Design of Maximally Decimated Near-Perfect-Reconstruction DFT Filter Banks with Allpass-Based Analysis Filters

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Abstract

In this paper we address a maximally decimated IIR DFT filter bank, where all polyphase components are replaced by allpass filters. Generally, a perfect reconstruction (PR) solution leads to a synthesis filter bank with unstable subband filters. As a novelty we show that by using closed-form FIR phase compensation filters as polyphase components in the synthesis filter bank we obtain a stable near-PR analysis-synthesis system. Furthermore, the remaining linear and aliasing distortions only depend on the overall system delay, and can be made arbitrarily small.

1. Introduction

Critically subsampled uniform M–channel filter banks can be used in many applications, for example in image and audio compression, where most design techniques deal with FIR-based systems. However, especially in real-time applications, computationally efficient solutions are often reasonable also for the filter bank system. For instance, a very low-complexity alternative for performing an M-channel subband decomposition can be obtained by a complex-modulated IIR filter bank, where all polyphase components are replaced by allpass filters [1]. However, an overall perfect reconstruction (PR) system can generally only be achieved with unstable synthesis subband filters. This problem can be solved by using anticausal filtering [2] and employing a double buffering schema as in [3, 4] for the processing of infinite-length signals. Unfortunately, this method requires extra information which grows with the order of the allpass filters in the analysis, since initial conditions for the synthesis allpasses have to be additionally transmitted.

In this paper we propose a novel design for the synthesis filter bank, which avoids anticausal filtering and leads to a near-PR solution for the critically subsampled analysis-synthesis system. The synthesis subband filters are of FIR-type and can be derived from simple closed-form expressions. Furthermore, all distortions in the reconstructed signal can be made arbitrarily small, and depend only on the overall system delay of the filter bank.

2. Allpass-based M-band DFT IIR Filter Banks

In the following we consider maximally decimated M-band DFT filter banks with $M \in \mathbb{N}$, where the corresponding analysis bank is shown in Fig. 1. Herein, $\mathbf{W}_M$ de-

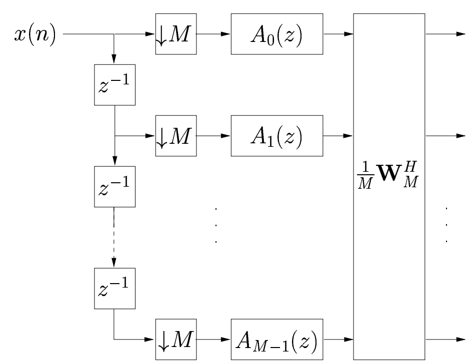

Figure 1. DFT IIR analysis filter bank in polyphase notation

notes the DFT matrix with $[\mathbf{W}_M]_{ki} = W_M^{ki} = e^{-j\frac{2\pi}{M}ki}$, $k, i = 0, \ldots, M-1$. The polyphase components $A_k(z)$ are allpass transfer functions, leading to an IIR prototype filter

$$P(z) = \sum_{k=0}^{M-1} z^{-k} A_k(z^M). \tag{1}$$

Since the analysis filters $H_i(z)$ are modulated versions of the lowpass prototype $P(z)$, they are also IIR filters and

*This work was supported in part by the German Research Foundation "Deutsche Forschungsgemeinschaft" (DFG Graduate College No. 357).

can be obtained from (1) according to

$$H_i(z) = P(zW_M^i) = \sum_{k=0}^{M-1} z^{-k}\, W_M^{-ki}\, A_k(z^M). \quad (2)$$

For the sake of simplicity we restrict ourselves in the following derivations to first-order allpass filters defined as

$$A_k(z) = \frac{z^{-1} + a_k}{1 + a_k z^{-1}}, \quad 0 < a_k < 1, \quad a_k \in \mathbb{R}. \quad (3)$$

However, later on we will also briefly address the higher order cases.

