Oversampled Cosine-Modulated Filter Banks with Arbitrary System Delay

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***Abstract*— In this paper, design methods for perfect reconstruction (PR) oversampled cosine-modulated filter banks with integer oversampling factors and arbitrary delay are presented. The system delay, which is an important parameter in real-time applications, can be chosen independently of the prototype lengths. Oversampling gives us additional freedom in the filter design process, which can be exploited to find FIR PR prototypes for oversampled filter banks with much higher stopband attenuations than for critically subsampled filter banks. It is shown that for a given analysis prototype, the PR synthesis prototype is not unique. The complete set of solutions is discussed in terms of the nullspace of a matrix operator. For example, oversampling allows the design of PR filter banks having unidentical prototypes (of equal and unequal lengths) for the analysis and synthesis stage. Examples demonstrate the increased design freedom due to oversampling. Finally, it is shown that PR prototypes being designed for the oversampled case can also serve as almost-PR prototypes for critically subsampled cosine-modulated pseudo QMF banks.**

I. Introduction

OVERSAMPLED filter banks have their applications in those areas of signal processing where one is interested in making modifications to signals in certain frequency bands. Examples are the simulation of room acoustics by filtering in subbands, noise reduction in the spectral domain, and equalization via fixed or dynamic (i.e. time-varying) filtering of subband signals. In this paper, we develop PR conditions for oversampled cosine-modulated filter banks with arbitrary system delay. Especially in real-time applications when there is some feedback involved (i.e., echo cancelation for mobile phones or hearing aids), it is extremely important that the overall system delay is as low as possible.

Cosine-modulated filter banks are a special subclass of the general M-channel filter bank depicted in Fig. 1, where the analysis and synthesis filters $H_k(z)$ and $F_k(z)$ are derived from prototypes $p(n)$ and $q(n)$, respectively, by cosine modulation. Previous work on PR cosine-modulated filter banks [1]–[9] has always addressed the critically subsampled case ($M = N$ in Fig. 1). This case is very attractive for subband coding because it leads to a minimum number of subband coefficients. However, if we simply want to implement some linear filtering by introducing gain factors in the subbands, critical subsampling is not useful. This is due to the fact that the main aliasing spectra will no longer cancel in the synthesis filter bank when the subband signals are modified. These main aliasing components are not present in the oversampled case, $M > N$, with sufficiently high oversampling.

Recently, perfect reconstruction (PR) conditions for oversampled DFT filter banks have been derived, and general relations between oversampled filter banks and frame theory have been investigated [10]–[13]. Due to the modulated nature of DFT and cosine-modulated filter banks, the requirements on the prototypes are somewhat related. As will be shown for the biorthogonal case, all PR prototypes for cosine-modulated filter banks also give PR in DFT banks. The opposite does not hold for all PR DFT prototypes.

This paper is organized as follows. In Section II, we present the polyphase representation for general (biorthogonal) oversampled filter banks and show how the polyphase matrices in the oversampled case with integer oversampling factors can be obtained from the critically subsampled version. Section III derives general PR conditions for biorthogonal cosine-modulated filter banks with integer oversampling rates and arbitrary overall system delays. We show that by solving the PR conditions in the oversampled case, we gain additional design freedom. The well-known critically subsampled case and the paraunitary oversampled case are regarded as special solutions of our general approach. Finally, relations between oversampled cosine-modulated and DFT filterbanks will be discussed. In Section IV, we establish the connection to pseudo QMF filter banks and show that PR solutions in the oversampled case have a partial aliasing cancelation property when applied to a critically subsampled filter bank. Section V refers to the prototype design, where examples are given.

Notation: Boldface letters indicate vectors and matrices. Given a matrix $\boldsymbol{A}$, its transpose is denoted as $\boldsymbol{A}^{\mathrm{T}}$ and its Hermitian or transpose-conjugate by $\boldsymbol{A}^{\mathrm{H}}$, respectively. The tilde on a (matrix-) function $\tilde{\boldsymbol{B}}(z)$ is defined as $\boldsymbol{B}_*^{\mathrm{T}}(z^{-1})$, where $*$ denotes complex conjugation of the polynomial coefficients in $\boldsymbol{B}(z)$. The matrices $\boldsymbol{I}_Q$ and $\boldsymbol{J}_Q$ stand for $Q \times Q$ identity and reverse identity matrices, respectively.

II. Polyphase Representation for Oversampled Filter Banks

Consider the polyphase representation of the filter bank in Fig. 2. Its oversampling ratio L is defined as $L = M/N$ and is restricted to be an integer in this paper. The analysis polyphase

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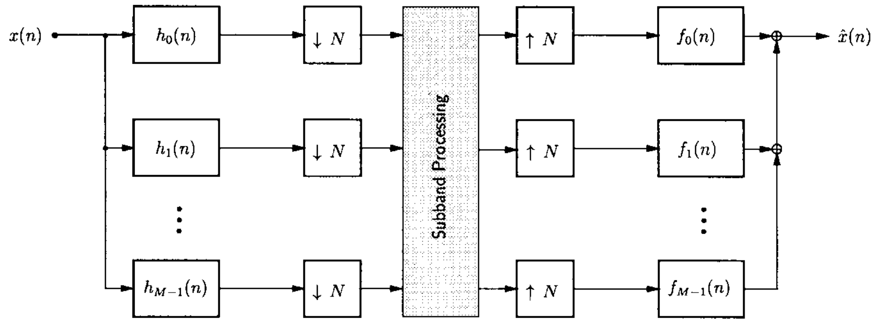


Fig. 1. General analysis and synthesis filter bank with subband processing.

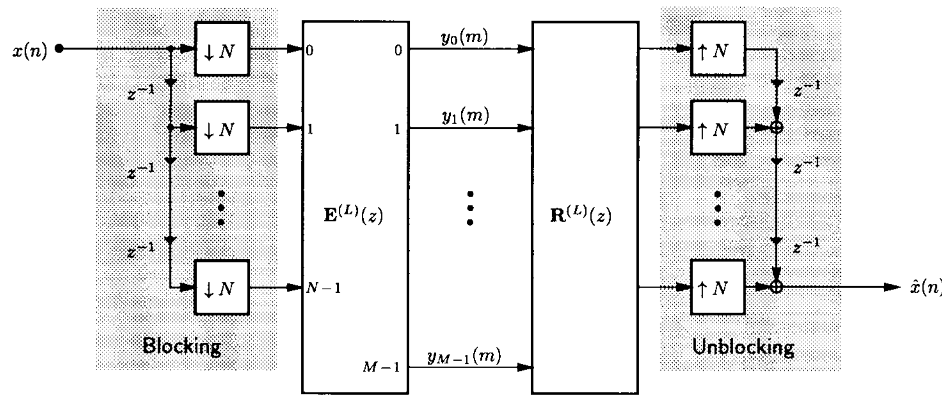


Fig. 2. Polyphase analysis and synthesis filter bank without subband processing.

matrix $\boldsymbol{E}^{(L)}(z)$ of size $M \times N$ in Fig. 2 is defined as

$$\boldsymbol{E}^{(L)}(z) = \begin{bmatrix} H_{0,0}(z) & H_{0,1}(z) & \cdots & H_{0,N-1}(z) \\ H_{1,0}(z) & H_{1,1}(z) & \cdots & H_{1,N-1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ H_{M-1,0}(z) & H_{M-1,1}(z) & \cdots & H_{M-1,N-1}(z) \end{bmatrix} \tag{1}$$

where $H_{k,j}(z), j = 0, 1, \cdots, N-1$, are the type-1 polyphase components of the kth analysis filter $h_k(n)$

$$H_{k,j}(z) = \sum_{\ell=-\infty}^{\infty} h_k(\ell N + j) z^{-\ell} \qquad k = 0, \cdots, M-1, \quad j = 0, \cdots, N-1.$$

