ $_r$ $1+\frac{h_2^2}{h_1^2}>\frac{P_r\sigma_r^2}{P_s\sigma_s^2}$ $h_1>2h_2$ $e^{-\frac{h_2^2P_s}{2\sigma_r^2}}$ $\frac{1}{2}e^{-\frac{h_2^2P_s}{2\sigma_r^2}}$ 1/2
- $_s>$ $_r$ $1+\frac{h_2^2}{h_1^2}<\frac{P_r\sigma_r^2}{P_s\sigma_s^2}$ $\frac{h_1}{h_2}<\sqrt{\frac{5P_s\sigma_s^2}{P_r\sigma_r^2}}$ $e^{-\frac{h_2^2P_s}{2\sigma_r^2}}$ $\frac{3}{4}e^{-\frac{h_2^2P_s}{2\sigma_r^2}}$ 3/4
- $_s>$ $_r$ $1+\frac{h_2^2}{h_1^2}<\frac{P_r\sigma_r^2}{P_s\sigma_s^2}$ $2>\frac{h_1}{h_2}>\sqrt{\frac{5P_s\sigma_s^2}{P_r\sigma_r^2}}$ $e^{-\frac{h_2^2P_s}{2\sigma_r^2}}$ $\frac{5}{8}e^{-\frac{h_2^2P_s}{2\sigma_r^2}}$ 5/8

h_1/h_2

fi $h_1=h_2=1$ fi fi $h(u)$

$$h(u_1+u_2)\neq h(u_1'+u_2),\quad \forall u_1\neq u_1'$$
$$h(u_1+u_2)\neq h(u_1+u_2'),\quad \forall u_2\neq u_2',\ u_i,u_i'\in\mathcal{Q},$$

$i=1,2$ $\mathcal{Q}$ fi

fi $h(u)$ fi $\mathcal{G}$ u_1+u_2 u_1+u_2 $u_1'+u_2$ $u_1'\neq u_1$ $h(u)$ $\mathcal{G}$ fi

$\mathcal{G}$ $h(u)$

$\mathcal{G}$

fi $h(u)$

$u_1 + u_2 = c_i$ $i = 1, \ldots, 2|\mathcal{V}| - 1$

$u_1, u_2 \in \mathcal{V}$ $\mathcal{V}$

c_i

$\mathcal{W}$ $|\mathcal{W}| = 2|\mathcal{V}| - 1$

$h(u)$ $\mathcal{W}$ $\mathcal{V}'$

$M \geq |\mathcal{V}|$

i

$k = (i \mod M)$ $\mathcal{V}'$ c_i

M $\mathcal{V}'$

$\mathcal{V} = \{-3, -1, 1, 3\}$

$\mathcal{W} = \{-6, -4, -2, 0, 2, 4, 6\}$ fi

$\mathcal{W}$ $\mathcal{V}' = \mathcal{V}$

$$h(-6) = -3,\ h(-4) = -1,\ h(-2) = 3,$$
$$h(0) = 1,\ h(2) = -3,\ h(4) = -1,\ h(6) = 3,$$

$$h(-6) = -3,\ h(-4) = -1,\ h(-2) = 1,$$
$$h(0) = 3,\ h(2) = -3,\ h(4) = -1,\ h(6) = 1.$$

$h(u)$

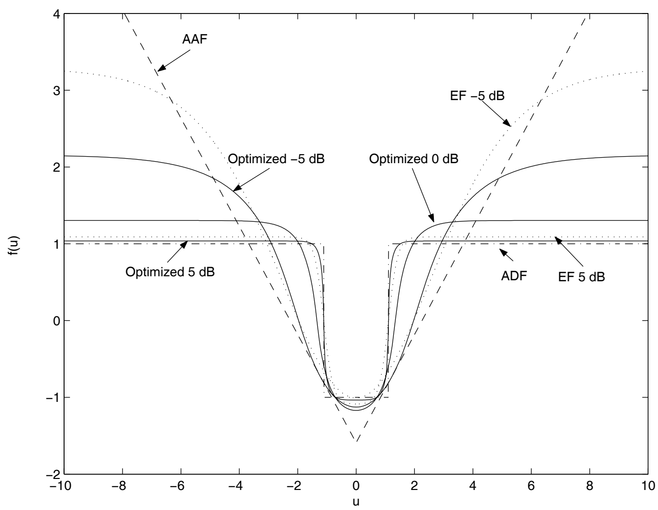
$$f(u) = \begin{cases} \beta(u+3), & u < -3, \\ \beta(u+5), & -2 > u \geq -3, \\ \beta(1-u), & 1 > u \geq -2, \\ \beta(-1-u), & 2 > u \geq 1, \\ \beta(u-5), & 5 > u \geq 2, \\ \beta(u-3), & u \geq 5, \end{cases}$$