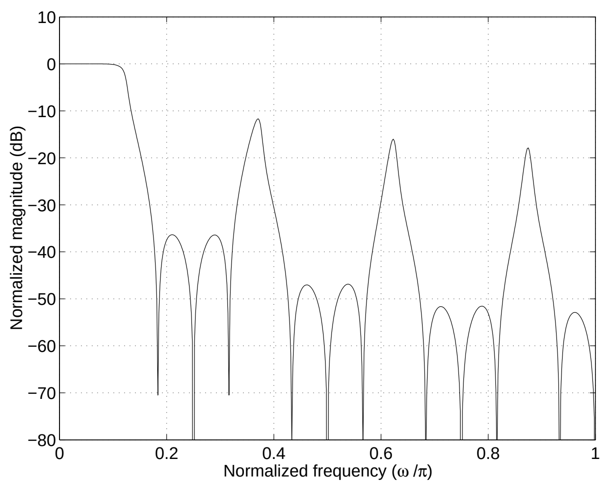

Figure 2. Normalized magnitude frequency response for the lowpass prototype ($M = 8$, $\omega_p = 0.6\,\pi/8$, first-order allpasses)

As an example a lowpass prototype is designed for $M = 8$ and a passband edge frequency of $\omega_p = 0.6\,\pi/8$ by using the Remez-type optimization approach from [5], where the normalized magnitude frequency response is depicted in Fig. 2. The side-lobes are inevitable [5], but they do not introduce any additional error into the reconstructed signal.

3. Compensation of the Phase Distortion

We now derive a closed-form expression for FIR filters, which approximately compensate the phase distortions introduced by first-order allpass filters up to a certain error [6].

Let us first consider the polynomial factorization relation

$$z^{-d} - (-1)^d \cdot a^d = (z^{-1} + a) \sum_{i=0}^{d-1} (-1)^i\, a^i\, z^{-(d-1-i)} \quad (4)$$

with $d \in \mathbb{N}$. Exploiting this relation, FIR filters $F_k(z)$, $k = 0, 1, \ldots, M-1$ can be defined according to

$$F_k(z) = (1 + a_k\, z^{-1}) \sum_{i=0}^{d_k-1} (-1)^i\, a_k^i\, z^{-(d_k-1-i)}, \quad (5)$$

where it can be shown from (3), (4) and (5) that

$$\begin{aligned} A_k(z)\, F_k(z) &= z^{-d_k} - (-1)^{d_k}\, a_k^{d_k} = \\ &= z^{-d_k} - (-1)^{d_k}\, \varepsilon(a_k, d_k), \quad d_k \in \mathbb{N}. \quad (6) \end{aligned}$$

Since $0 < a_k < 1$ we can select the order d_k of the filter $F_k(z)$ such that the error $\varepsilon(a_k, d_k)$ can be made arbitrarily small at the expense of additional delay. Then (6) can be approximated as $A_k(z)\, F_k(z) \approx z^{-d_k}$.

4. Synthesis Filter Bank Design

4.1. Basic idea

Based on the result from the last section we are now able to construct a synthesis filter bank, where the phase-compensation filters $F_k(z)$ from (5) are used as (type-2) polyphase components. The resulting critically subsampled analysis-synthesis filter bank is shown in Fig. 3 with $d_{\max} = \max_k(d_k)$ for $k = 0, 1, \ldots, M-1$. When an approximately equal phase-compensation error $\varepsilon(a_k, d_k)$ in each polyphase branch k is desired, it is necessary to introduce the extra delay blocks $z^{-(d_{\max} - d_k)}$ due to different coefficients a_k. The synthesis filters $G_i(z)$ in Fig. 3 are modulated versions of the synthesis prototype

$$Q(z) = \sum_{k=0}^{M-1} z^{-(M-1-k)}\, F_k(z^M)\, z^{-M(d_{\max} - d_k)},$$

and can be written as

$$\begin{aligned} G_i(z) &= Q(zW_M^i) = \\ &= \sum_{k=0}^{M-1} z^{-(M-1-k)}\, F_k(z^M)\, W_M^{ki}\, z^{-M(d_{\max} - d_k)}. \quad (7) \end{aligned}$$

Note that, due to $|F_k(e^{j\omega})| \approx |A_k(e^{j\omega})|$, $Q(z)$ has almost the same normalized magnitude frequency response as the analysis prototype, which will be demonstrated with a design example in Section 5.