The synthesis polyphase matrix $\boldsymbol{R}^{(L)}(z)$ of size $N \times M$ is defined as

$$\boldsymbol{R}^{(L)}(z) = \begin{bmatrix} \overline{F}_{0,0}(z) & \overline{F}_{1,0}(z) & \cdots & \overline{F}_{M-1,0}(z) \\ \overline{F}_{0,1}(z) & \overline{F}_{1,1}(z) & \cdots & \overline{F}_{M-1,1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ \overline{F}_{0,N-1}(z) & \overline{F}_{1,N-1}(z) & \cdots & \overline{F}_{M-1,N-1}(z) \end{bmatrix} \tag{2}$$

where the polyphase components $\overline{F}_{k,j}(z)$ of the kth synthesis filter $f_k(n)$ are

$$\overline{F}_{k,j}(z) = \sum_{\ell=-\infty}^{\infty} f_k(N(\ell+1) - j - 1) z^{-\ell} \qquad k = 0, \cdots, M-1, \quad j = 0, \cdots, N-1.$$

Note that type-2 polyphase components (marked with a bar) are used on the synthesis side. However, for the description of oversampled modulated filter banks, it will be more convenient to describe the synthesis filter bank with type-1 polyphase components $F_{k,j}(z)$. The relation between type-1 and type-2 components can be stated as $F_{k,N-1-j}(z) = \overline{F}_{k,j}(z)$, $j = 0, \cdots, N-1$.

For critical subsampling, we achieve quadratic polyphase matrices $\boldsymbol{E}^{(1)}(z)$ and $\boldsymbol{R}^{(1)}(z)$ of size $M \times M$, whereas in the oversampled case, the matrices have a rectangular shape of size $M \times N$ and $N \times M$, respectively.

Generally, an L-times oversampled filter bank has the perfect reconstruction property if

$$\boldsymbol{R}^{(L)}(z)\boldsymbol{E}^{(L)}(z) = z^{-D_0^{(L)}} \boldsymbol{I}_N \tag{3}$$

holds, where $D_0^{(L)}$ denotes the delay in the polyphase domain. Under consideration of the blocking and unblocking delay chains in Fig. 2, we have an overall delay of

$$D = N - 1 + D_0^{(L)} N \tag{4}$$

samples between $x(n)$ and $\hat{x}(n)$.

We will now derive relations between the polyphase matrices $\boldsymbol{E}^{(L)}(z)$ and $\boldsymbol{R}^{(L)}(z)$ of a given filter bank for different

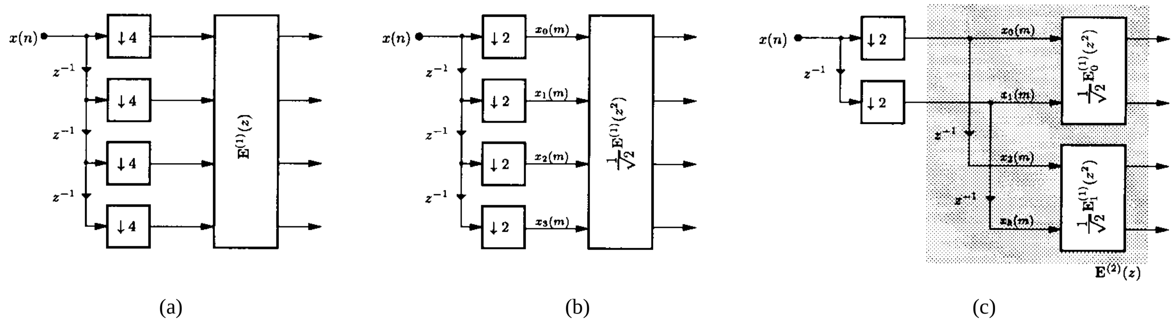


Fig. 3. Polyphase analysis filter bank ($M = 4$). (a) Critically subsampled. (b) Oversampled by factor 2 and scaled by $1/\sqrt{2}$. (c) Equivalent structure to (b) (see text).

oversampling ratios L, whereas L is restricted to be an integer. For this, let us have a look at the example depicted in Fig. 3. In Fig. 3(a), a critically sampled four-channel analysis filter bank is shown. Fig. 3(b) shows the same filter bank but with oversampling by the factor $L = 2$ and a prefactor $1/\sqrt{2}$, which is introduced in order to compensate the increased amplification of the complete analysis–synthesis system. Note that the polyphase matrix $\boldsymbol{E}^{(1)}(z)$ from Fig. 3(a) has been replaced by the matrix $\boldsymbol{E}^{(1)}(z^2)$ in Fig. 3(b), which means that upsampled versions of the polyphase filters are used. Partitioning the polyphase matrix according to $\boldsymbol{E}^{(1)}(z^2) = [\boldsymbol{E}_0^{(1)}(z^2), \boldsymbol{E}_1^{(1)}(z^2)]$ and rearranging the delay chain and the subsamplers yields the structure in Fig. 3(c), which is equivalent to the one in Fig. 3(b). The system in the gray box is represented by the polyphase matrix of the oversampled system

$$\boldsymbol{E}^{(2)}(z) = \frac{1}{\sqrt{2}}[\boldsymbol{E}_0^{(1)}(z^2), \boldsymbol{E}_1^{(1)}(z^2)]\begin{bmatrix}\boldsymbol{I}_2\\ z^{-1}\boldsymbol{I}_2\end{bmatrix}. \tag{5}$$

Likewise, the synthesis polyphase matrix can be given as

$$\boldsymbol{R}^{(2)}(z) = \frac{1}{\sqrt{2}}[z^{-1}\boldsymbol{I}_2, \boldsymbol{I}_2]\underbrace{\begin{bmatrix}\boldsymbol{R}_0^{(1)}(z^2)\\ \boldsymbol{R}_1^{(1)}(z^2)\end{bmatrix}}_{\boldsymbol{R}^{(1)}(z^2)}. \tag{6}$$

When we assume that the critically subsampled filter bank has the PR property with $\boldsymbol{R}^{(1)}(z)\boldsymbol{E}^{(1)}(z) = \boldsymbol{I}_4$, the oversampled system also has the PR property

$$\boldsymbol{R}^{(2)}(z)\boldsymbol{E}^{(2)}(z) = \tfrac{1}{2}\,[z^{-1}\boldsymbol{I}_2, \boldsymbol{I}_2]\begin{bmatrix}\boldsymbol{I}_2\\ z^{-1}\boldsymbol{I}_2\end{bmatrix} = z^{-1}\boldsymbol{I}_2. \tag{7}$$

Since we neither changed the filters (except by factor $1/\sqrt{2}$) nor introduced an additional delay into the system, the overall delay of the system must be preserved for all oversampling ratios. However, the delay caused by the delay chain for the oversampled case in Fig. 3(c) is reduced by two taps compared with Fig. 3(a). Thus, the delay introduced by multiplication of the polyphase analysis and synthesis matrix has to be increased by the same amount, which explains the additional delay of z^{-1} in (7).

