On the Distance Growth Properties of Double Serially Concatenated Convolutional Codes

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***Abstract*—It has recently been shown that the minimum distance of the ensemble of repeat multiple accumulate codes grows linearly with block length. In this paper we present a method to obtain the minimum distance growth rate coefficient of double-serially concatenated codes and determine the growth coefficient of the rate 1/2 double serially concatenated code consisting of an outer memory-one convolutional code followed by two accumulators.**

I. Introduction

We address the minimum distance of double-serially concatenated codes (DSCCs), which were introduced in 1998 by Benedetto et al. [1]. In addition to distance properties superior to those of single-serially concatenated codes (SSCCs), DSCCs also have a simple encoding and decoding structure and, in many cases, can be decoded using relatively few iterations. Among DSCCs, the ensemble of repeat-accumulate-accumulate (RAA) codes is the one that has been studied most extensively. In [2] it was shown that there exist RAA codes having a minimum distance growing linearly with block length [2]. In [3] a similar result for generalized RAA codes is presented, showing that linear distance growth can be achieved with any outer code having a minimum distance larger than or equal to two. A new proof of linear distance growth, allowing the exact calculation of the growth rate coefficient for repeat-multiple accumulate code ensembles, was the focus of a recent paper by the authors [4]. In the following we present a method to investigate the distance growth properties of DSCCs and to obtain the growth rate coefficients of DSCC ensembles that exhibit linear distance growth with block length.

II. Minimum distance properties of DSCCs

In a DSCC, a codeword of the outer encoder $\mathcal{C}_1$ with weight h_1 is permuted by the first interleaver π_1 into the input word of the middle encoder $\mathcal{C}_2$, which then generates a codeword having weight h_2, that in turn is permuted by the second interleaver π_2 to form the input word for the inner encoder $\mathcal{C}_3$. We assume that both the middle and the inner encoder are recursive, so that the minimum distance of the single-serial concatenation of these two encoders grows sublinearly with codeword length N, as shown in [5]. The ensemble average weight enumerating function (WEF) of a DSCC is given by [1]

$$\bar{A}_h^{\mathcal{C}_{DSCC}} = \sum_{w=1}^{K} \sum_{h_1=1}^{N_1} \sum_{h_2=1}^{N_2} \underbrace{\frac{A_{w,h_1}^{\mathcal{C}_1} \cdot A_{h_1,h_2}^{\mathcal{C}_2} \cdot A_{h_2,h}^{\mathcal{C}_3}}{\binom{R_2 R_3 N}{h_1}\binom{R_3 N}{h_2}}}_{\bar{A}_{w,h_1,h_2,h}}, \qquad (1)$$

where the overall output weight is h, $A_{w',h'}^{\mathcal{C}_k}$ represents the input-output weight enumerating function (IOWEF) for the code $\mathcal{C}_k$, $k = 1, 2, 3$, and $\bar{A}_{w,h_1,h_2,h}$ is the overall ensemble average input-output weight enumerator.

We can upper bound the summation in (1) as follows:

$$\bar{A}_h^{\mathcal{C}_{DSCC}} \leq R_1 R_2^2 R_3^3 N^3 \max_{0<w\leq K} \max_{0<h_1\leq N_1} \max_{0<h_2\leq N_2} \bar{A}_{w,h_1,h_2,h}^{\mathcal{C}_{DSCC}}, \qquad (2)$$

where R_k, $k = 1, 2, 3$, denotes the code rate of the k-th constituent encoder. We now normalize the weights $\alpha = w/K$, $\beta = h_1/N_1$, $\gamma = h_2/N_2$, and $\rho = h/N$, and define the function $f^{\mathcal{C}_k}$ as the asymptotic behavior of the IOWEF of a code $\mathcal{C}_k$ as

$$f^{\mathcal{C}_k}(\lambda, \varsigma) = \lim_{N\to\infty} \frac{\log A_{\lambda N_i, \varsigma N_o}^{\mathcal{C}_k}}{N}, \qquad (3)$$