4.2. Linear and aliasing distortions

In the following we derive expressions for the linear and aliasing transfer functions for the proposed analysis-synthesis system in Fig. 3. The analysis polyphase matrix $\mathbf{E}(z)$ can be given as

$$\mathbf{E}(z) = \frac{1}{M} \cdot \mathbf{W}_M^H\, \mathrm{diag}\,[A_0(z),\, A_1(z),\, \ldots,\, A_{M-1}(z)], \quad (8)$$

whereas the synthesis polyphase matrix $\mathbf{R}(z)$ is written as

$$\begin{aligned} \mathbf{R}(z) = \mathrm{diag}\,[&F_0(z)\, z^{-(d_{\max} - d_0)},\, F_1(z)\, z^{-(d_{\max} - d_1)},\, \ldots \\ &\ldots,\, F_{M-1}(z)\, z^{-(d_{\max} - d_{M-1})}]\, \mathbf{W}_M. \quad (9) \end{aligned}$$

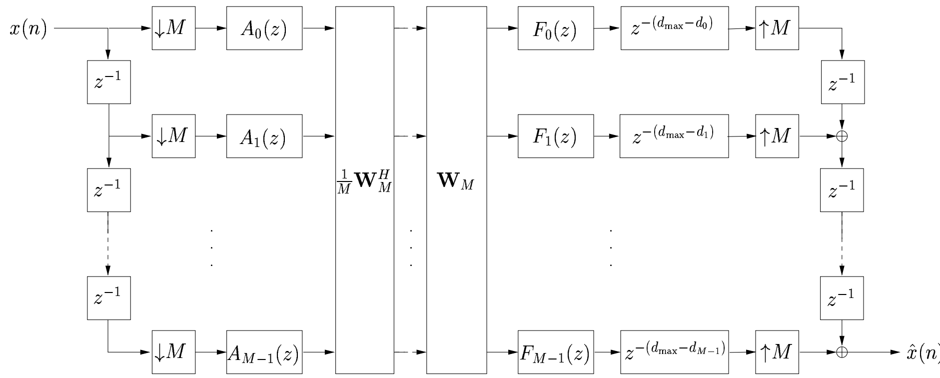

Figure 3. Critically subsampled M-band analysis-synthesis IIR / FIR DFT filter bank

With (8), (9) and $\mathbf{W}_M^H \mathbf{W}_M = M\,\mathbf{I}$ the product $\mathbf{S}(z) = \mathbf{R}(z)\,\mathbf{E}(z)$ finally yields the diagonal matrix

$$\mathbf{S}(z) = \text{diag}\,[A_0(z)\,F_0(z)\,z^{-(d_{\max}-d_0)}, \ldots \\ \ldots,\, A_{M-1}(z)\,F_{M-1}(z)\,z^{-(d_{\max}-d_{M-1})}], \quad (10)$$

where the corresponding simplified system is depicted in Fig. 4. If all diagonal elements in $\mathbf{S}(z)$ are equal (which is a special case of a pseudocirculant matrix) it is a well known fact that the filter bank is free from aliasing [7]. Furthermore, if those elements are equal to a delay, the filter bank is a PR system. However, since all diagonal elements of $\mathbf{S}(z)$ in (10) are almost equal and represent approximate delays, we can say that the filter bank is close to PR, or with other words, it represents a near-PR system.

Using (10), the input-output relation can now be obtained as

$$\begin{aligned}\hat{X}(z) &= \frac{z^{-(M-1)}}{M} \sum_{l=0}^{M-1} X(zW_M^l)\cdot \\ &\quad \cdot \sum_{k=0}^{M-1} A_k(z^M)\,F_k(z^M)\,z^{-M(d_{\max}-d_k)}\,W_M^{-kl} \\ &= \frac{z^{-(M-1)}}{M} \sum_{l=0}^{M-1} X(zW_M^l)\cdot \\ &\quad \cdot \sum_{k=0}^{M-1} (z^{-M\,d_k} - (-1)^{d_k} a_k^{d_k}) z^{-M(d_{\max}-d_k)}\,W_M^{-kl}.\end{aligned} \quad (11)$$

The linear distortion transfer function $T_{\text{lin}}(z)$ can be derived from (11) for $l = 0$ according to

$$T_{\text{lin}}(z) = z^{-M\,d_{\max}-M+1} + \\ + \underbrace{\frac{z^{-(M-1)}}{M} \sum_{k=0}^{M-1} (-1)^{d_k+1} a_k^{d_k}\, z^{-M\,(d_{\max}-d_k)}}_{=:\,E_{\text{lin}}(z)}, \quad (12)$$

