Minimum Distance Bounds for Multiple-Serially Concatenated Code Ensembles

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***Abstract*—It has recently been shown that the minimum distance of the ensemble of repeat multiple accumulate codes grows linearly with block length. In this paper, we present a method to obtain the distance growth rate coefficient of multiple-serially concatenated code ensembles and determine the growth rate coefficient of the rate 1/2 double-serially concatenated code consisting of an outer memory one convolutional code followed by two accumulators. We compare both the growth rate of the minimum distance, as well as the convergence behavior, of this code with rate 1/2 repeat multiple accumulate codes, and we show that repeat multiple accumulate codes have better minimum distance growth but worse performance in terms of convergence.**

I. Introduction

Turbo codes are a practical solution to the problem of communicating reliably at rates close to channel capacity. Since their invention in 1993 [1], turbo codes have become a focus of research and have inspired the creation of several other turbo like codes consisting of relatively simple component codes separated by interleavers. Among these are serially concatenated codes (SCC), introduced by Benedetto et al. in 1998 [2]. While the general class of SCCs are powerful codes, even the simple structure of repeat-accumulate (RA) codes was shown to have a vanishing word error probability as the interleaver length tends to infinity [3]. While parallel concatenated codes achieve thresholds close to capacity, SCC schemes in general exhibit a lower error floor due to their better minimum distance properties. Yet upper bounds on the minimum distance of single-serially concatenated codes (SSCCs) show that the minimum distance cannot grow linearly with block length [4], [5].

For RA codes, Pfister and Siegel [6] showed that increasing the number of serially concatenated accumulators increases the minimum distance of the code. Every additional interleaver adds randomness to the code structure and the minimum distance reaches the Gilbert-Varshamov bound (GVB) as the number of interleavers approaches infinity.

This result motivates us to look at the minimum distance of multiple-serially concatenated codes (MSCCs), with particular emphasis on double-serially concatenated codes (DSCCs), also introduced in 1998 by Benedetto et al. [7]. Apart from distance properties superior to those of SSCCs, DSCCs also have a simple encoding and decoding structure and, in many cases, can be decoded using relatively few iterations. Among DSCCs, the ensemble of repeat-accumulate-accumulate (RAA) codes is the one that has been studied most extensively. Bazzi, Mahdian, and Spielman were the first to show that there exist RAA codes having a minimum distance growing linearly with block length [5] by using a similar bounding technique as in [4], but they did not compute a growth rate coefficient. Pfister presented a similar result for generalized RAA codes in his PhD thesis [8], showing that linear distance growth can be achieved with any outer code having a minimum distance larger than or equal to two. He also derived bounds on the asymptotic behavior of the weight enumerator and conjectured a value of the distance growth rate coefficient for several different code ensembles. A new proof of linear distance growth, allowing the exact calculation of the growth rate coefficient for repeat-multiple accumulate code ensembles, was the focus of a recent paper by the authors [9].

In this paper we present a method to investigate the distance growth properties of MSCCs and to obtain the growth rate coefficients of DSCC and triple serially-concatenated code (TSCC) ensembles that exhibit linear distance growth with block length. This method is then used to compute the distance growth rate coefficient of the DSCC (TSCC) consisting of a memory one, rate 1/2 convolutional code followed by two (three) accumulators, and we compare these results to the growth rate coefficients of other DSCCs and TSCCs calculated in [9].

II. Weight enumerator and spectral shape

To analyze the minimum distance of a concatenated code, we must consider the weight distribution of its component codes. Following the notation of [2] and [7], let $A^{\mathcal{C}}_{w,h}$ denote the Input-Output Weight Enumerating Function (IOWEF), which specifies the number of codewords in the code $\mathcal{C}$, truncated to length N, with input weight w and output weight h. Also, let $A^{\mathcal{C}}_{h}$ denote the Weight Enumerating Function (WEF), the number of codewords with output weight h. For a code ensemble $\mathcal{C}^N$ of length N, we then write the average CWEF as

$$\bar{A}^{\mathcal{C}}_h = \frac{1}{|\mathcal{C}^N|}\sum_{\mathcal{C}\in\mathcal{C}^N} A^{\mathcal{C}}_h. \quad (1)$$

In the same way, we can define the ensemble average of the IOWEF, $\bar{A}^{\mathcal{C}}_{w,h}$. Finally, we define the asymptotic spectral shape

This work was partly supported by NSF grants CCR02-05310 and CCF05-15012, NASA grants NNG05GH73G and NNX07AK536, German Research Foundation (DFG) grant KL 1080/3-1, and the University of Notre Dame Faculty Research Program.