As can be easily verified (same arguments as above), the generalizations of (5) and (6) to M-channel filter banks being oversampled by $L \geq 2, L \in \mathbb{N}$ are

$$\boldsymbol{E}^{(L)}(z) = \boldsymbol{E}^{(1)}(z^L) \cdot \boldsymbol{S}^{(L)}(z) \tag{8}$$

$$\boldsymbol{R}^{(L)}(z) = z^{-(L-1)} \cdot \tilde{\boldsymbol{S}}^{(L)}(z) \cdot \boldsymbol{R}^{(1)}(z^L) \tag{9}$$

with

$$\boldsymbol{S}^{(L)}(z) = 1/\sqrt{L} \cdot [\boldsymbol{I}_N, z^{-1}\boldsymbol{I}_N, \cdots, z^{-(L-1)}\boldsymbol{I}_N]^{\mathrm{T}}. \tag{10}$$

If the critically sampled filter bank has the PR property

$$\boldsymbol{R}^{(1)}(z)\boldsymbol{E}^{(1)}(z) = z^{-D_0^{(1)}}\boldsymbol{I}_M \tag{11}$$

then the corresponding oversampled filter bank obviously also guarantees PR, and (3) holds. This can be easily seen from (8), (9), and (11). We have

$$\begin{aligned}\boldsymbol{R}^{(L)}(z)&\boldsymbol{E}^{(L)}(z)\\ &= z^{-(L-1)}\tilde{\boldsymbol{S}}^{(L)}(z)\boldsymbol{R}^{(1)}(z^L)\boldsymbol{E}^{(1)}(z^L)\boldsymbol{S}^{(L)}(z)\\ &= z^{-(L-1)-D_0^{(1)}L}\tilde{\boldsymbol{S}}^{(L)}(z)\boldsymbol{S}^{(L)}(z)\\ &= z^{-D_0^{(L)}}\boldsymbol{I}_N.\end{aligned}$$

Here, the delay (in the polyphase domain) introduced by the multiplication of the analysis and synthesis polyphase matrix amounts to $D_0^{(L)} = L - 1 + D_0^{(1)}L$ samples for a given oversampling ratio L. The overall delay D remains unchanged for all L, and we have $D = M - 1 + D_0^{(1)}M$.

III. Cosine-Modulated Filter Banks with Arbitrary Subsampling Rate

Unless otherwise noted, we consider biorthogonal cosine-modulated filter banks where the analysis filters $h_k(n), k = 0, \cdots, M-1$ are derived from an FIR prototype $p(n)$ and the synthesis filters $f_k(n), k = 0, \cdots, M-1$ from an FIR prototype $q(n)$ according to

$$\begin{aligned}h_k(n) &= 2p(n)\cos\left[\frac{\pi}{M}\left(k+\frac{1}{2}\right)\left(n-\frac{D}{2}\right)+\phi_k\right]\\ &\qquad n = 0, \cdots, L_p - 1\\ f_k(n) &= 2q(n)\cos\left[\frac{\pi}{M}\left(k+\frac{1}{2}\right)\left(n-\frac{D}{2}\right)-\phi_k\right]\\ &\qquad n = 0, \cdots, L_q - 1.\end{aligned}$$

The length of the analysis prototype $p(n)$ is L_p, and the length of the synthesis prototype $q(n)$ is L_q. D denotes the overall delay of the analysis-synthesis system, where we will later show that D normally varies between the minimal delay of $2M - 1$ and a delay of $L_p - 1$ samples. The latter case corresponds to linear-phase prototypes with $f_k(n) = h_k(L_p - n - 1), L_q = L_p$, which are discussed in Section III-D. A suitable choice for ϕ_k is given as $\phi_k = (-1)^k\pi/4$ [2], [3].

In this paper, we restrict ourselves to even M and analysis and synthesis prototype lengths of $L_p = 2mM, m \in \mathbb{N}$, and $L_q = 2m'M, m' \in \mathbb{N}$. In this case, all polyphase components of a given prototype have the same length. Note that we consider a general approach with different lengths for the analysis and synthesis prototype.

A. Analysis and Synthesis Polyphase Matrices

1) Critically Subsampling: Let $P_j(z), j = 0, \cdots, 2M-1$, denote the type-1 polyphase components of the analysis prototype $p(n)$ according to

$$P_j(z) = \sum_{\ell=0}^{m-1} p(2\ell M + j) z^{-\ell}.$$

The analysis polyphase matrix for the critically sampled case $(L = 1)$ can be written as [7], [9]

$$\boldsymbol{E}^{(1)}(z) = \boldsymbol{T}_a \begin{bmatrix} \boldsymbol{P}_0(z^2) \\ z^{-1}\boldsymbol{P}_1(z^2) \end{bmatrix} = \boldsymbol{T}_a \boldsymbol{P}(z) \qquad (12)$$

where

$$[\boldsymbol{T}_a]_{k,j} = 2\cos\left[\frac{\pi}{M}\left(k+\frac{1}{2}\right)\left(j - \frac{D}{2}\right) + \phi_k\right]$$
$$k = 0, \cdots, M-1, \quad j = 0, \cdots, 2M-1 \qquad (13)$$

and

$$\boldsymbol{P}_0(z^2) = \mathrm{diag}\,[P_0(-z^2), P_1(-z^2), \cdots, P_{M-1}(-z^2)]$$
$$\boldsymbol{P}_1(z^2) = \mathrm{diag}\,[P_M(-z^2), P_{M+1}(-z^2), \cdots, P_{2M-1}(-z^2)]. \qquad (14)$$

In the same way, the synthesis prototype $q(n)$ is decomposed into the polyphase filters

$$Q_j(z) = \sum_{\ell=0}^{m-1} q(2\ell M + j) z^{-\ell}.$$

Using these type-1 polyphase components, the synthesis polyphase matrix can now be written as

$$\boldsymbol{R}^{(1)}(z) = [z^{-1}\boldsymbol{J}_M \boldsymbol{Q}_1(z^2)\boldsymbol{J}_M, \boldsymbol{J}_M \boldsymbol{Q}_0(z^2)\boldsymbol{J}_M]\boldsymbol{T}_s^{\mathrm{T}}$$
$$= \boldsymbol{Q}(z)\boldsymbol{T}_s^{\mathrm{T}} \qquad (15)$$

where the diagonal matrices $\boldsymbol{Q}_0(z^2)$ and $\boldsymbol{Q}_1(z^2)$ are defined in the same way as $\boldsymbol{P}_0(z^2)$ and $\boldsymbol{P}_1(z^2)$ in (14), and

$$[\boldsymbol{T}_s]_{k,j} = 2\cos\left[\frac{\pi}{M}\left(k+\frac{1}{2}\right)\left(2M-1-j-\frac{D}{2}\right) - \phi_k\right] \qquad (16)$$

for $k = 0, \cdots, M-1, j = 0, \cdots, 2M-1$.

For completeness, we also give $\boldsymbol{R}^{(1)}(z)$ in terms of type-2 components

$$\boldsymbol{R}^{(1)}(z) = [z^{-1}\overline{\boldsymbol{Q}}_0(z^2), \overline{\boldsymbol{Q}}_1(z^2)]\boldsymbol{T}_s^{\mathrm{T}}$$

where

$$\overline{\boldsymbol{Q}}_0(z^2) = \mathrm{diag}\,[\overline{Q}_0(-z^2), \overline{Q}_1(-z^2), \cdots, \overline{Q}_{M-1}(-z^2)]$$
$$\overline{\boldsymbol{Q}}_1(z^2) = \mathrm{diag}\,[\overline{Q}_M(-z^2), \overline{Q}_{M+1}(-z^2), \cdots, \overline{Q}_{2M-1}(-z^2)].$$

2) Oversampled Case: We use the description of the polyphase matrices for the critically sampled case and insert (12) into the relation (8), thus obtaining a more general expression for the analysis polyphase matrix in the oversampled case. With the definition of diagonal matrices

$$\boldsymbol{p}_\ell(z^{2L}) = \mathrm{diag}\,[P_{\ell N}(-z^{2L}), P_{\ell N+1}(-z^{2L}), \cdots$$
$$P_{\ell N+(N-1)}(-z^{2L})], \quad \ell = 0, \cdots, 2L-1 \qquad (17)$$

this yields for the analysis polyphase matrix

$$\boldsymbol{E}^{(L)}(z) = \boldsymbol{T}_a \boldsymbol{P}(z^L)\boldsymbol{S}^{(L)}(z)$$
$$= \frac{1}{\sqrt{L}} \cdot \boldsymbol{T}_a \begin{bmatrix} \boldsymbol{p}_0(z^{2L}) \\ z^{-1}\boldsymbol{p}_1(z^{2L}) \\ \vdots \\ z^{-(2L-1)}\boldsymbol{p}_{2L-1}(z^{2L}) \end{bmatrix}. \qquad (18)$$

