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Tuned Turbo Codes

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Abstract

As implied by previous studies, there exists a fundamental trade-off between the minimum distance and the iterative decoding convergence behavior of a turbo code. While capacity achieving code ensembles typically are asymptotically bad in the sense that their minimum distance does not grow linearly with block length and they therefore exhibit an error floor at medium to high signal to noise ratios, asymptotically good codes usually converge further away from channel capacity. In this paper we present so-called tuned turbo codes, a family of asymptotically good hybrid concatenated code ensembles, where minimum distance growths and convergence thresholds can be traded-off using a tuning parameter λ. By decreasing λ, the asymptotic minimum distance growth rate coefficient is reduced for the sake of improved iterative decoding convergence behavior, and thus the code performance can be tuned to fit the desired application.

1. INTRODUCTION

Turbo codes [1] and multiple parallel concatenated codes (MPCCs) can perform very close to the Shannon limit, but the corresponding code ensembles are asymptotically bad in the sense that their minimum distance does not grow linearly with block length [2]. As a result, their minimum distance may not be sufficient to yield very low bit error rates at moderate to high signal to noise ratios (SNRs), and an error floor can occur.

On the other hand, multiple serially concatenated code (MSCC) ensembles with three or more component encoders can be asymptotically good. This has been shown for repeat multiple accumulate codes in [3] and [4], where the method in [4] allows the exact calculation of the distance growth rate coefficient. MSCCs in general exhibit good error floor performance due to their large minimum distance, but they have the drawback of converging at an SNR further from capacity than parallel concatenated codes. While the distance growth rate coefficient of MSCCs can be made arbitrarily close to the Gilbert Varshamov Bound (GVB) by adding more concatenation stages [4], the iterative decoding convergence behavior of the resulting code ensembles becomes worse, making codes with more than three concatenation stages impractical.

An alternative to the above discussed schemes are hybrid concatenated codes (HCCs) [5]. They combine the features of parallel and serially concatenated codes and thus offer more freedom in code design. It has been demonstrated in [6] that HCCs can be designed which perform closer to capacity than MSCCs while still maintaining a minimum distance that grows linearly with block length. In particular, small memory one component encoders are sufficient to yield asymptotically good code ensembles for such schemes. The resulting codes provide low complexity encoding and decoding, and, in many cases, can be decoded using relatively few iterations.

The HCCs presented in [6] consist of an outer MPCC serially concatenated with an inner accumulator. In this paper, we further elaborate on this code structure to create a family of codes where the mini-

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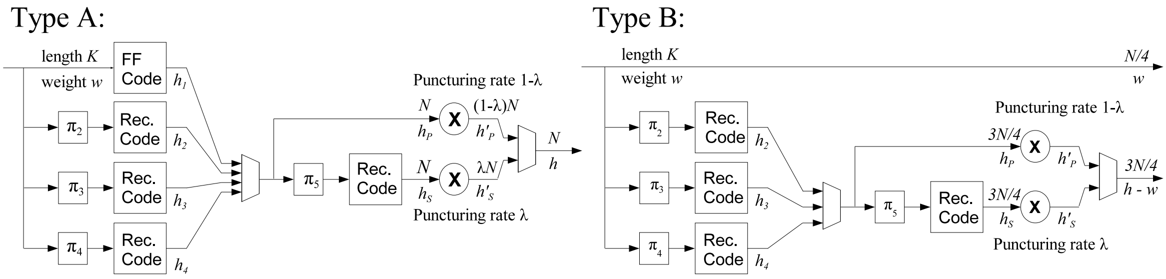

Figure 1: Encoder structure for rate $R = 1/4$ tuned turbo codes with rate 1 feedforward and recursive convolutional component encoders.

mum distance growth rate and the convergence threshold can be adjusted by varying a tuning parameter λ. In particular, we replace a fraction $1-\lambda$ of the bits at the output of the inner accumualtor with bits taken from the output of the outer MPCC. This leads to a smaller distance growth rate coefficient for decreasing λ but also to a better iterative decoding threshold. The resulting code ensemble remains asymptotically good over the range of all positive values of λ. We call this family of codes *tuned turbo codes*.