978-1-4244-2571-6/08/$25.00 ©2008 IEEE

as

$$r(\rho) = \lim_{N\to\infty} \frac{\log \bar{A}^{\mathcal{C}}_{\rho N}}{N}, \qquad (2)$$

where $\rho = \frac{h}{N}$ is the normalized codeword weight. Clearly, when $r(\rho) < 0$, the average number of codewords with normalized weight ρ goes exponentially to zero as N gets large.

III. The Typical Minimum Distance of Single-Serially Concatenated Codes

The encoder of an SSCC is shown in Fig. 1. A binary input sequence u_1, with length K and weight w, enters the outer encoder C_1 and generates a codeword v_1 with weight h_1. The interleaver π_1 maps v_1 into the input sequence u_2 of the second encoder. At the output of the inner decoder we obtain a codeword of weight h_2.

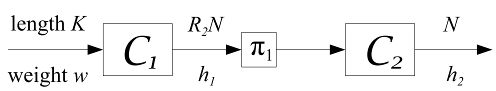

Fig. 1. Encoder for a serially concatenated code. The variables above the graph represent the length of each block and the variables below the corresponding weight.

To consider the ensemble over all possible interleavers we use the uniform interleaver approach ([2], [7]) in our analysis. The uniform interleaver of length N maps an input of weight w into all of its $\binom{N}{w}$ possible permutations with equal probability. Thus the average IOWEF $\bar{A}^{\mathcal{C}_{SSCC}}_{w,h_2}$ of an SSCC over the ensemble of all interleavers is given by

$$\bar{A}^{\mathcal{C}_{SSCC}}_{w,h_2} = \sum_{h_1=d^{\mathcal{C}_1}_{min}}^{R_2 N} \frac{A^{\mathcal{C}_1}_{w,h_1} A^{\mathcal{C}_2}_{h_1,h_2}}{\binom{R_2 N}{h_1}}, \qquad (3)$$

where $A^{\mathcal{C}_1}_{w,h_1}$ and $A^{\mathcal{C}_2}_{h_1,h_2}$ are the IOWEFs of the outer and the inner codes, respectively, and R_2 is the rate of the inner code $\mathcal{C}_2$.

The minimum distance of SSCCs was investigated by Kahale and Urbanke in [4]. Although their theorem was originally stated for the concatenation of two rate 1/2 recursive systematic convolutional codes, their lower bound on the minimum distance also holds for more general SSCCs, as long as the inner code remains recursive, as stated in the following theorem.

Theorem 1. *For a single-serially concatenated code ensemble with an outer code having minimum distance $d^{\mathcal{C}_1}_{min} \geq 3$, the minimum distance of almost all codes in the ensemble is lower bounded by*

$$d^{SSCC}_{min} \geq N^{1-\frac{2}{d^{\mathcal{C}_1}_{min}}-\epsilon}$$

as the block length N goes to infinity, where ϵ can be made arbitrarily small.

An upper bound on the minimum distance of an SSCC with an inner recursive code was derived by Perotti and Benedetto in [10] and is restated in the following theorem.

Theorem 2. *The minimum distance of a single-serially concatenated code with a recursive inner code is upper bounded by*

$$d^{SSCC}_{min} < O(N^{1-\frac{1}{d^{\mathcal{C}_1}_{min}}})$$

as the block length N goes to infinity, where $d^{\mathcal{C}_1}_{min} \geq 2$.

Since the minimum distance of the outer code is a positive constant greater than one, the minimum distance of an SSCC cannot grow linearly with block length. From Theorems 1 and 2, however, we conclude that the minimum distance of almost all SSCCs having an outer code with $d_{min} \geq 3$ grows with the block length N as $O(N^\nu)$, where $1-\frac{2}{d^{\mathcal{C}_1}_{min}}-\epsilon \leq \nu \leq 1-\frac{1}{d^{\mathcal{C}_1}_{min}}$.