The synthesis polyphase matrix can be constructed in the same way. From (15) and (9), we get

$$\boldsymbol{R}^{(L)}(z) = z^{-(L-1)}\tilde{\boldsymbol{S}}^{(L)}(z)\boldsymbol{Q}(z^L)\boldsymbol{T}_s^{\mathrm{T}}$$
$$= \frac{1}{\sqrt{L}} \cdot [z^{-(2L-1)}\boldsymbol{J}_N \boldsymbol{q}_{2L-1}(z^{2L})\boldsymbol{J}_N$$
$$z^{-(2L-2)}\boldsymbol{J}_N \boldsymbol{q}_{2L-2}(z^{2L})\boldsymbol{J}_N, \cdots$$
$$\cdots, \boldsymbol{J}_N \boldsymbol{q}_0(z^{2L})\boldsymbol{J}_N]\boldsymbol{T}_s^{\mathrm{T}} \qquad (19)$$

where the matrices $\boldsymbol{q}_\ell(z^{2L})$ containing the type-1 polyphase components of the synthesis prototype are defined as $\boldsymbol{p}_\ell(z^{2L})$ in (17).

Note that (compared with critical subsampling) the matrices $\boldsymbol{T}_a$ and $\boldsymbol{T}_s$, respectively, are unchanged in the oversampled case so that the original phase offset in (13) and (16) is preserved. The matrix $\boldsymbol{P}(z^L)\boldsymbol{S}^{(L)}(z)$ is of size $2M \times N$, where $N = M/L$, and therefore, the polyphase matrices $\boldsymbol{E}^{(L)}(z)$ and $\boldsymbol{R}^{(L)}(z)$ are of size $M \times N$ and $N \times M$, respectively.

B. PR Conditions

In the integer oversampled case, we have two ways of satisfying the PR constraint (3).

1) The PR prototypes for the critically subsampled filter bank are also used in the oversampled case, and thus, (3) is met for all $L \geq 1$ up to the factor L.
2) When the L-times oversampled filter bank is designed in such a way that (3) is satisfied and (11) is not, PR up to a constant factor is obtained for all oversampling ratios $L_1 = cL, c \in \mathbb{N}$. The filter bank designed this way also ensures almost PR for all $L_2 \in \mathbb{N}$ satisfying $L = cL_2, c \in \mathbb{N}$; see Section IV.

The latter case is the interesting one for the design of oversampled PR filter banks because we have more degrees of freedom for the design process, and thus, the conditions on both prototypes $p(n)$ and $q(n)$ can be relaxed. Therefore, we focus on the discussion of this point.

First, we derive conditions for the polyphase components in order to satisfy (3) for arbitrary oversampling ratios L.

Inserting (18) and (19) into (3) yields

$$\begin{aligned}\boldsymbol{R}^{(L)}(z)\boldsymbol{E}^{(L)}(z) &= z^{-(L-1)}\tilde{\boldsymbol{S}}^{(L)}(z)\boldsymbol{Q}(z^L)\boldsymbol{T}_s^{\mathrm{T}}\boldsymbol{T}_a\boldsymbol{P}(z^L)\boldsymbol{S}^{(L)}(z)\\ &\stackrel{!}{=} z^{-D_0^{(L)}}\boldsymbol{I}_N \end{aligned} \quad (20)$$

where we use the type-1 polyphase description on the synthesis side. Choosing $\boldsymbol{T}_a$ as in (13), $\boldsymbol{T}_s$ as in (16), and writing the delay D as

$$D = 2M \cdot (D_1 + 1) - 1, \quad \text{with} \quad D_1 \in \mathbb{N} \quad (21)$$

leads to the expression for the product $\boldsymbol{T}_s^{\mathrm{T}}\boldsymbol{T}_a$ [9], [14]

$$\boldsymbol{T}_s^{\mathrm{T}}\boldsymbol{T}_a = (-1)^{D_1} 2M \cdot \boldsymbol{I}_{2M} + 2M \cdot \begin{bmatrix} \boldsymbol{J}_M & \mathbf{o} \\ \mathbf{o} & -\boldsymbol{J}_M \end{bmatrix}. \quad (22)$$

Inserting this product into (20), we get

$$\begin{aligned}&(-1)^{D_1} 2M \cdot z^{-(L-1)}\tilde{\boldsymbol{S}}^{(L)}(z)\boldsymbol{Q}(z^L)\boldsymbol{P}(z^L)\boldsymbol{S}^{(L)}(z)\\ &\quad + 2M \cdot z^{-(L-1)}\tilde{\boldsymbol{S}}^{(L)}(z)\boldsymbol{Q}(z^L)\begin{bmatrix} \boldsymbol{J}_M & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{J}_M \end{bmatrix}\\ &\quad \cdot \boldsymbol{P}(z^L)\boldsymbol{S}^{(L)}(z) \stackrel{!}{=} z^{-D_0^{(L)}}\boldsymbol{I}_N. \end{aligned} \quad (23)$$

For achieving the PR property, the second term of this equation, which contains the antidiagonal terms, must be zero. With (18) and (19), this can be written as

$$\begin{aligned}&\sum_{\ell=0}^{L-1} z^{-L-2\ell}\boldsymbol{J}_N\boldsymbol{q}_{L+\ell}(z^{2L})\boldsymbol{p}_\ell(z^{2L})\\ &\quad - \sum_{\ell=0}^{L-1} z^{-L-2\ell}\boldsymbol{J}_N\boldsymbol{q}_\ell(z^{2L})\boldsymbol{p}_{L+\ell}(z^{2L}) \stackrel{!}{=} \mathbf{0}. \end{aligned} \quad (24)$$

When this condition is satisfied, we still must constrain the first term of (23) to be a simple delay. With (18), (19), and (23), we then can state the requirement

$$\begin{aligned}&(-1)^{D_1} 2N \cdot z^{-(2L-1)} \sum_{\ell=0}^{2L-1} \boldsymbol{J}_N\boldsymbol{q}_{2L-1-\ell}(z^{2L})\boldsymbol{J}_N\boldsymbol{p}_\ell(z^{2L})\\ &\quad \stackrel{!}{=} z^{-D_0^{(L)}}\boldsymbol{I}_N. \end{aligned} \quad (25)$$

We now use (17) and express the two PR conditions (24) and (25) directly with the polyphase components $P_j(z)$ and $Q_j(z), j = 0, \cdots, 2M-1$. This gives the conditions

$$\begin{aligned}&\sum_{\ell=0}^{L-1} z^{-2\ell}[P_{k+\ell N}(-z^{2L})Q_{M+k+\ell N}(-z^{2L})\\ &\quad - P_{M+k+\ell N}(-z^{2L})Q_{k+\ell N}(-z^{2L})] \stackrel{!}{=} 0 \end{aligned} \quad (26)$$

and

$$\begin{aligned}&\sum_{\ell=0}^{2L-1} P_{k+\ell N}(-z^{2L})Q_{2M-1-k-\ell N}(-z^{2L})\\ &\quad \stackrel{!}{=} (-1)^{D_1} \cdot \frac{z^{-D_0^{(L)}+2L-1}}{2N} \end{aligned} \quad (27)$$

which have to be fulfilled for $k = 0, \cdots, N-1$.