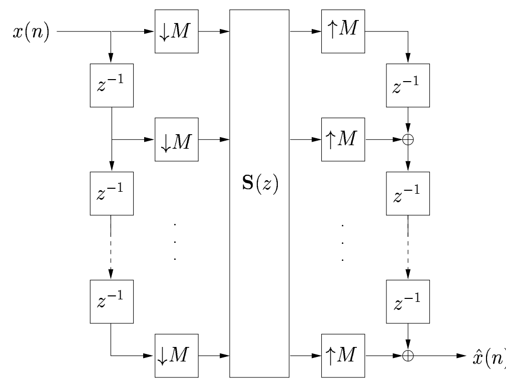

Figure 4. Simplified system from Fig. 3

where $E_{\text{lin}}(z)$ denotes the transfer function of the remaining linear distortion error. By appropriate selection of the parameters d_k the linear distortion error tends to zero, i.e. $\varepsilon(a_k, d_k) \approx 0$. Besides, we can see from (11) that $T_{\text{lin}}(z)$ has approximately linear phase, and the overall system delay is $D = M\,d_{\max} + M - 1$.

The transfer function $U_l(z)$, $l = 1, 2, \ldots, M-1$ of the l-th aliasing component $X(zW_M^l)$ can be likewise obtained from (11) as

$$U_l(z) = \frac{z^{-(M-1)}}{M} \sum_{k=0}^{M-1} (-1)^{d_k+1} a_k^{d_k}\cdot \\ \cdot\, z^{-M\,(d_{\max}-d_k)} W_M^{-kl}. \quad (13)$$

It is evident that the amount of aliasing distortion only depends on the parameters a_k and the appropriate choice of the delays d_k. This choice is arbitrary and we can select the d_k such that the aliasing frequency responses approach zero at all frequencies.

Thus, all distortions in the reconstructed signal are due to the phase-compensation error $\varepsilon(a_k, d_k)$ in (6). Amplitude and phase distortion as well as aliasing can be made arbitrarily small and can be exchanged for an increase of the overall system delay and vice versa.

The resulting distortions can be further reduced when we use the maximal order $d_{\max}$ for all FIR phase-compensation filters, i.e. $d_k = d_{\max}$ for all $k = 0, 1 \ldots, M-1$. In this case eq. (6) writes

$$A_k(z)\, F_k(z) = z^{-d_{\max}} - \underbrace{(-1)^{d_{\max}}\, a_k^{d_{\max}}}_{=\varepsilon(a_k, d_{\max})}$$

with $|\varepsilon(a_k, d_{\max})| \leq |\varepsilon(a_k, d_k)|$. Furthermore, the extra delay blocks in the synthesis filter bank (Fig. 3) vanish.

4.3 Higher order allpasses

The proposed phase-compensation approach can be extended to higher-order allpass filters. An N-th order allpass function can be in general written as a product of N first order allpass functions [7]. Therefore, we now choose the k-th polyphase component of the analysis filter bank as an allpass filter of N_k-th order according to

$$A_k(z) = c_k \cdot \prod_{\rho=1}^{N_k} \frac{z^{-1} + a_{k,\rho}}{1 + a_{k,\rho}\, z^{-1}}, \quad c_k \neq 0, \quad 0 < a_{k,\rho} < 1, \tag{14}$$

where c_k is some constant, and $a_{k,\rho}$ denotes the coefficient of the ρ-th first order allpass filter in the k-th polyphase branch. For simplicity reasons we restrict ourselves to real valued coefficients $a_{k,\rho}$. By cascading the corresponding filters from (5) in the synthesis filter bank we derive an FIR filter $F_k(z)$ of order $\sum_{\rho=1}^{N_k} d_{k,\rho}$ according to

$$F_k(z) = \frac{1}{c_k} \cdot \prod_{\rho=1}^{N_k} F_{k,\rho}(z) \tag{15}$$

with $d_{k,\rho}$ denoting the order of the individual FIR filters $F_{k,\rho}(z)$. By multiplying (14) and (15) we finally get

$$\begin{aligned} A_k(z)\, F_k(z) &= \prod_{\rho=1}^{N_k} A_{k,\rho}(z)\, F_{k,\rho}(z) \\ &= z^{-\sum_{\rho=1}^{N_k} d_{k,\rho}} + \varepsilon_k(a_{k,1}, \ldots, a_{k,N_k}, d_{k,1}, \ldots, d_{k,N_k}). \end{aligned} \tag{16}$$

Herein, $\varepsilon_k(a_{k,1}, \ldots, a_{k,N_k}, d_{k,1}, \ldots, d_{k,N_k})$ specifies the error caused by the non-ideal phase compensation. The expressions for the linear distortion and aliasing transfer functions can be derived in a straightforward way as in the first-order case discussed above.