IV. The Typical Minimum Distance of Multiple-Serially Concatenated Codes

As an example, the encoder of a DSCC is shown in Fig. 2. Compared to an SSCC, it has an additional concatenated encoder. A codeword of the outer encoder $\mathcal{C}_1$, having weight h_1, is permuted by the first interleaver π_1 into the input word of the middle encoder $\mathcal{C}_2$, generating a codeword having weight h_2, which in turn is permuted by the second interleaver π_2 to generate the input word for the inner encoder $\mathcal{C}_3$. We assume that both the middle and the inner code are recursive, so that the minimum distance of the first cascade grows as N^ν. The total codeword length is N and the resulting codeword weight is h. The WEF of a DSCC is given by [7]

$$\bar{A}^{\mathcal{C}_{DSCC}}_{h} = \sum_{w=1}^{K}\sum_{h_1=1}^{N_1}\sum_{h_2=1}^{N_2} \underbrace{\frac{A^{\mathcal{C}_1}_{w,h_1}\cdot A^{\mathcal{C}_2}_{h_1,h_2}\cdot A^{\mathcal{C}_3}_{h_2,h}}{\binom{R_2R_3N}{h_1}\binom{R_3N}{h_2}}}_{\bar{A}_{w,h_1,h_2,h}}, \qquad (4)$$

where, when the output weights of each component encoder are fixed, we call the quantity $\bar{A}_{w,h_1,h_2,h}$ the average weight path enumerator.

A. The asymptotic spectral shape of an MSCC

To analyze MSCCs, we use the same method used to analyze the repeat-multiple-accumulate ensemble in [9]. Here we summarize the method for DSCCs. If we randomly pick a code from an ensemble, the expected number of codewords of weight h is equal to the ensemble average WEF for weight h, given in (4). We can upper bound the summation in (4) by taking the maximum of the (triple) sum times the number of elements, i.e.,

$$\bar{A}^{\mathcal{C}_{DSCC}}_{h} \leq R_1R_2^2R_3^3N^3 \max_{0<w\leq K}\max_{0<h_1\leq N_1}\max_{0<h_2\leq N_2} \bar{A}^{\mathcal{C}_{DSCC}}_{w,h_1,h_2,h}. \qquad (5)$$

Since we are interested in the asymptotic behavior of the DSCC, the quantity of interest is the asymptotic spectral shape. If it is negative for a given $\rho = h/N$, then the number of codewords with normalized weight ρ grows slower than the block length, resulting in the probability of the code having codewords with normalized weight ρ going to zero as $N \to \infty$. Thus, the important property of $r(\rho)$, in terms of minimum distance, can be stated as follows: if the function is negative for all ρ, $0 < \rho < \rho_0$, then crosses zero and is positive for $\rho > \rho_0$, it follows that, for almost all codes in the ensemble, the minimum distance is lower bounded by $d_{\min} \geq \rho_0 N$ as the block length N tends to infinity, and ρ_0 is called the distance growth rate coefficient of the ensemble.

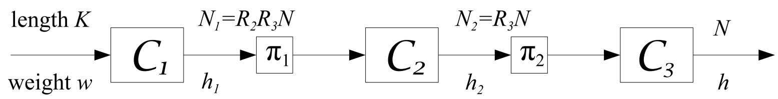

Fig. 2. Encoder for a double-serially concatenated code

We normalize the weights $\alpha = w/K$, $\beta = h_1/N_1$, $\gamma = h_2/N_2$ and $\rho = h/N$ and define the function $f^{\mathcal{C}_i}$ as the asymptotic behavior of the IOWEF of a code $\mathcal{C}_i$, i.e.,

$$f^{\mathcal{C}_i}(\lambda,\varsigma) = \lim_{N\to\infty} \frac{\log A^{\mathcal{C}_i}_{\lambda N_i, \varsigma N_o}}{N}, \tag{6}$$

where λ is the normalized input weight (normalized w.r.t the input block length N_i) and ς is the normalized output weight (normalized w.r.t the output block length N_o) of code $\mathcal{C}_i$, and N is the total block length of the DSCC. Combining (2), (4), (5), and (6), and using the following limiting expression [11] for the binomial coefficients in the denominator of (4)

$$\binom{n}{k} \overset{N\to\infty}{\longrightarrow} e^{n\mathbb{H}(k/n)}, \tag{7}$$