2. ENCODER STRUCTURE AND WEIGHT ENUMERATORS

The proposed tuned turbo code constructions are shown in Fig. 1, where two different encoder structures (Types A and B), consisting of an outer MPCC serially concatenated with an inner rate-1 recursive convolutional encoder, are considered. At the output of the inner serially concatenated encoder puncturing is applied. We denote the puncturing ratio by λ, which represents the fraction of bits that survive after puncturing. The fraction $1-\lambda$ of punctured bits from the inner code are replaced by an equal number of bits that survive after puncturing the output of the the outer MPCC. In this way, the code rate R is maintained at 1/4. Clearly, for $\lambda = 1$, the code will be equal to the HCC in [6], while for $\lambda = 0$, the code is the outer MPCC. In order to perform the asymptotic minimum distance analysis, the outputs of the MPCC and the inner code are punctured randomly and independently.

In the case of the Type A code ensemble, the first component encoder is a feedforward convolutional encoder and all the branches from the outer MPCC enter the inner recursive encoder, while in the case of the Type B code ensemble only the outputs from the three recursive encoders enter the inner encoder, leading to a fully systematic code. In [6], only 2-state convolutional encoders were considered, with the recursive encoder being the accumulator with generator polynomial $[1/3]_8$ in octal notation, where the most significant bit corresponds to the largest power of the delay variable D, and the feedforward encoder having generator polynomial $[3]_8$. In addition, in this paper we consider 4-state component encoders with generator polynomial $[5/7]_8$ for the recursive encoders and $[5]_8$ for the feedforward encoder. In the 2-state case, the inner encoder is also the accumulator, while in the 4-state case it is a $[1/7]_8$ encoder.

While the Type B ensemble with 2-state component encoders has the best convergence behavior among the HCCs discussed in [6], the outer MPCC of the Type A ensemble (with 2-state encoders), introduced in [7], exhibits an excellent iterative decoding threshold due to the presence of the feedforward encoder (see [8]).

The weight spectrum of a code ensemble is described by its ensemble-average weight enumerating function (WEF), obtained by using the uniform interleaver analysis [9]. The input-output WEF (IOWEF) $\bar{A}^{\mathcal{C}}_{w,h}$ represents the expected number of codewords of weight h resulting from an input block of weight w if a code is randomly chosen from the ensemble $\mathcal{C}$.

The IOWEF of an HCC can be obtained in the following most general way [6]. Let an HCC consist of L component encoders, which represent the different subcomponents of its structure, and $L-1$ interleavers. After termination, the lth component code is an (N_l, K_l) linear block code and every encoder $\mathcal{C}_l$, except $\mathcal{C}_1$, which is directly connected to the input, is preceded by a uniform random interleaver π_l. Without loss of generality, we assume that encoder $\mathcal{C}_L$ is always connected to the channel. Finally, we take the set $\{1, 2, \cdots, L-1\}$ and separate it into two disjoint sets: the set $\mathcal{S}_O$ of all indices l for which component encoder $\mathcal{C}_l$ is connected to the channel and the set $\bar{\mathcal{S}_O}$, its complement. The average IOWEF of an HCC is then given by

$$\begin{aligned}\bar{A}^{\mathcal{C}_{hyb}}_{w,h} &= \sum_{h_1=1}^{N_1} \cdots \sum_{h_{L-1}=1}^{N_{L-1}} A^{\mathcal{C}_1}_{w,h_1} \frac{A^{\mathcal{C}_L}_{w_L, h-\sum_{l\in\mathcal{S}_O} h_l}}{\binom{K_L}{w_L}} \prod_{l=2}^{L-1} \frac{A^{\mathcal{C}_l}_{w_l,h_l}}{\binom{K_l}{w_l}} \\ &= \sum_{h_1=1}^{N_1} \cdots \sum_{h_{L-1}=1}^{N_{L-1}} \bar{A}_{w,h_1,\cdots,h_{L-1},h}, \end{aligned} \tag{1}$$