β fi

fi u

$h(u)$

$$f(u) = \begin{cases} -3\beta, & u < -5, \\ -\beta, & -3 > u \geq -5, \\ 3\beta, & -1 > u \geq -3, \\ \beta, & 1 > u \geq -1, \\ -3\beta, & 3 > u \geq 1, \\ -\beta, & 5 > u \geq 3, \\ 3\beta, & u \geq 5. \end{cases}$$



$f(u)$

$\sigma_r^2 = \sigma_s^2$ $h_1 = h_2 = 1$ $P_r = P_s = 1$

fi $g(u)$

$$g(u) = \arg\min_{g'(u)} E\left\{ \left| h(x_1 + x_2) - g'(u) \right|^2 \Big| \, u \right\}.$$

$f(u)$ $g(u)$ $f(u) = \beta\, g(u)$

$\beta \geq 0$

v v

$$\hat{x}_2 = \arg\min_{\tilde{x}_2 \in \mathcal{Q}} |y_1 - f(x_1 + \tilde{x}_2)|^2.$$

$\mathcal{W} = \{-6, -4, -2, 0, 2, 4, 6\}$

$\mathcal{V}'$ $\mathcal{V}' =$

$\{-4, -2, 0, 2, 4\}$

$$h(-6) = -4,\ h(-4) = -2,\ h(-2) = 0,$$
$$h(0) = 2,\ h(2) = 4,\ h(4) = -2,\ h(6) = -4,$$

$\mathcal{V}' = \{-5, -3, -1, 1, 3, 5\}$

$$h(-6) = -5,\ h(-4) = -3,\ h(-2) = -1,$$
$$h(0) = 1,\ h(2) = 3,\ h(4) = 5,\ h(6) = -5.$$

$\mathcal{V}' = \mathcal{W}$ $h(u) = u$

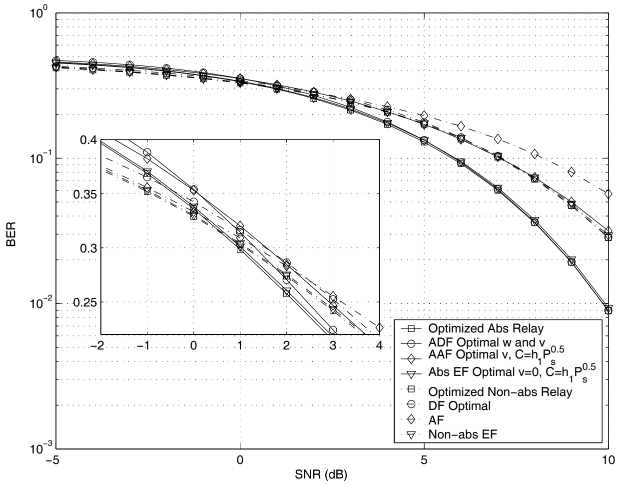
$\sigma_r^2 = \sigma_s^2$ $P_r = P_s = 1$