we can write the asymptotic spectral shape of a DSCC as

$$\begin{aligned} r(\rho) &= \max_{\alpha,\beta,\gamma} f^{\mathcal{C}_{DSCC}}(\alpha,\beta,\gamma,\rho) \\ &= \max_{0<\alpha\le 1}\max_{0<\beta\le 1}\max_{0<\gamma\le 1} f^{\mathcal{C}_1}(\alpha,\beta) + f^{\mathcal{C}_2}(\beta,\gamma) \\ &\quad + f^{\mathcal{C}_3}(\gamma,\rho) - R_2R_3\mathbb{H}(\beta) - R_3\mathbb{H}(\gamma), \end{aligned} \tag{8}$$

where $\mathbb{H}(\cdot)$ is the binary entropy function with natural logarithms, and we have used the upper bound on the WEF (5) in (2) to obtain (8). We can also lower bound the WEF by approximating the (triple) sum by its maximum term only, and using this lower bound in (2) results in the same expression for the asymptotic spectral shape. Thus, since the limiting expression of the upper bound equals the limiting expression of the lower bound, the expression for the asymptotic spectral shape given by (8) is exact.

A maximum for $f^{\mathcal{C}_{DSCC}}$ can either occur on the boundaries or in the region $\{0<\alpha<1, 0<\beta<1, 0<\gamma<1\}$. For a maximum within the region, a necessary condition is that the partial derivatives $\partial f/\partial\alpha$, $\partial f/\partial\beta$, and $\partial f/\partial\gamma$ are all zero at the maximum point. Because of the sequential nature of the encoder, the analysis can be performed sequentially, one maximization after the other.

This method is not restricted to DSCCs, but can also be applied to general MSCCs. For the outer code, the necessary condition on β is

$$\frac{d}{d\alpha} f^{\mathcal{C}_1}(\alpha,\beta) = 0, \tag{9}$$

and for a code at concatenation stage j ($j \ge 2$) with normalized input weight λ_j, normalized output weight ς_j, and a total of L concatenation stages, the necessary condition on ς_j is

$$\begin{aligned} &\frac{\partial}{\partial\lambda_j} f^{\mathcal{C}_j}(\lambda_j,\varsigma_j) + \frac{\partial}{\partial\lambda_j} f^{\mathcal{C}_{j-1}}(\lambda_{j-1},\lambda_j) \\ &\qquad + R_{[j,L]} \log\left(\frac{\lambda_j}{1-\lambda_j}\right) = 0, \end{aligned} \tag{10}$$

where the constant $R_{[j,L]}$ is defined as

$$R_{[j,L]} = \prod_{k=j}^{L} R_k. \tag{11}$$

B. Computation of the growth rate coefficient

To compute distance growth rate coefficients, we must know the asymptotic weight enumerators of the component codes. We now give examples of asymptotic weight enumerators for some simple component codes. The expressions given can then be used to obtain the asymptotic spectral shape in (8).

Example 1. *Rate 1/q repetition code.*
The relationship between the input weight and the output weight of a repetition code is deterministic and is given by the repetition factor q, and thus the normalized input weight equals the normalized output weight. Hence, the number of variables to be maximized is reduced by one, and we obtain

$$f^{Rep}(\alpha) = \frac{R_{[2,L]}}{q}\mathbb{H}(\alpha). \tag{12}$$

Example 2. *Rate 1 accumulate code.*
The asymptotic behavior of the accumulate code at stage j in an L-stage serial concatenation is

$$\begin{aligned} &f^{acc_j}(\lambda_j,\varsigma_j) = \\ &\quad R_{[j,L]}\left((1-\varsigma_j)\mathbb{H}\left(\frac{\lambda_j}{2(1-\varsigma_j)}\right) + \varsigma_j\mathbb{H}\left(\frac{\lambda_j}{2\varsigma_j}\right)\right). \end{aligned} \tag{13}$$