5. Design Example

In the following design example we use the first-order allpass-based analysis prototype from Section 2 with parameters $M = 8$ and $\omega_p = 0.6\,\pi/8$ and the magnitude frequency response from Fig. 2. By choosing the synthesis polyphase components according to (5) with $d_{\max} = 37$ we obtain the results depicted in Fig. 5. Fig. 5(a) shows the overall magnitude amplitude distortion $|T(e^{j\omega})|$ of the resulting analysis-synthesis system, Fig. 5(b) the peak aliasing distortion

$$E_a(\omega) = \sqrt{\sum_{l=1}^{M-1} |U_l(e^{j\omega})|^2},$$

and Fig. 5(c) the group delay, respectively. We can see that the linear distortion transfer function has almost linear phase with an average system delay of $D = 303$ samples. In this example the choice of the d_k reflects a good

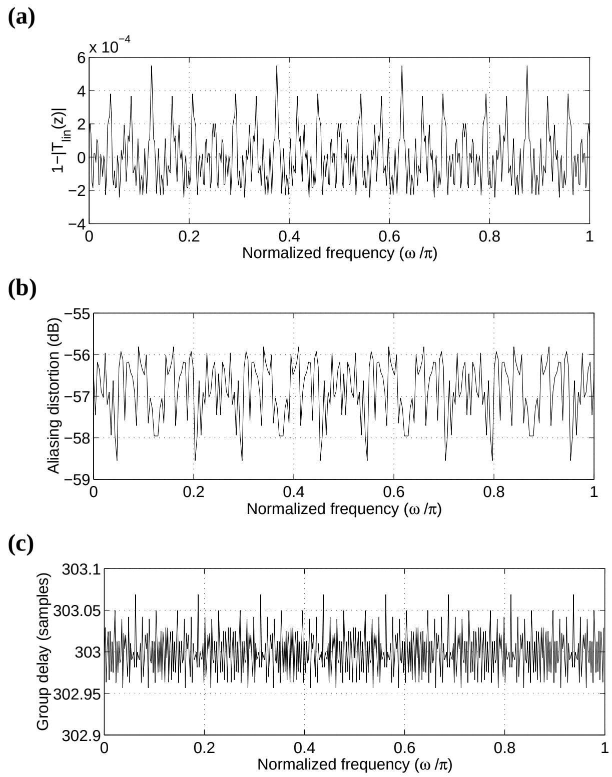

Figure 5. Overall analysis-synthesis system ($M = 8$, $d_{\max} = 37$): (a) amplitude distortion, (b) peak aliasing distortion, (c) group delay

compromise between low complexity on the one hand, and sufficient suppression of aliasing and linear distortions on the other hand.

The normalized magnitude frequency response of the lowpass synthesis prototype $Q(z)$ is shown in Fig. 6. Note that this frequency response is strongly similar to the one of the analysis prototype in Fig. 2.

In the second example displayed in Fig. 7 the same allpass-based analysis prototype as in the example from Fig. 5 is used, but the synthesis polyphase components are now obtained from (5) with $d_k = d_{\max} = 37$ for all k. It can be observed that all distortions are reduced, whereas the overall delay remains the same. All aliasing magnitude frequency responses $|U_l(e^{j\omega})|$ are constant functions, which

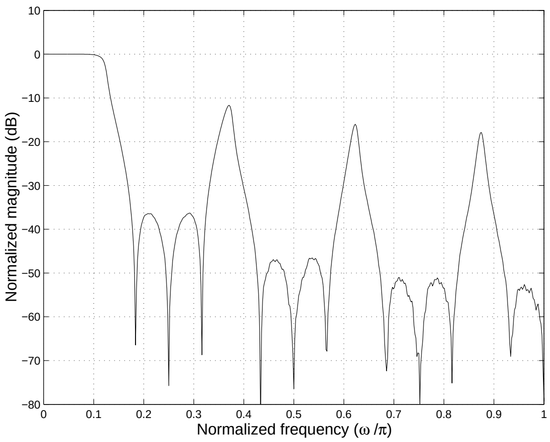

Figure 6. Normalized magnitude frequency response for the lowpass synthesis prototype ($M = 8$, $d_{\max} = 37$)

can be shown from (13) in a straightforward way.