where we call the quantity $\bar{A}_{w,h_1,\cdots,h_{L-1},h}$, with the output weights of each component encoder fixed, the average conditional weight enumerating function (CWEF). In the case of tuned turbo codes, the component encoders may be punctured, and we have to apply (1) recursively. The IOWEF of a randomly punctured component code $\mathcal{C}$ with input weight w and codeword weight h' after puncturing is given by

$$\bar{A}^{\mathcal{C}_{punct.}}_{w,h'} = \sum_{h=h'}^{N} \bar{A}^{\mathcal{C}}_{w,h} \frac{\binom{h}{h'}\binom{N-h}{\lambda N-h'}}{\binom{N}{\lambda N}}, \tag{2}$$

where N is the block length before puncturing, h is the codeword weight before puncturing, and λ is the fraction of bits that survive puncturing.

As an example, when computing the CWEF of the Type B tuned turbo code using (1), we consider 3 component encoders. The systematic branch is the first component encoder $\mathcal{C}_1$, with its IOWEF given by $A^{\mathcal{C}_1}_{w,w} = \binom{K}{w}$ and zero for all output weights $h \neq w$. The punctured outer MPCC is the second component encoder, and its average IOWEF $\bar{A}^{MPCC-punct.}_{w,h'_p}$ is obtained from (2), where the IOWEF $\bar{A}^{MPCC}_{w,h_p}$ of the unpunctured MPCC is obtained by using (1). (To simplify the notation, we denote the output weight of the outer MPCC as $h_p = h_2 + h_3 + h_4$.) The third component encoder is the punctured inner serially concatenated code. Its average IOWEF is also obtained from (2). For 2-state component encoders, the unpunctured IOWEFs are given in closed form as [10]

$$A^{\frac{1}{1+D}}_{w,h} = A^{1+D}_{h,w} = \binom{N-h}{\lfloor w/2 \rfloor}\binom{h-1}{\lceil w/2 \rceil - 1}. \tag{3}$$

The systematic branch, the punctured MPCC, and the punctured inner serially concatenated code are all connected to the channel, so the set $\bar{\mathcal{S}_O} = \{1, 2\}$. In summary, the CWEF of the 2-state type B ensemble is given by

$$\bar{A}^{\mathcal{C}_{typeB}}_{w,\cdots,h} = \frac{\prod_{l=2}^{4}\binom{K-h_l}{\lfloor w/2 \rfloor}\binom{h_l-1}{\lceil w/2 \rceil-1}}{\binom{K}{w}^2} \cdot \frac{\binom{h_p}{h'_p}\binom{3K-h_p}{3(1-\lambda)K-h'_p}}{\binom{3K}{3(1-\lambda)K}} \cdot \frac{\binom{3K-h_s}{\lfloor h_p/2 \rfloor}\binom{h_s-1}{\lceil h_p/2 \rceil-1}}{\binom{3K}{h_p}} \cdot \frac{\binom{h_s}{h-h'_p-w}\binom{3K-h_s}{3\lambda K-h+h'_p+w}}{\binom{3K}{3\lambda K}}. \tag{4}$$

The ensemble-average WEF can be used to upper bound the average bit error rate (BER) of a code using the union bound,

$$P_b \leq \frac{1}{2}\sum_{h=1}^{N}\sum_{w=1}^{K}\frac{w}{K}\bar{A}^{\mathcal{C}_{hyb}}_{w,h}\,\mathrm{erfc}\left(\sqrt{\frac{hRE_b}{N_0}}\right), \tag{5}$$

where E_b/N_0 is the SNR of an additive white Gaussian noise (AWGN) channel, and we have assumed BPSK modulation.