The resulting maximum is given by [9]

$$\varsigma_j^{max} = \frac{1}{2} - \frac{1-\lambda_j}{2}\sqrt{1-e^{-\frac{2}{R_{[j,L]}}\frac{\partial}{\partial\varsigma_j} f^{\mathcal{C}_{j-1}}(\lambda_{j-1},\lambda_j)}}, \tag{14}$$

and the derivative w.r.t the output weight, needed to solve for the maximum of the subsequent code in (10), is given by

$$\begin{aligned} &\frac{\partial}{\partial\varsigma_j} f^{acc_j}(\lambda_j,\varsigma_j) = \\ &\quad R_{[j,L]}\left(\log\left(\frac{\varsigma_j}{1-\varsigma_j}\right) - \log\left(\frac{2\varsigma_j-\lambda_j}{2-2\varsigma_j-\lambda_j}\right)\right). \end{aligned} \tag{15}$$

Example 3. *Rate 1/2 convolutional code.*
We can write the asymptotic spectral shape of the rate 1/2, memory one, convolutional code with generator matrix $[1, \frac{1}{1+D}]$ in closed form as

$$\begin{aligned} &f^{[1,\frac{1}{1+D}]_j}(\lambda_j,\varsigma_j) = \\ &\quad = R_{[j,L]}(1-2\varsigma_j+\lambda_j)\mathbb{H}\left(\frac{\lambda_j}{2(1-2\varsigma_j+\lambda_j)}\right) \\ &\quad + R_{[j,L]}(2\varsigma_j-\lambda_j)\mathbb{H}\left(\frac{\lambda_j}{2(2\varsigma_j-\lambda_j)}\right), \end{aligned} \tag{16}$$

and its derivative $\frac{\partial}{\partial\varsigma_j}$ is given by

$$\begin{aligned} &\frac{\partial}{\partial\varsigma_j} f^{[1,\frac{1}{1+D}]_j}(\lambda_j,\varsigma_j) = \\ &\quad 2R_{[j,L]}\left(\log\left(\frac{2-4\varsigma_j+\lambda_j}{2-4\varsigma_j+2\lambda_j}\right) - \log\left(\frac{4\varsigma_j-3\lambda_j}{4\varsigma_j-2\lambda_j}\right)\right). \end{aligned} \tag{17}$$

In the following we will focus on the rate 1/2 DSCC consisting of an outer rate 1/2, memory one, convolutional code with generator matrix $[1, \frac{1}{1+D}]$ followed by two rate 1 accumulators. The asymptotic behavior of this code is given by

$$\begin{aligned} f^{\mathcal{C}_{DSCC}}(\alpha,\beta,\gamma,\rho) = \frac{1-2\beta+\alpha}{2}\mathbb{H}\left(\frac{\alpha}{2(1-2\beta+\alpha)}\right) \\ +\frac{2\beta-\alpha}{2}\mathbb{H}\left(\frac{\alpha}{2(2\beta-\alpha)}\right)+(1-\gamma)\mathbb{H}\left(\frac{\beta}{2(1-\gamma)}\right) \\ +\gamma\mathbb{H}\left(\frac{\beta}{2\gamma}\right)-\mathbb{H}(\beta)+(1-\rho)\mathbb{H}\left(\frac{\gamma}{2(1-\rho)}\right) \\ +\rho\mathbb{H}\left(\frac{\gamma}{2\rho}\right)-\mathbb{H}(\gamma), \end{aligned} \tag{18}$$

and the maximizing condition is

$$\begin{aligned} \frac{\partial}{\partial\alpha}f^{\mathcal{C}_1}(\alpha,\beta) = \frac{1}{2}\log\frac{4\beta-3\alpha}{4\beta-2\alpha}+ \\ \frac{1}{4}\log\frac{4\beta-3\alpha}{2-4\beta+\alpha}-\frac{1}{2}\log\frac{\alpha}{2-4\beta+2\alpha}=0. \end{aligned} \tag{19}$$

For α in the range $0 < \alpha < 1$, we are not able to obtain a closed form solution for β in (19), but a solution for β can be found numerically. For the DSCC ensemble obtained when two accumulators follow the $[1, \frac{1}{1+D}]$ code, we can solve for γ using (17) in (14) (for the middle accumulator) and then solve for ρ using (15) in (14) (for the outer accumulator). The resulting spectral shape is shown in Fig. 3. It has a zero crossing at $\rho^* = 0.0832$, corresponding to the asymptotic growth rate coefficient of the minimum distance. So, for large N, we expect a code in this ensemble to have minimum distance $d_{min}(N) \sim 0.0832 \cdot N$. This also confirms the growth rate coefficient conjectured in [8].