In order to simplify the expressions (26) and (27), we first write (26) as

$$\begin{aligned}&\sum_{\ell=0}^{L-1} z^{-\ell}[P_{k+\ell N}(-z^{L})Q_{M+k+\ell N}(-z^{L})\\ &\quad - P_{M+k+\ell N}(-z^{L})Q_{k+\ell N}(-z^{L})] \stackrel{!}{=} 0. \end{aligned} \quad (28)$$

Note that (28) has the form of a polyphase decomposition of some filter (which is identically zero here), and therefore, we can conclude that (28) can be written as a set of L independent equations

$$\begin{aligned}&P_{k+\ell N}(z)Q_{M+k+\ell N}(z) - P_{M+k+\ell N}(z)Q_{k+\ell N}(z)\\ &\quad \stackrel{!}{=} 0 \quad \text{for} \quad k = 0, \cdots, N-1; \ell = 0, \cdots, L-1. \end{aligned} \quad (29)$$

Here, $-z^L$ was replaced by z. Equation (27) can be simplified by replacing the argument $-z^{2L}$ with z and by writing the delay $D_0^{(L)}$ with (4) and (21) as

$$D_0^{(L)} = 2LD_1 + 2L - 1 \quad (30)$$

thus resulting in

$$\begin{aligned}&\sum_{\ell=0}^{2L-1} P_{k+\ell N}(z)Q_{2M-1-k-\ell N}(z)\\ &\quad \stackrel{!}{=} \frac{z^{-D_1}}{2N} \quad \text{for} \quad k = 0, \cdots, N-1. \end{aligned} \quad (31)$$

To achieve a PR filter bank, we have to fulfill (29) and find a feasible solution that also satisfies (31). We will see in the next sections that we have different solutions in the critically sampled and the oversampled case, where in the latter case, we have less severe constraints for the analysis and synthesis prototypes, which will give us some benefits for the prototype filter design.

The overall length of filters of the type $P_j(z) \cdot Q_k(z)$ in (31) amounts to $m + m' - 1$ taps, which allows the delay parameter D_1 to vary between 0 and $m + m' - 2$. Thus, D_1 can be understood to be a design parameter that allows us to choose the overall delay of the analysis-synthesis system independently of L. We can achieve minimum delay prototypes for $D_1 = 0$ and linear-phase prototypes (with $m' = m$) for $D_1 = m - 1$.

C. General Solutions for the Design of PR Prototypes

We can see that (29) is always fulfilled if the synthesis prototype and the analysis prototype are related as $q(n) = c \cdot p(n - 2Mn_0)$ with $c \in \mathbb{R}, n_0 \in \mathbb{N}$; they differ only by a constant factor and an additional delay. However, in general, we have more solutions and more design freedom. In order to describe the complete set of solutions for the oversampled case, we rewrite the linear set of (29) and (31) as

$$\boldsymbol{A} \cdot \boldsymbol{q} = \boldsymbol{b}. \quad (32)$$

The matrix $\boldsymbol{A}$ of dimension $N(L+1) \times 2M$ has a block diagonal structure

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{A}_1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{A}_{N-1} \end{bmatrix}.$$

Because of this block structure, we can split the whole system into N smaller linear sets of equations

$$\boldsymbol{A}_k \boldsymbol{q}_k = \boldsymbol{b}_k, \quad \text{for} \quad k = 0, \cdots, N-1. \tag{33}$$

The following partitioning of the vectors $\boldsymbol{q}$ and $\boldsymbol{b}$ is used.

$$\boldsymbol{q} = [\boldsymbol{q}_0, \boldsymbol{q}_1, \cdots, \boldsymbol{q}_{N-1}]^{\mathrm{T}}, \quad \text{where}$$
$$\boldsymbol{q}_k = [Q_k(z), Q_{k+N}(z), \cdots, Q_{k+(2L-1)N}(z)]^{\mathrm{T}}$$

and

$$\boldsymbol{b} = [\boldsymbol{b}_0, \boldsymbol{b}_1, \cdots, \boldsymbol{b}_{N-1}]^{\mathrm{T}}, \quad \text{where}$$
$$\boldsymbol{b}_k = [z^{-D_1}/(2N), \underbrace{0, 0, \cdots, 0}_{L \text{ zeros}}]^{\mathrm{T}}, \quad k = 0, \cdots, N-1.$$

The submatrices $\boldsymbol{A}_k, k = 0, \cdots, N-1$ have the dimension $(L+1) \times 2L$. With the abbreviation $S = 2M - 1$, they can be written as (34), shown at the bottom of the page. Note that the first row of (33) is due to (31), whereas the remaining L rows are due to (29).

Since the matrices $\boldsymbol{A}_k$ are rectangular of size $(L+1) \times 2L$, we have N underdetermined sets of equations with at least $2L - (L+1) = L-1$ free parameters. All solutions to (32) can be obtained by taking one solution of the inhomogeneous system $\boldsymbol{A}\boldsymbol{q} = \boldsymbol{b}$ and adding linear combinations of all other solutions from the nullspace of $\boldsymbol{A}$ (these are the solutions of $\boldsymbol{A}\boldsymbol{q} = \mathbf{0}$). For the sake of simplicity, let us assume that the matrixes $\boldsymbol{A}_k$ have maximal rank, that is, $\mathrm{rank}\,(\boldsymbol{A}_k) = L+1$ (otherwise, we would have polyphase components $P_j(z)$ with zero coefficients). This means we obtain $L-1$ linearly independent basis vectors $\boldsymbol{n}_k^{(i)}, i = 0, \cdots, L-2$ for each nullspace $\mathcal{N}\{\boldsymbol{A}_k\}$. Such a set of basis vectors for the nullspace will be denoted as

$$\boldsymbol{n}_k^{(i)} = [N_k^{(i)}(z), N_{k+N}^{(i)}(z), \cdots, N_{k+(2L-1)N}^{(i)}(z)]^{\mathrm{T}}, \quad \text{with}$$
$$k = 0, \cdots, N-1$$

where $N_j^{(i)}(z), j = 0, \cdots, 2M-1$ stands for the polyphase components of the corresponding nullspace filter.

When we design the analysis prototype $P(z)$ in such a way that it satisfies (31) with $Q_j(z) = P_j(z)$, its polyphase components serve as a special solution to (33) because for identical analysis and synthesis prototypes, (29) is always satisfied. Then, all solutions can be characterized by

$$\boldsymbol{q}_k = \boldsymbol{p}_k + C_k(z) \cdot \sum_{i=0}^{L-2} \alpha_k^{(i)} \boldsymbol{n}_k^{(i)} \text{ for } \quad k = 0, \cdots, N-1 \tag{35}$$

where the coefficients $\alpha_k^{(i)}$ denote the $N(L-1)$ free parameters in the above problem. The terms $C_k(z)$ are arbitrary (FIR or IIR) transfer functions. Additionally, we may introduce further delay such that $Q_{\mathrm{new}}(z) = z^{-2Mn_0} Q(z), n_0 \in \mathbb{N}$.

Note that the role of the analysis and synthesis prototype can be interchanged. This can be seen from the PR constraints in (29) and (31), which still hold when $Q_k(z)$ plays the role of $P_k(z)$ and vice versa. Thus, instead of (35), we can also write

$$\boldsymbol{p}_k = \boldsymbol{q}_k + C_k(z) \cdot \sum_{i=0}^{L-2} \alpha_k^{(i)} \boldsymbol{n}_k^{(i)} \quad \text{for} \quad k = 0, \cdots, N-1.$$

These considerations show that we can construct PR filter banks with different analysis and synthesis prototypes. The synthesis filters can be longer or shorter as the analysis filters, and in special cases, we can also have unidentical prototypes of the same length. This is demonstrated in the example below.