3. ASYMPTOTIC ANALYSIS AND CONVERGENCE THRESHOLD

The performance of a turbo code in the moderate to high SNR region is determined by its minimum distance. To investigate the asymptotic minimum distance properties of tuned turbo codes as the block length N tends to infinity we use the spectral shape function [11]. The spectral shape of a code ensemble is defined as

$$r(\rho) = \lim_{N\to\infty}\frac{\log \bar{A}^{\mathcal{C}}_{\rho N}}{N}, \tag{6}$$

where $\rho = \frac{h}{N}$ is the normalized codeword weight. When $r(\rho) < 0$, the average number of codewords with normalized weight ρ goes exponentially to zero as N tends to infinity. We now focus on 2-state component encoders since their unpunctured IOWEFs can be given in closed form, as noted in (3). For large block lengths N, we can approximate the binomial coefficients using Stirling's approximation,

$$\binom{n}{k} \overset{n\to\infty}{\longrightarrow} e^{n\mathbb{H}(k/n)}, \tag{7}$$

where $\mathbb{H}(x) = -x\ln x - (1-x)\ln(1-x)$ denotes the binary entropy function with the natural logarithm. Using (7) we can then write the CWEF of a 2-state tuned turbo code as

$$\bar{A}_{w,h_1,\cdots,h'_p,h,} = \exp\left\{f(\alpha,\beta_1,\cdots,\rho'_p,\rho)\,N + o(N)\right\}, \tag{8}$$

where $\alpha = \frac{w}{K}$ is the normalized input weight, $\beta_l = \frac{h_l}{N_l}$ denotes the normalized output weights of the component encoders, and $\rho'_p = \frac{h'_p}{N}$ is the normalized weight of the outer MPCC after puncturing. The spectral shape function can now be written as

$$r(\rho) = \sup_{0<\alpha,\beta_1,\cdots,\rho'_p\leq 1} f(\alpha,\beta_1,\cdots,\rho'_p,\rho). \tag{9}$$

If the function $f(\cdots)$ is strictly negative for all possible parameters $0 < \alpha, \beta_1, \cdots, \rho'_p \leq 1$, then almost all codes in the ensemble do not contain a codeword with normalized weight ρ. Further, if $f(\cdots)$ is stricly negative for all ρ, $0 < \rho < \rho_0$, and has a positive supremum for $\rho > \rho_0$, it follows that almost all codes in the ensemble have a minimum distance of at least $\rho_0 N$ as the block length N tends to infinity. Thus, the analysis of the asymptotic minimum distance properties of 2-state tuned turbo code ensembles is an optimization problem, where one must maximize the asymptotic form of the WEF over the normalized weights.

While the minimum distance of a turbo code ensemble determines its error floor behavior at moderate to high SNRs, it provides no information about the code performance in the waterfall region of the BER curve. To determine the iterative decoding threshold we employ an extrinsic information transfer (EXIT) chart-based analysis [12]. To generate EXIT charts that are

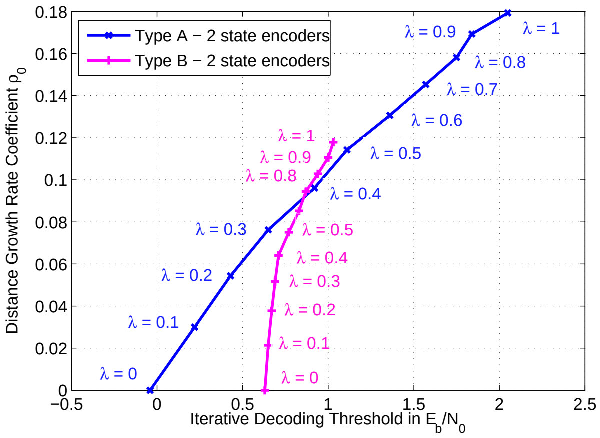

Figure 2: Asymptotic minimum distance growth rate coefficient ρ_0 versus the iterative decoding convergence threshold for the code ensembles in Fig. 1 with 2-state component encoders.

easy to interpret, the EXIT functions of the component encoders of the outer MPCCs in Fig. 1 can be combined to obtain the EXIT function of the MPCC. Thus, the behavior of the code structures considered in this paper can be determined by using a two-dimensional EXIT chart, displaying in a single figure the EXIT functions of the outer MPCC and the inner recursive encoder. To compute the EXIT function of the MPCCs, we follow the approach suggested in [13].