6. Conclusion

We have proposed a new near-PR design approach for the allpass- / FIR-based critically subsampled DFT filter bank leading to stable subband filters. The novel synthesis filter bank only employs FIR polyphase components, which almost eliminate the phase distortion introduced by the allpass polyphase components on the analysis side. Furthermore, the reconstruction error only depends on the maximally allowable overall system delay. The proposed compensation technique can also be extended to higher order allpass filters in the analysis bank by concatenating allpass-filters on the analysis and the corresponding FIR phase-compensation filters on the synthesis side, respectively. Another advantage is that by exchanging the role of the analysis and synthesis filter bank our phase-compensation approach can also be applied to the analysis side. We then obtain a filter bank system having an efficient allpass-based synthesis filter bank, which is especially well suited for low-complexity decoder-only applications.

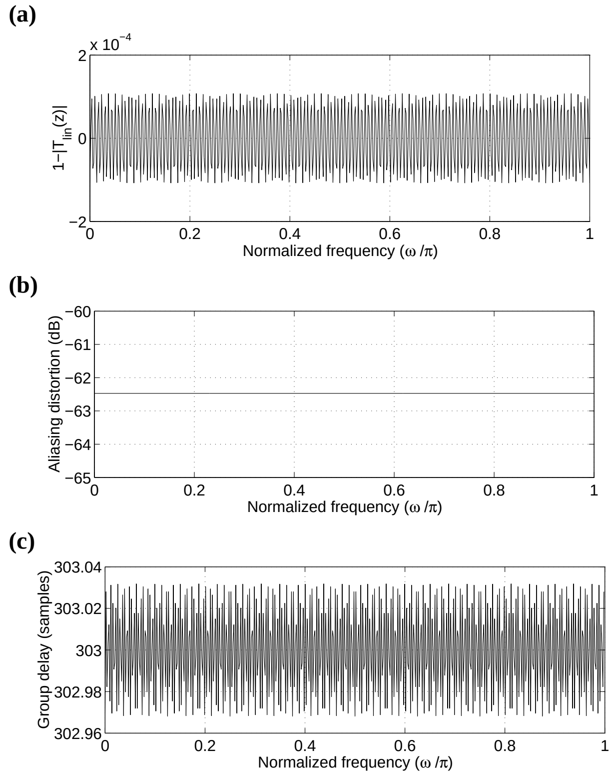

Figure 7. Overall analysis-synthesis system with $d_k = d_{\max} = 37$, $k = 0, 1, \ldots, M-1$: (a) amplitude distortion, (b) peak aliasing distortion, (c) group delay

References

[1] J. H. Husøy and T. A. Ramstad. Application of an efficient parallel IIR filter bank to image subband coding. *Signal Processing*, 20(4):279–292, August 1990.

[2] T. Chen and P. P. Vaidyanathan. General theory of time-reversed inversion for perfect reconstruction filter banks. In *Proc. 26th Asilomar Conference on Signals, Systems and Computers*, pages 821–825, Pacific Grove, CA, USA, October 1992.

[3] T. A. Ramstad. IIR filterbank for subband coding of images. In *Proc. IEEE Int. Sympos. Circuits and Systems*, pages 827–830, Helsinki, Finland, June 1988.

[4] S. K. Mitra, C. D. Creusere, and H. Babic. A novel implementation of perfect reconstruction QMF banks using IIR filters for infinite length signals. In *Proc. IEEE Int. Sympos. Circuits and Systems*, pages 2312–2315, San Diego, USA, May 1992.

[5] M. Renfors and T. Saramäki. Recursive Nth-band digital filters-Part I: Design and properties. *IEEE Trans. on Circuits and Systems*, CAS-34(1):24–37, January 1987.

[6] E. Galijašević and J. Kliewer. Non-uniform near-perfect-reconstruction oversampled DFT filter banks based on allpass-transforms. In *Proc. Ninth IEEE DSP Workshop (DSP 2000)*, Hunt, TX, USA, October 2000.

[7] P. P. Vaidyanathan. *Multirate Systems and Filter Banks*. Prentice Hall, Englewood Cliffs, 1993.