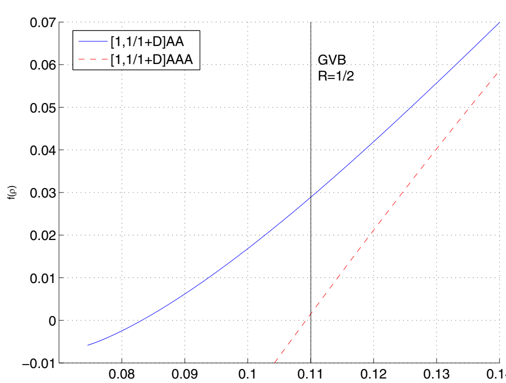

Fig. 3. Asymptotic spectral shape for a rate 1/2 DSCC ensemble

C. Random puncturing and the spectral shape

The asymptotic behavior of higher rate codes can be analyzed using random puncturing. In this case, the average number of codewords in the punctured code of weight h' is given by the summation of the WEF over all codewords of weight h, $h' \le h \le N$, times the probability that weight h is punctured to weight h', i.e.,

$$\bar{A}_{h'}^{\mathcal{C}_{DSCC_P}} = \sum_{h=h'}^{N} \bar{A}_{h}^{\mathcal{C}_{DSCC}} \mathbb{P}\left[h'|h, N'\right], \tag{20}$$

where N' is the length of the code after puncturing. Thus the analysis of the spectral shape of a randomly punctured code involves one more maximization step. The conditional probability that a codeword of weight h is punctured to weight h' is given by

$$\mathbb{P}\left[h'|h, N'\right] = \frac{\binom{h}{h'}\binom{N-h}{N'-h'}}{\binom{N}{N'}}. \tag{21}$$

We can then normalize the output weight $\rho' = h'/N'$ and denote the fraction of bits left after puncturing by $r = N'/N$. The asymptotic behavior of random puncturing is now given by

$$f^{\mathbb{P}[h'|h,N']}(\rho,\rho') = \rho\mathbb{H}\left(\frac{r\rho'}{\rho}\right)+(1-\rho)\mathbb{H}\left(\frac{r(1-\rho')}{1-\rho}\right)-\mathbb{H}(r). \tag{22}$$

Random puncturing introduces an additional level of randomness into the code, and as codes are punctured to obtain higher rates, their distance growth rate coefficients approach the GVB. For example, for the rate 1/2 RAA ensemble with a repeat-by-two outer code, denoted as the R^2AA ensemble, we did not observe a zero crossing of the spectral shape function, but only positive values for $\rho > 0.0286$ (the growth rate coefficient conjectured in [8]), as shown in Fig. 4. For the R^2AA ensemble punctured to rate 4/5, however, we are able to observe a zero crossing at $\rho_0 = 0.0262$, implying that the minimum distance of the randomly punctured R^2AA ensemble grows linearly with block length.

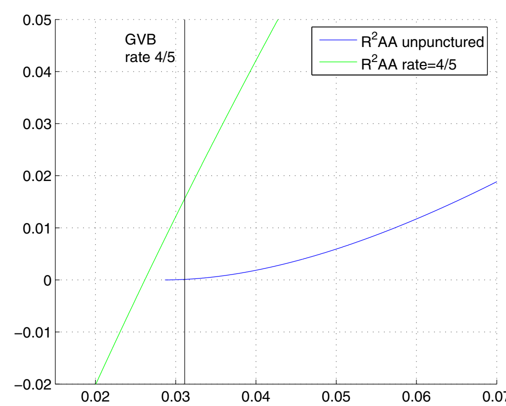

Fig. 4. Asymptotic spectral shape for the unpunctured rate 1/2 R^2AA ensemble and the same ensemble punctured to rate 4/5.

Table I lists the distance growth rate coefficients of different rate 1/2 DSCC ensembles (see also [9]). For RAA codes with repetition factors 3 or greater, we can achieve rate 1/2 by puncturing. As noted earlier, the more a code is punctured, the closer its distance growth rate coefficient is to the GVB (0.11 for rate 1/2). The strongly punctured rate 1/5 mother code exhibits the largest growth rate coefficient, followed by the punctured rate 1/4 code, and so on. The only unpunctured rate 1/2 DSCC ensemble exhibits the lowest growth rate, but it is still not far away from the GVB.