For 2-state component encoders, Fig. 2 shows the asymptotic minimum distance growth rate coefficient ρ_0 versus the iterative decoding convergence threshold for specific values of the parameter λ. For $\lambda = 1$, the Type A ensemble exhibits a distance growth rate coefficient of 0.1793 and a threshold of $E_b/N_0 = 2.05\,\text{dB}$. Decreasing λ leads to better convergence properties, but also to a reduction of the growth rate coefficient. In the extreme case of $\lambda = 0$, the code is equal to the outer MPCC consisting of a parallel concatenation of three accumulators and one feedforward encoder. It can be seen that the presence of the feedforward encoder results in a low iterative decoding threshold of $E_b/N_0 = -0.04\,\text{dB}$. However, the minimum distance of an MPCC grows sublinearly in N [2], and thus the asymptotic distance growth rate coefficient is zero.

It is interesting to note that we did not observe a specific λ for which linear distance growth property breaks down, i.e., even a small fraction of bits in the overall code originating from the HCC is sufficient to maintain an asymptotically good code ensemble.

For $\lambda = 1$, the Type B ensemble exhibits a better threshold ($E_b/N_0 = 1.03\,\text{dB}$) but a lower distance growth rate coefficient (0.1179) than the Type A ensemble. On the other hand, the threshold of the outer MPCC in the Type B ensemble is $E_b/N_0 = 0.63\,\text{dB}$ for $\lambda = 0$, which is much worse than the threshold of the Type A ensemble. Also, we can see from Fig. 2 that the dynamic range over which the Type B ensemble can be adjusted is thus only 0.4 dB, whereas the Type A ensemble can be tuned over a range of more than 2 dB. This indicates that in the design of tuned turbo codes it is important to use an outer MPCC with very good convergence properties.

Table 1: Iterative decoding thresholds for the Type A and B code constructions of Fig. 1 using 4-state component encoders.

λ	Type A	Type B
0	0.92 dB	-0.03 dB
0.1	1.35 dB	0.36 dB
0.2	1.95 dB	0.53 dB
0.3	2.66 dB	0.87 dB
0.4	3.28 dB	1.19 dB
0.5	3.84 dB	1.50 dB
0.6	4.48 dB	1.80 dB
0.7	4.98 dB	2.10 dB
0.8	5.38 dB	2.24 dB
0.9	5.48 dB	2.52 dB
1	5.61 dB	2.65 dB

The iterative decoding thresholds for tuned turbo codes with 4-state component encoders are given in Table 1. Clearly, for 4-state component encoders, the feedforward encoder does not have the same effect as in the 2-state version of the Type A ensemble, and thus this structure is unpractical due to its poor iterative decoding convergence threshold. However, the systematic Type B ensemble exhibits very good iterative decoding convergence behavior and a wide dynamic range. We were not able to perform an asymptotic minimum distance analysis in the case of 4-state component encoders, though, so we relied on a finite length minimum distance analysis.

4. FINITE LENGTH MINIMUM DISTANCE ANALYSIS

The minimum distance of a code ensemble for a finite block length N can also be analyzed using the average WEF. The probability that a randomly chosen code from the ensemble has minimum distance $d_{\min} < d$ is upper bounded by

$$\mathbb{P}\left[d_{\min} < d\right] \leq (\bar{A}_0^{\mathcal{C}} - 1) + \sum_{h=1}^{d-1} \bar{A}_h^{\mathcal{C}}. \qquad (10)$$