h_1 h_2

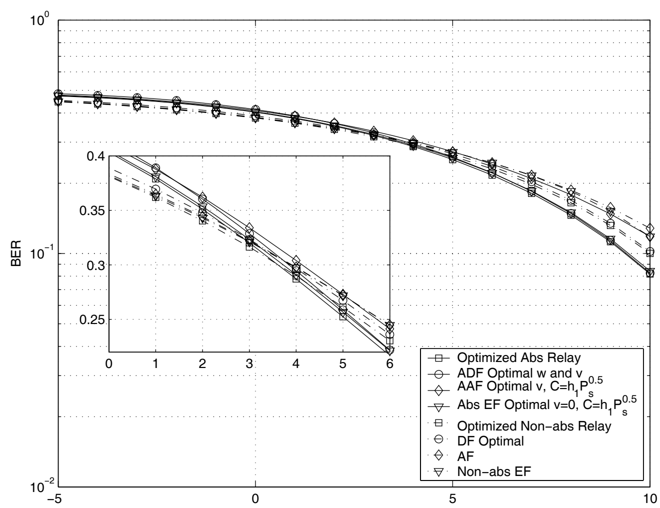
$f(u)$

$C = \sqrt{P_s} + \sigma_r / \sqrt{2}$

fi

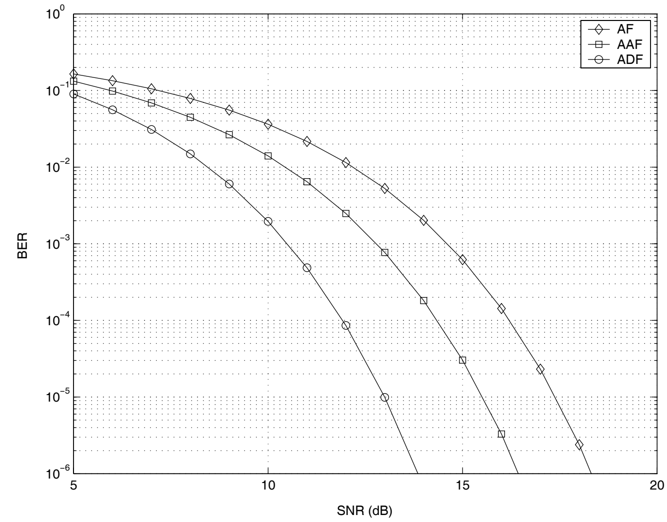


$h_1 = 1$ $h_2 = 0.8$ $P_r = P_s = 1$ fi

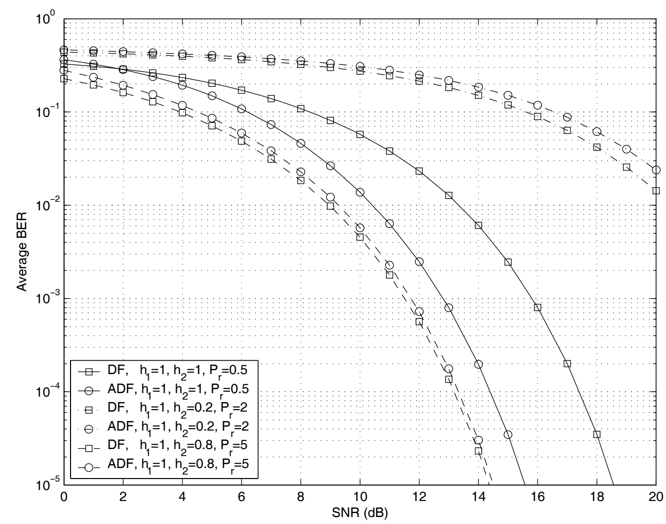


$h_1 = 1$ $h_2 = 0.5$ $P_r = P_s = 1$ fi

$h_1 = 1$ $h_2 = 0.8$



$\sigma_r^2 = \sigma_s^2$ $h_1 = h_2$ $P_r = P_s = 1$



a priori

$h_1 = 1$ $h_2 = 0.5$
$h_1 = 1$ $h_2 = 0.8$

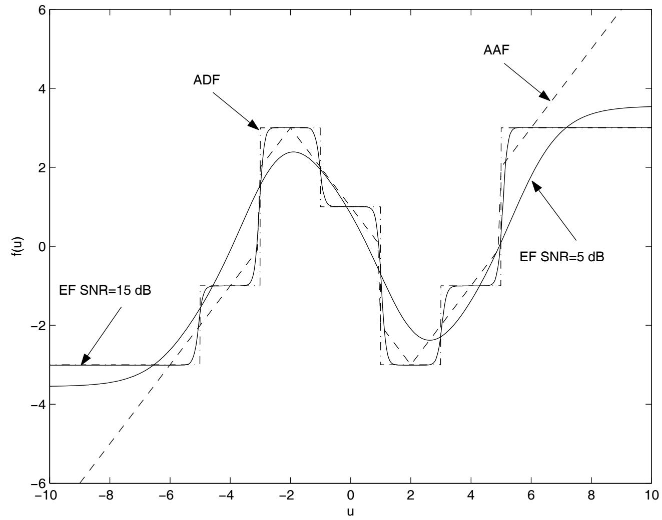
fi

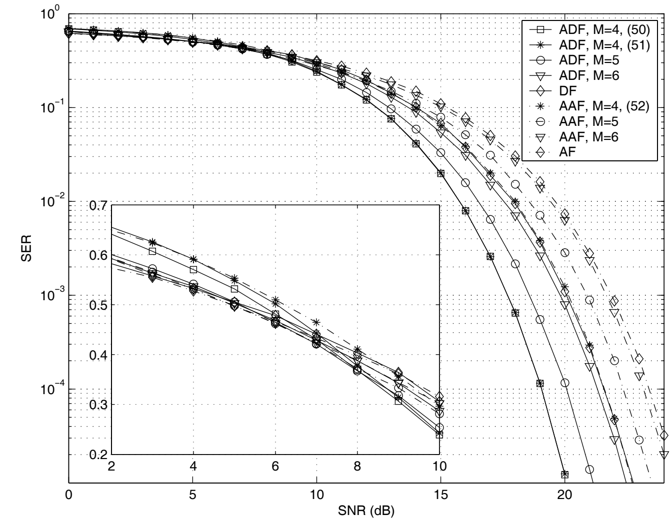
$\sigma_r^2 = \sigma_s^2$ $h_1 = h_2$

$P_r = P_s = 1$ $C = 1$

2

10^{-8}





$\sigma_r^2 = \sigma_s^2$

$M = |\mathcal{V}'| = 4, 5, 6, 7$ $\sigma_r^2 = \sigma_s^2$ fi

2.7

10^{-8}

$f(u)$ $\sigma_r^2 = \sigma_s^2$

$M = |\mathcal{V}'| = 4, 5, 6, 7$

$5/\sigma_r^2$

M

M M

M

fi

M

$M = 7$

$= 0$ $M = 4,\ 5,\ 6,\ 7$ 0.6904 0.6472

0.6428 0.6146

M

fi

fi

Lemma 2: Z

$p_U(\mu)$ $p_V(\mu)$ U

V $p_U(\mu) - p_V(\mu)$

$\mu \geq t$ t

$p_{U+Z}(\nu) - p_{V+Z}(\nu)$ $\nu \geq t'$

t'

Proof: σ^2 Z

$\frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{\nu^2 - 2t\nu}{2\sigma^2}\right\} > 0$

ν ■

Proof of Theorem 1: $a \triangleq h_1\sqrt{P_s} + h_2\sqrt{P_s}$

$b \triangleq h_1\sqrt{P_s} - h_2\sqrt{P_s}$

$x_1 = \sqrt{P_s}$

$x_2 = \sqrt{P_s}$

$p_{f(a+N)+Z_1}(y_1) - p_{f(b+N)+Z_1}(y_1) \geq 0$ $x_2 = -\sqrt{P_s}$

$f(U)$ U

$U = f(a+N)$ $V = f(b+N)$

$Z = Z_1$

$$
\begin{aligned}
p_{U+Z}(\nu) - p_{V+Z}(\nu) &= \int_{-\infty}^{\infty} p_Z(\nu-\mu)p_U(\mu)d\mu - \int_{-\infty}^{\infty} p_Z(\nu-\mu)p_V(\mu)d\mu \\
&= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\nu-\mu)^2}{2\sigma^2}\right\}(p_U(\mu)-p_V(\mu))\,d\mu \\
&= \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ \frac{-\nu^2+2t\nu}{2\sigma^2}\right\}\left(\int_{-\infty}^{t} \exp\left\{\frac{2\nu(\mu-t)-\mu^2}{2\sigma^2}\right\}(p_U(\mu)-p_V(\mu))\,d\mu \right.\\
&\quad \left. + \int_{t}^{\infty} \exp\left\{\frac{2\nu(\mu-t)-\mu^2}{2\sigma^2}\right\}(p_U(\mu)-p_V(\mu))\,d\mu\right)
\end{aligned}
$$

$$x_1 = \sqrt{P_s}$$

$$x_2 = \sqrt{P_s}$$

$p_{f(|a+N|)+Z_1}(y_1) - p_{f(|b+N|)+Z_1}(y_1) \geq 0$ $x_2 = -\sqrt{P_s}$

$$
\begin{aligned}
&p_{|a+N|}(\mu) - p_{|b+N|}(\mu) \\
=&p_{a+N}(\mu) + p_{a+N}(-\mu) - p_{b+N}(\mu) - p_{b+N}(-\mu) \\
=&C(\mu)\,(D(\mu)-1)
\end{aligned}
$$

$C(\mu) = \left(\exp\{-(\mu-b)^2/2\sigma^2\} + \exp\{-(-\mu-b)^2/2\sigma^2\}\right)/\sigma\sqrt{2\pi} > 0$

$$
\begin{aligned}
D(\mu) &= \frac{\exp\{-(\mu-a)^2/2\sigma^2\} + \exp\{-(-\mu-a)^2/2\sigma^2\}}{\exp\{-(\mu-b)^2/2\sigma^2\} + \exp\{-(-\mu-b)^2/2\sigma^2\}} \\
&= \exp\left\{\frac{-a^2+b^2}{2\sigma^2}\right\} \frac{\exp\{\mu a/\sigma^2\} + \exp\{-\mu a/\sigma^2\}}{\exp\{\mu b/\sigma^2\} + \exp\{-\mu b/\sigma^2\}}
\end{aligned}
$$

$\mu \geq 0$ $p_{|a+N|}(\mu) - p_{|b+N|}(\mu)$ $\mu \geq t$ t $f(|U|)$ $|U|$ $U = f(|a+N|)$ $V = f(|b+N|)$ $Z = Z_1$

$x_1 = -\sqrt{P_s}$

1 2 ■

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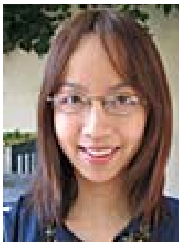
Elements of Information Theory

Tao Cui



Tracey Ho

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