To compare the distance growth rate coefficients of the DSCCs in Table I with TSCC ensembles, we list the growth rate coefficients of two unpunctured rate 1/2 TSCC ensembles in Table II. We see that adding one more concatenation stage

TABLE I
DISTANCE GROWTH RATE COEFFICIENT ρ_0 FOR DIFFERENT RATE 1/2 DSCCS.

Code	ρ_0
$[1, \frac{1}{1+D}]AA$	0.0832
R^3AA-punct.	0.1036
R^4AA-punct.	0.1091
R^5AA-punct.	0.1098
random codes (GVB)	0.1100

to the ensemble improves the growth rate coefficient and that both ensembles in Table II have growth rates very close to the GVB. We also see that the growth rate coefficient increases much more in case of the R^2AA ensemble, than in the case of the $[1, \frac{1}{1+D}]AA$ ensemble, since in the latter case the coefficient is already close to the GVB.

TABLE II
DISTANCE GROWTH RATE COEFFICIENT ρ_0 FOR DIFFERENT RATE 1/2 CODE ENSEMBLES WITH THREE INTERLEAVERS.

Code	ρ_0
R^2AAA	0.1034
$[1, \frac{1}{1+D}]AAA$	0.1092
random codes (GVB)	0.1100

Finally, we recall that the minimum distance of a code impacts only one aspect of its performance, i.e., its expected behavior in the error floor region of the bit error rate (BER) curve. Hence we have also investigated the convergence behavior of DSCCs using an EXIT chart approach. Fig. 5 shows

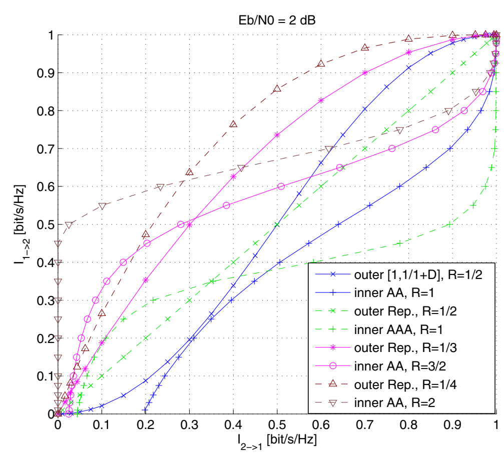

Fig. 5. Exit charts for the rate 1/2 ensembles in Tables I and II.

the pairs of EXIT functions for the rate 1/2 code ensembles given in Tables I and II at an SNR of $E_b/N_0 = 2$ dB, where the individual EXIT characteristics for the middle and inner constituent decoders are combined into one single joint EXIT characteristic. The curves for the punctured inner codes were created using a uniform puncturing pattern. We observe from Fig. 5 that 2 dB represents the convergence threshold for the $[1, \frac{1}{1+D}]AA$ ensemble. Also, while puncturing a low rate mother code results in a distance growth rate coefficient close to the GVB, the convergence behavior of these codes is poor. Further, we see that moving from a DSCC to a TSCC degrades convergence behavior. Similarly, the convergence behavior of the $[1, \frac{1}{1+D}]AAA$ code is much worse than the $[1, \frac{1}{1+D}]AA$ code, which offers a good compromise between minimum distance growth and convergence behavior. Thus, as noted in many other studies, we observe that convergence behavior is inversely related to distance growth.

V. CONCLUSIONS

In contrast to SSCCs, MSCCs can be asymptotically good in the sense that their minimum distance grows linearly with the block length as the block length tends to infinity. If the asymptotic IOWEF of the component encoders is known, we can use the method presented in this paper to calculate the distance growth rate coefficients of different MSCC ensembles. In this paper, we calculated the growth rate coefficient of the DSCC given by an outer rate 1/2 code with generator matrix $[1, 1/1 + D]$ followed by two accumulators, and our result agrees with the conjecture given in [8]. The results in [12] will be useful to compute the asymptotic IOWEFs of other component encoders. While the minimum distance growth rate can be made arbitrarily close to the GVB by either puncturing a low rate code or by adding more concatenation stages, these ensembles have convergence thresholds that are far from capacity. Since DSCCs can be asymptotically good, ensembles with more serially concatenated component codes, such as TSCCs, may not be a good choice in practice.

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