By fixing $\mathbb{P}\left[d_{\min} < d\right] = 1/2$ and evaluating (10), we expect at least half of the codes in the ensemble to have a minimum distance of at least $d_{\min}$. For 2-state component encoders, Fig. 3 shows the resulting lower bound on the minimum distance of the Type A and Type B code ensembles for codeword lengths up to $N = 1000$

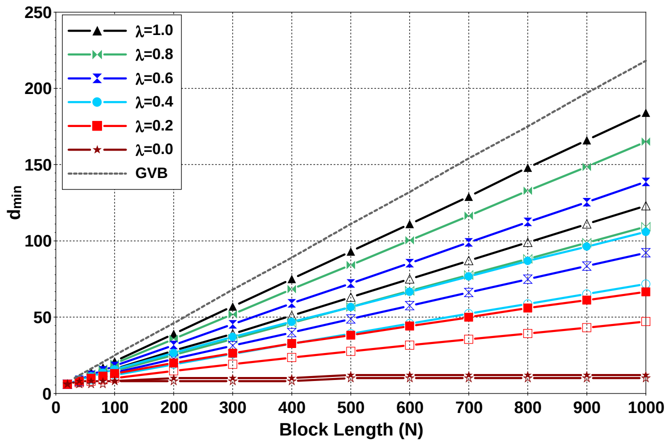

Figure 3: Lower bound on the minimum distance of Type A (filled markers) and Type B (empty markers) tuned turbo codes with 2-state component encoders for different values of the tuning parameter λ.

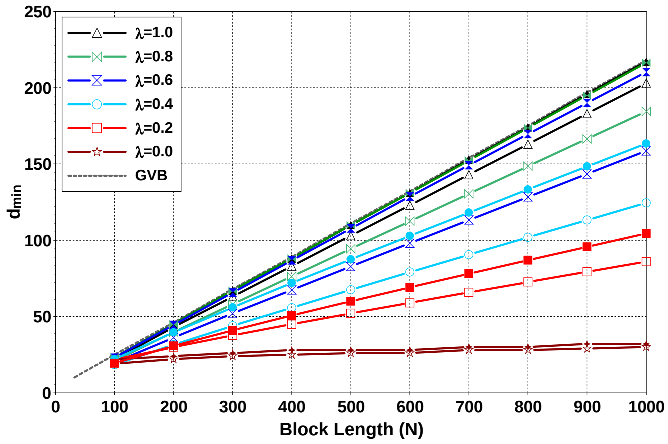

Figure 4: Lower bound on the minimum distance of Type A (filled markers) and Type B (empty markers) tuned turbo codes with 4-state component encoders for different values of the tuning parameter λ.

and several values of λ. The finite length GVB is also plotted for reference. The results are consistent with the asymptotic analysis in Section 3 and show increasing minimum distance growth rates for both code constructions with increasing λ. Also, for a given value of the tuning parameter λ, the minimum distance of the Type A code ensemble is larger than for the Type B code ensemble.

The lower bounds on the minimum distances for 4-state component encoders are shown in Fig. 4. For a given value of λ, the minimum distances for the 4-state ensembles are larger than the minimum distances of their 2-state counterparts. The 4-state Type B ensemble also has a larger minimum distance than the 2-state Type A ensemble, while both ensembles have similar convergence thresholds for small values of λ.

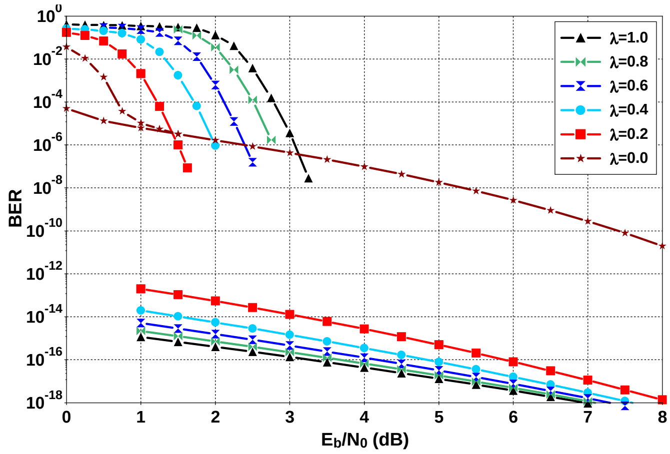

Figure 5: Bit error rate performance of Type A tuned turbo codes with 2-state component encoders for different values of the parameter λ.