Obviously, if we decompose a signal with our analysis filter bank and we only use a nullspace filter as a synthesis prototype, the output signal will be identically zero. If we construct the synthesis prototype according to (35), the output will be independent of the parameters $\alpha_k^{(i)}$ and of the filters $C_k(z)$. However, it is important to notice that the output signal only is independent of the nullspace component as long as the subband signals are directly derived from the analysis filter bank. If the subband signals are modified, which is the case in real applications, the nullspace component will also influence the output. Therefore, the nullspace filters can be used in order to modify and optimize the prototypes.

Example: Let us consider the case $M = 4, N = 2$. The matrix $\boldsymbol{A}$ is then given by the equation at the bottom of the next page, and the basis vectors for the nullspace of $\boldsymbol{A}$ can be obtained as

$$\boldsymbol{n}_0^{(0)} = \begin{bmatrix} -P_0(z)[P_2(z)P_5(z) + P_1(z)P_6(z)] \\ P_2(z)[P_3(z)P_4(z) + P_0(z)P_7(z)] \\ -P_4(z)[P_2(z)P_5(z) + P_1(z)P_6(z)] \\ P_6(z)[P_3(z)P_4(z) + P_0(z)P_7(z)] \end{bmatrix}$$
$$\boldsymbol{n}_1^{(0)} = \begin{bmatrix} -P_1(z)[P_3(z)P_4(z) + P_0(z)P_7(z)] \\ P_3(z)[P_2(z)P_5(z) + P_1(z)P_6(z)] \\ -P_5(z)[P_3(z)P_4(z) + P_0(z)P_7(z)] \\ P_7(z)[P_2(z)P_5(z) + P_1(z)P_6(z)] \end{bmatrix}.$$

$$\boldsymbol{A}_k = \begin{bmatrix} P_{S-k}(z) & P_{S-k-N}(z) & \cdots & P_{S-k-(L-1)N}(z) & P_{S-k-LN}(z) & P_{S-k-(L+1)N}(z) & \cdots & P_{S-k-(2L-1)N}(z) \\ -P_{M+k}(z) & 0 & \cdots & 0 & P_k(z) & 0 & \cdots & 0 \\ 0 & -P_{M+k+N}(z) & \cdots & 0 & 0 & P_{k+N}(z) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -P_{M+k+(L-1)N}(z) & 0 & 0 & \cdots & P_{k+(L-1)N}(z) \end{bmatrix} \tag{34}$$

From this expression, we see that the nullspace polyphase components $N_j^{(i)}(z)$ have a maximum length of $3m-2$ when the analysis polyphase components are of length m. With (35), this leads to analysis and synthesis prototypes of unequal length. However, if

$$P_2(z)P_5(z)+P_1(z)P_6(z)=c \quad \text{and}$$
$$P_3(z)P_4(z)+P_0(z)P_7(z)=c, \quad c\in\mathbb{R}$$

we have the same lengths for both prototypes, but the analysis and synthesis prototypes do not have to be identical. These conditions can be satisfied by designing the analysis prototype for the critically subsampled case [according to (31) with $Q_j(z)=P_j(z), N=M=4, D_1=0$]. When we scale this analysis prototype by $1/\sqrt{2}$ and apply it to an oversampled filter bank with $N=2$, we can construct the corresponding PR synthesis prototypes via (35). In this case, the two nullspace basis vectors reduce to $\boldsymbol{n}_0^{(0)}=c\cdot[-P_0(z),P_2(z),-P_4(z),P_6(z)]^{\mathrm{T}}$ and $\boldsymbol{n}_0^{(1)}=c\cdot[-P_1(z),P_3(z),-P_5(z),P_7(z)]^{\mathrm{T}}$ and with $C_0(z)=C_1(z)=1, \alpha_0^{(0)}\neq 0, \alpha_1^{(0)}\neq 0$, we obtain unidentical analysis and synthesis prototypes of the same length.

Fig. 4 shows an example for the nullspace solution where the analysis prototype of length $L_p=48$ is designed for $D_1=0$. The magnitude frequency response of the analysis prototype is depicted in Fig. 4(a). For the choice $\alpha_0^{(0)}=20, \alpha_1^{(0)}=20$, and $C_0(z)=C_1(z)=1$, the frequency response of the nullspace filter is shown in Fig. 4(b). Thus, the nullspace impulse response has a length of $2M\cdot(3m-2)=128$. Note that the passband of the nullspace filter is located within the stopband of the analysis prototype. This gives a vivid explanation of the properties of a nullspace filter. However, since we have the free choice for $\alpha_k^{(i)}$ and $C_k(z)$, such a behavior must not be found in all nullspace filters.

A synthesis prototype was obtained as the sum of the analysis and the nullspace filter. Fig. 4(c) depicts the corresponding magnitude frequency response, which qualitatively looks like the sum of the responses in Fig. 4(a) and (b). A more constructive solution is depicted in Fig. 4(d). In this case, the polyphase components of the analysis and the nullspace filters are linearly combined in such a way that the stopband attenuation of the resulting synthesis filter is increased.

D. PR Solution for Special Cases

1) Critically Subsampled Case: In the following, it is shown how the results for the critically subsampled case as stated in literature [7], [9] are related to the general solution presented in the last section.

With $L=1$ and $M=N$, (31) and (29) can be written as

$$P_k(z)Q_{2M-1-k}(z)+P_{M+k}(z)Q_{M-1-k}(z)\overset{!}{=}\frac{z^{-D_1}}{2M} \quad (36)$$
$$P_k(z)Q_{M+k}(z)-P_{M+k}(z)Q_k(z)\overset{!}{=}0 \quad (37)$$

for $k=0,\cdots,M-1$. In (36), two products of analysis and synthesis prototype polyphase components always have to add up to a delay, which is more restrictive than (31) in the oversampled case.

The submatrices $\boldsymbol{A}_k, k=0,\cdots,M-1$ in (34) now have the dimension 2×2, and the M individual systems are given as

$$\underbrace{\begin{bmatrix} P_{2M-1-k}(z) & P_{M-1-k}(z) \\ -P_{M+k}(z) & P_k(z)\end{bmatrix}}_{\boldsymbol{A}_k}\underbrace{\begin{bmatrix} Q_k(z) \\ Q_{k+M}(z)\end{bmatrix}}_{\boldsymbol{q}_k} = \underbrace{\frac{1}{2M}\begin{bmatrix} z^{-D_1} \\ 0\end{bmatrix}}_{\boldsymbol{b}_k}.$$

Since $\boldsymbol{A}_k$ is now quadratic and is assumed to have full rank, the nullspace $\mathcal{N}\{\boldsymbol{A}_k\}$ is just the null vector. This means that given the analysis prototype and the delay parameter D_1, we have only one solution for the synthesis prototype, which can be written in closed form as

$$\begin{bmatrix} Q_k(z) \\ Q_{k+M}(z)\end{bmatrix} = \frac{z^{-D_1}}{2M\cdot\det\{\boldsymbol{A}_k\}}\begin{bmatrix} P_k(z) \\ P_{k+M}(z)\end{bmatrix}.$$

An FIR solution for the synthesis filter is obtained if

$$\det\{\boldsymbol{A}_k\}=P_{2M-1-k}(z)P_k(z)+P_{M+k}(z)P_{M-1-k}(z)$$
$$\overset{!}{=}\frac{z^{-D_1}}{2M}, \qquad k=0,\cdots,[M/2]-1$$

which is exactly the same condition as (36) with $Q_j(z)=P_j(z)$. In other words, an FIR solution for the case $L=1$ essentially requires analysis and synthesis prototypes with poyphase components being equal up to a scaling factor.