where λ is the normalized input weight (normalized w.r.t. the input block length N_i) and ς is the normalized output weight (normalized w.r.t. the output block length N_o) of code $\mathcal{C}_k$, and N is the total block length of the DSCC. We also define the asymptotic spectral shape as $r(\rho) = \lim_{N\to\infty} \log \bar{A}_{\rho N}^{\mathcal{C}}/N$. By employing Stirling's approximation for the terms in the denominator of (1) and by combining the result with (2), we arrive at the following expression for $r(\rho)$:

$$\begin{aligned} r(\rho) &\leq \max_{\alpha,\beta,\gamma} f^{\mathcal{C}_{DSCC}}(\alpha, \beta, \gamma, \rho) \\ &= \max_{0<\alpha\leq 1} \max_{0<\beta\leq 1} \max_{0<\gamma\leq 1} f^{\mathcal{C}_1}(\alpha, \beta) + f^{\mathcal{C}_2}(\beta, \gamma) \qquad (4) \\ &\quad + f^{\mathcal{C}_3}(\gamma, \rho) - R_2 R_3 \mathbb{H}(\beta) - R_3 \mathbb{H}(\gamma), \end{aligned}$$

with $\mathbb{H}(x) = -x \ln(x) - (1-x) \ln(1-x)$. It can be shown that for $N \to \infty$ the bound in (4) is tight.

This work was partly supported by NSF grants CCR02-05310 and CCF05-15012, NASA grants NNG05GH73G and NNX07AK536, German Research Foundation (DFG) grant KL 1080/3-1, and the University of Notre Dame Faculty Research Program.

A maximum for $f^{\mathcal{C}_{DSCC}}$ can either occur on the boundaries or in the region $\{0 < \alpha < 1, 0 < \beta < 1, 0 < \gamma < 1\}$. For a maximum within the region, a necessary condition is that the partial derivatives $\partial f/\partial\alpha$, $\partial f/\partial\beta$, and $\partial f/\partial\gamma$ are all zero at the maximum point. Because the encoder is serially concatenated, the analysis can be performed sequentially, one maximization after the other. This concept can be extended to arbitrary concatenation depths in a straightforward way.

We now give some examples of the asymptotic weight enumerators of simple component codes, where L denotes the number of constituent encoders in the serial concatenation. We define $R_{[j,L]} := \prod_{k=j}^{L} R_k$. The expressions derived in the following can then be used as constituent codes in (4) for $L = 3$.

Example 1. *Rate 1/q repetition code.*
In this case the relationship between the input weight and the output weight is deterministic, and therefore we obtain

$$f^{Rep}(\alpha) = \frac{R_{[2,L]}}{q}\,\mathbb{H}(\alpha)\,. \tag{5}$$

Example 2. *Rate 1 accumulate code.*
The asymptotic behavior of the accumulate code at stage j in a serial concatenation is given as

$$f^{acc_j}(\lambda_j, \varsigma_j) = R_{[j,L]}\left((1-\varsigma_j)\mathbb{H}\left(\frac{\lambda_j}{2(1-\varsigma_j)}\right) + \varsigma_j\mathbb{H}\left(\frac{\lambda_j}{2\varsigma_j}\right)\right). \tag{6}$$

Example 3. *Rate 1/2 convolutional code $[1, \frac{1}{1+D}]$.*
Here, we obtain

$$f^{[1,\frac{1}{1+D}]_j}(\lambda_j, \varsigma_j) = R_{[j,L]}(1-2\varsigma_j+\lambda_j)\mathbb{H}\left(\frac{\lambda_j}{2(1-2\varsigma_j+\lambda_j)}\right) + R_{[j,L]}(2\varsigma_j-\lambda_j)\mathbb{H}\left(\frac{\lambda_j}{2(2\varsigma_j-\lambda_j)}\right). \tag{7}$$