The minimum distances of the 4-state Type A ensemble in Fig. 4 are very high, and they are very close to the GVB for $\lambda \geq 0.6$. However, as discussed in the previous section, this ensemble suffers from a poor iterative decoding threshold.

Since the finite block length minimum distance of the code ensembles increases when we replace 2-state component encoders by more complex 4-state component encoders, we strongly conjecture that HCC code ensembles with the same structure using more powerful component encoders are also asymptotically good.

5. SIMULATION RESULTS

In Fig. 5, BER curves for Type A codes with 2-state component encoders are displayed for parameters $\lambda \in [0, 1]$, together with the union bounds on error probability. The information block length is $K = 1024$ bits and random interleavers are employed. The Type A code with $\lambda = 1$ shows the worst convergence, but according to the minimum distance analysis it has the best minimum distance growth, resulting in the lowest error floor. On the other hand, the Type A code with $\lambda = 0$ performs best in the waterfall region, but it has a high error floor due to its poor minimum distance. In this case, the code is equivalent to the MPCC in [7]. By tuning λ, we can obtain any behavior in between these two extreme cases; by decreasing λ the convergence of the code is improved (the curves get closer to the curve of the MPCC), but the error floor is higher. A similar behavior is observed in Figure 6, where BER curves and union bounds on error probability for Type B codes with 4-state component encoders are shown. Again, by varying the tuning paramenter λ, we can obtain any behavior in between the outer MPCC ($\lambda = 0$), which has the best iterative decoding convergence

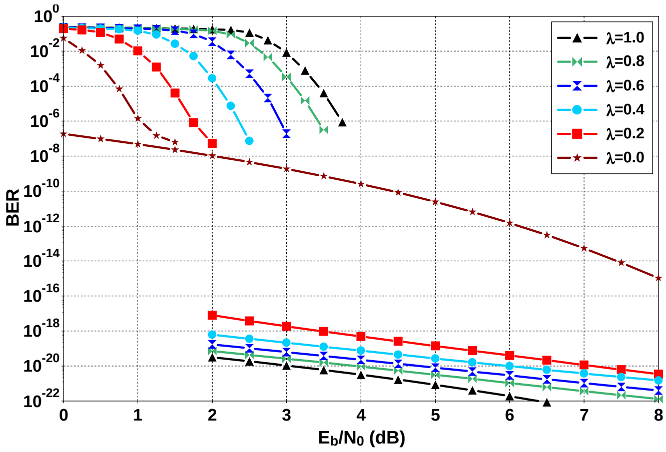

Figure 6: Bit error rate performance of Type B tuned turbo codes with 4-state component encoders for different values of the parameter λ.

behavior, and the HCC ($\lambda = 1$) which has the lowest error floor. Comparing the 4-state Type B code in Fig. 6 to the 2-state Type A code in Fig. 5, both codes show similar convergence behavior for small values of λ, but for larger values of λ a degradation of $0.4 - 0.6$ dB is observed in the waterfall region of the 4-state code. However, the error floors of the 4-state code are significantly lower than for the 2-state code.

6. CONCLUSIONS

In this paper, we have introduced a family of hybrid concatenated codes where the tradeoff between minimum distance growth rate and convergence behavior can be tuned by varying a single parameter λ. By decreasing λ, the convergence behavior of the code is improved at the expense of a smaller minimum distance and a worse error-floor performance, and vice versa. Another advantage of the proposed tuned turbo code constructions is that they are asymptotically good for a large range of values of λ, so that even small values of λ are sufficient to ensure linear distance growth with block length.

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