2) Paraunitary Case: The paraunitary case is characterized by the fact that the sum of the energies of all subband signals is equal to the energy of the input signal. This may be expressed as $||\boldsymbol{y}||=||\boldsymbol{x}||\ \forall\boldsymbol{x}$ with $||\boldsymbol{x}||<\infty$, where $\boldsymbol{x}(z)$ is the polyphase vector of a finite-energy input signal, and $\boldsymbol{y}(z)=\boldsymbol{E}^{(L)}(z)\boldsymbol{x}(z)$ is the vector of subband signals. $||\cdot||$ denotes the Euclidean norm.

As can be easily verified, filter banks (oversampled and critically sampled) are paraunitary if

$$\tilde{\boldsymbol{E}}^{(L)}(z)\boldsymbol{E}^{(L)}(z)=\boldsymbol{I}_N \quad (38)$$

$$\boldsymbol{A}=\begin{bmatrix} P_7(z) & P_5(z) & P_3(z) & P_1(z) & 0 & 0 & 0 & 0 \\ -P_4(z) & 0 & P_0(z) & 0 & 0 & 0 & 0 & 0 \\ 0 & -P_6(z) & 0 & P_2(z) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_6(z) & P_4(z) & P_2(z) & P_0(z) \\ 0 & 0 & 0 & 0 & -P_5(z) & 0 & P_1(z) & 0 \\ 0 & 0 & 0 & 0 & 0 & -P_7(z) & 0 & P_3(z)\end{bmatrix}$$

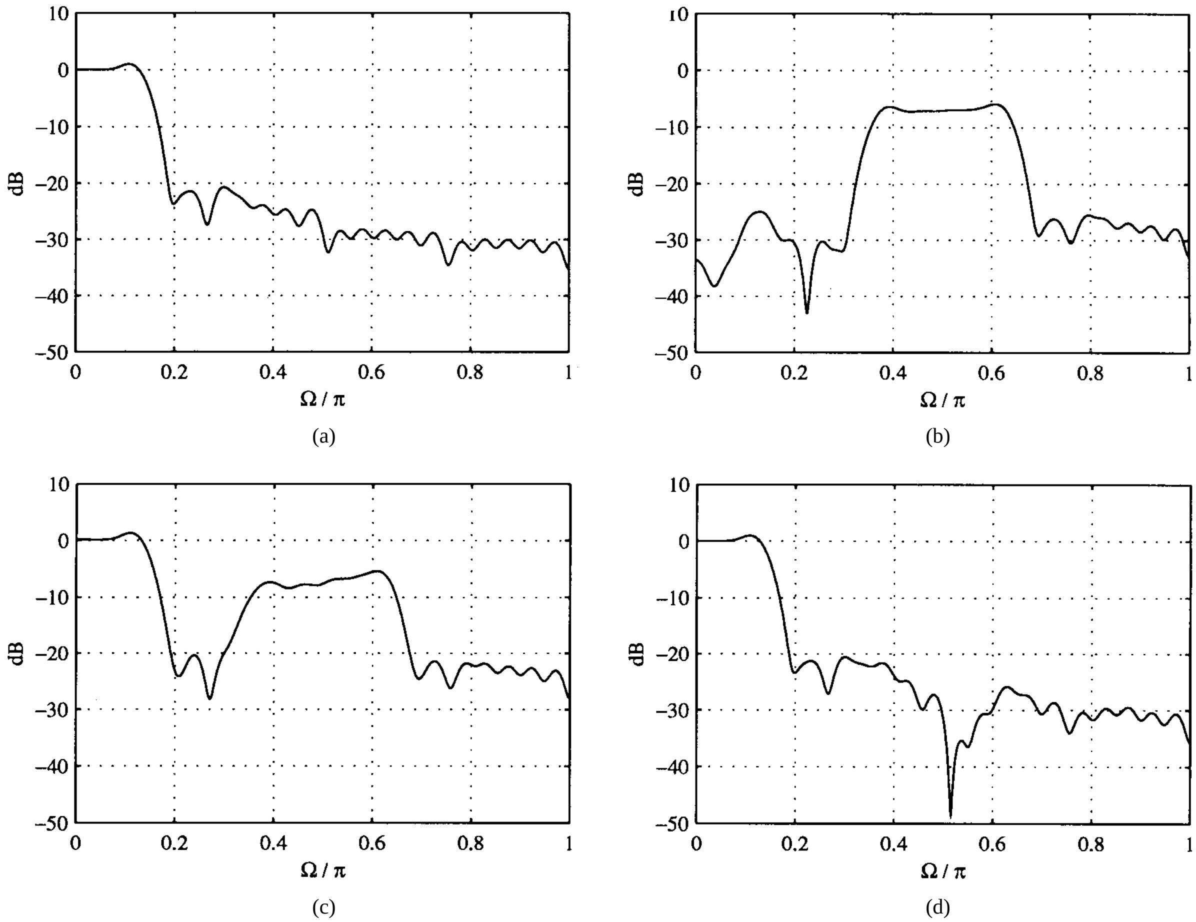


Fig. 4. Examples for magnitude frequency responses ($M = 4, N = 2, D_1 = 0$). (a) Analysis prototype. (b) Nullspace filter ($\alpha_0^{(0)} = \alpha_1^{(0)} = 20$, $C_0(z) = C_1(z) = 1$). (c) Synthesis prototype according to (35) with $\alpha_0^{(0)} = \alpha_1^{(0)} = 20, C_0(z) = C_1(z) = 1$. (d) Synthesis prototype according to (35) with $\alpha_0^{(0)} = -1, \alpha_1^{(0)} = 1, C_0(z) = C_1(z) = 1$.

holds. From this equation, we may conclude that

$$\boldsymbol{R}^{(L)}(z) = z^{-(2mL-1)}\tilde{\boldsymbol{E}}^{(L)}(z) + \boldsymbol{N}^{(L)}(z), \quad \text{where} \quad \boldsymbol{N}^{(L)}(z)\boldsymbol{E}^{(L)}(z) = \boldsymbol{0} \tag{39}$$

yields a PR synthesis filter bank. The delay in (39) is introduced in order to achieve causal synthesis filters. The polyphase matrix $\boldsymbol{N}^{(L)}(z)$ contains the solutions from the nullspace, which have been introduced in Section III-C.

In the following, we restrict ourselves to $\boldsymbol{N}^{(L)}(z) = \boldsymbol{0}$. From (39), we then see that the synthesis filters are time-reversed versions of the analysis filters, that is, $f_k(n) = h_k(L_p - n - 1)$. As in the critically sampled case [14], this is fulfilled when we choose linear-phase prototypes $p(n) = p(L_p - n - 1)$ and $q(n) = p(n)$. For the overall delay D, this means that D is fixed to $D = L_p - 1 = 2mM - 1, m \in \mathbb{N}$. The delay parameters D_1 and $D_0^{(L)}$ can be identified as $D_1 = m - 1$ and $D_0^{(L)} = 2mL - 1$, respectively.

Due to the linear-phase property of $p(n)$ and its length being an integer multiple of $2M$, two polyphase components are always related as

$$P_j(z) = z^{-(m-1)}\tilde{P}_{2M-j-1}(z) \quad \text{for} \quad j = 0, \cdots, M-1. \tag{40}$$

For identical linear-phase analysis and synthesis prototypes, $Q_j(z) = P_j(z)$, from (40) and (31), it follows that

$$\sum_{\ell=0}^{2L-1} \tilde{P}_{k+\ell N}(z) P_{k+\ell N}(z) \overset{!}{=} \frac{1}{2N} \quad \text{for } k = 0, \cdots, \left\lceil \frac{N}{2} \right\rceil - 1. \tag{41}$$

This result was already established in [15].