In the following we consider the rate 1/2 DSCC consisting of an outer rate-1/2 memory-one convolutional code with generator matrix $[1, \frac{1}{1+D}]$ followed by two rate-1 accumulators. From the above examples we conclude that the asymptotic behavior of this code is given by

$$\begin{aligned} f^{\mathcal{C}_{DSCC}}(\alpha,\beta,\gamma,\rho) &= \frac{1-2\beta+\alpha}{2}\mathbb{H}\left(\frac{\alpha}{2(1-2\beta+\alpha)}\right) \\ &+\frac{2\beta-\alpha}{2}\mathbb{H}\left(\frac{\alpha}{2(2\beta-\alpha)}\right) + (1-\gamma)\mathbb{H}\left(\frac{\beta}{2(1-\gamma)}\right) \\ &+\gamma\mathbb{H}\left(\frac{\beta}{2\gamma}\right) - \mathbb{H}(\beta) + (1-\rho)\mathbb{H}\left(\frac{\gamma}{2(1-\rho)}\right) \\ &+\rho\mathbb{H}\left(\frac{\gamma}{2\rho}\right) - \mathbb{H}(\gamma)\,. \end{aligned} \tag{8}$$

The maximization of $f^{\mathcal{C}_{DSCC}}(\alpha,\beta,\gamma,\rho)$ with respect to α, β, and γ yields the asymptotic spectral shape function shown in Fig. 1. It has a zero crossing at $\rho = 0.0832$, corresponding to the asymptotic growth rate coefficient of the minimum distance. So, for large N, we expect a code in this ensemble to have minimum distance $d_{min}(N) \sim 0.0832 \cdot N$. Finally, we note that the growth rate coefficient of this DSCC ensemble is not far from the Gilbert-Varshamov bound (GVB).

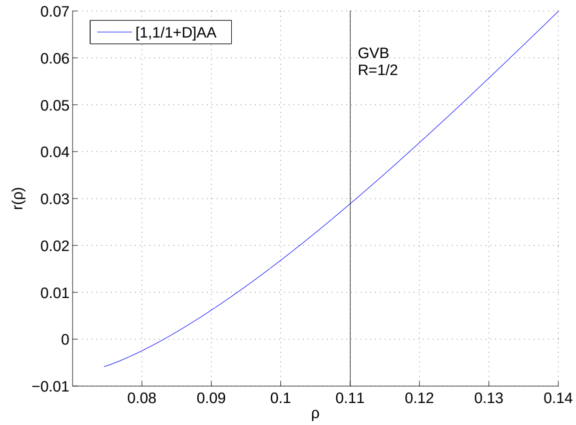

Fig. 1. Asymptotic spectral shape for a rate 1/2 DSCC.

Table I lists the distance growth rate coefficients of different rate-1/2 DSCC ensembles. For the RAA code with repetition factors of three or greater, we have achieved rate-1/2 code ensembles by random puncturing [4], which results in growth rate coefficients remarkably close to the GVB.

TABLE I
DISTANCE GROWTH RATE COEFFICIENT ρ_0 FOR DIFFERENT RATE 1/2 DSCCS.

Code	ρ_0
$[1, \frac{1}{1+D}]AA$	0.0832
R^3AA-punct.	0.1036
R^4AA-punct.	0.1091
R^5AA-punct.	0.1098
random codes (GVB)	0.1100

III. CONCLUSIONS

In contrast to SSCCs, DSCCs can be asymptotically good in the sense that the minimum distance grows linearly with block length as the block length tends to infinity. If the asymptotic IOWEF of the component encoders is known, we can use the method presented in this paper to determine the distance growth rate coefficients of different DSCC ensembles. Based on this analysis we obtained the minimum distance growth rate coefficient of the DSCC given by an outer rate-1/2 code with generator matrix $[1, 1/1+D]$ followed by two accumulators.

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