Equation (41) states that $2L$ polyphase components always have to be power complementary in the oversampled case. For increasing L, this condition on the prototype becomes less restrictive, and the design freedom increases. Note that (41) is the only condition on the analysis prototype for paraunitary oversampled filter banks because with $q(n) = p(n)$, the condition (29) is always fulfilled.

E. Relation to DFT Filter Banks

In DFT filter banks, the analysis and synthesis filters, $h_k(n)$ and $f_k(n)$, respectively, are obtained by complex modulation from given prototypes. The literature [10], [11], [13] covers only the paraunitary case, where the same prototype is used for analysis and synthesis. However, for relating DFT filter banks to the cosine-modulated banks derived in this paper, it is useful to consider two different prototypes ($p(n)$ and $q(n)$)

also for DFT banks. Thus, $h_k(n) = p(n)e^{j(\pi/M)k(n-D/2)}$ and $f_k(n) = q(n)e^{j(\pi/M)k(n-D/2)}, k = 0, \cdots, 2M-1$. Note that we consider $2M$-band DFT and M-band cosine-modulated filter banks so that the subbands are of equal spectral width in both cases.

In the critically subsampled case, the analysis and synthesis polyphase matrices for the DFT filter bank can be written with D as in (21) and the diagonal matrix $\boldsymbol{K}$ with elements $[\boldsymbol{K}]_{kk} = e^{j(\pi/M)kD/2}$ as

$$\begin{aligned}\boldsymbol{E}_D^{(1)}(z) &= \boldsymbol{K}^{\mathrm{H}}\boldsymbol{W}^{\mathrm{H}}\,\mathrm{diag}\,[P_0(z), P_1(z), \cdots, P_{2M-1}(z)] \\ &= \boldsymbol{K}^{\mathrm{H}}\boldsymbol{W}^{\mathrm{H}}\boldsymbol{P}_D(z) \end{aligned} \tag{42}$$

$$\begin{aligned}\boldsymbol{R}_D^{(1)}(z) &= \mathrm{diag}\,[Q_{2M-1}(z), \cdots, Q_1(z), Q_0(z)]\boldsymbol{W}\boldsymbol{K} \\ &= \boldsymbol{Q}_D(z)\boldsymbol{W}\boldsymbol{K}. \end{aligned} \tag{43}$$

Here, $\boldsymbol{W}$ denotes the $2M \times 2M$ DFT matrix and the $P_j(z), Q_j(z), j = 0, \cdots, 2M-1$, the type-1 polyphase components of the analysis and synthesis prototype, respectively.

Similar to the derivation for the cosine-modulated filter bank in Section III-B, the PR constraints for the $2L$-times oversampled DFT filter bank with $L = M/N$ can be obtained by inserting (42) into (8), (43) into (9), and replacing L by $2L$. Hence, the PR condition in (3) can be written as

$$\begin{aligned}&\boldsymbol{R}_D^{(2L)}(z)\boldsymbol{E}_D^{(2L)}(z) \\ &\quad = z^{-(2L-1)}\tilde{\boldsymbol{S}}^{(2L)}(z)\boldsymbol{Q}_D(z^{2L})\boldsymbol{W}\boldsymbol{W}^{\mathrm{H}}\boldsymbol{P}_D(z^{2L})\boldsymbol{S}^{(2L)}(z) \\ &\quad \stackrel{!}{=} z^{-D_0^{(2L)}}\boldsymbol{I}_N \end{aligned} \tag{44}$$

which is a similar expression as (20) in the cosine-modulated case for half the oversampling ratio. However, since $\boldsymbol{W}\boldsymbol{W}^{\mathrm{H}} = 2M \cdot \boldsymbol{I}_{2M}$, it turns out that it is sufficient to satisfy only the following PR condition similar to (31) in order to achieve PR for an $2L$-times oversampled DFT filter bank.

$$\sum_{\ell=0}^{2L-1} P_{k+\ell N}(z)Q_{2M-1-k-\ell N}(z) \stackrel{!}{=} \frac{z^{-D_1}}{N} \quad \text{for} \quad k = 0, \cdots, N-1.$$

Due to the absence of antidiagonal terms in the product of the analysis and synthesis transform matrices [compare (22)], the additional PR conditions for the cosine-modulated case in (29) are not relevant in the oversampled DFT case. Thus, all PR solutions for the L-times oversampled cosine-modulated case including unidentical analysis and synthesis prototypes can also be used as solutions in the $2L$-times oversampled DFT case, but they only represent a subset of all possible solutions. All considerations regarding the system delay, the link to pseudo QMF filter banks, and the design issues in the next sections can be applied to oversampled DFT filter banks as well.

IV. Relation to Pseudo QMF Filter Banks

The approaches presented in the previous sections can not only be used to design PR L-times oversampled filter banks. As we will show in the following, it is also possible to use a prototype that satisfies the PR constraints for a given oversampling ratio L_o as an almost PR solution for $L < L_o$ with $L_o = cL, c \in \mathbb{N}$. These solutions have the interesting property that every L_o/Lth aliasing spectrum is canceled.

1) Partial Aliasing Cancelation: We first state some general results and consider an arbitrary M-channel filter bank with oversampling ratio L and subband sampling rate N, as it is depicted in Fig. 1. The analysis modulation (which is also called the aliasing component) matrix of size $N \times M$ is defined as

$$\boldsymbol{H}(z) = \begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{M-1}(z) \\ H_0(zW_N) & H_1(zW_N) & \cdots & H_{M-1}(zW_N) \\ \vdots & \vdots & \ddots & \vdots \\ H_0(zW_N^{N-1}) & H_1(zW_N^{N-1}) & \cdots & H_{M-1}(zW_N^{N-1}) \end{bmatrix} \tag{45}$$

where $W_N = e^{-j2\pi/N}$.

We now define the transfer functions

$$A_\ell(z) = \frac{1}{N}\sum_{k=0}^{M-1} H_k(zW_N^\ell)F_k(z)$$

which denote the aliasing contribution of the analysis filter $H_k(z)$ shifted by $2\pi\ell/N$ for $\ell = 1, \cdots, N-1$ and (for $\ell = 0$) the overall transfer function of the nonsubsampled filter bank. The aliasing component vector $\boldsymbol{a}(z) = [A_0(z), \cdots, A_{N-1}(z)]^{\mathrm{T}}$ can be obtained by

$$\boldsymbol{a}(z) = \frac{1}{N} \cdot \boldsymbol{H}(z)\boldsymbol{f}(z) \tag{46}$$

with the synthesis filter vector $\boldsymbol{f}(z) = [F_0(z), \cdots, F_{M-1}(z)]^{\mathrm{T}}$. On the other hand, we can express the aliasing component matrix (45) in terms of the analysis polyphase matrix (1) according to [16] as

$$\boldsymbol{H}(z) = \boldsymbol{W}^{\mathrm{H}}\boldsymbol{D}(z)\boldsymbol{E}^{(L)\mathrm{T}}(z^N) \tag{47}$$

where $\boldsymbol{D}(z) = \mathrm{diag}\,[1, z^{-1}, \cdots, z^{-(N-1)}]$, and $\boldsymbol{W}$ denotes the $N \times N$ DFT matrix.

The synthesis filters $F_k(z)$ can be constructed from their type-2 polyphase components according to $F_k(z) = \sum_{\ell=0}^{N-1} z^{-(N-1-\ell)}\overline{F}_{k,\ell}(z^N)$. Writing this with $\boldsymbol{f}(z)$ and the synthesis polyphase matrix (2) yields

$$\boldsymbol{f}(z) = z^{-(N-1)}\boldsymbol{R}^{(L)\mathrm{T}}(z^N)\boldsymbol{e}(z) \tag{48}$$

where $\boldsymbol{e}(z) = [1, z^1, \cdots, z^{(N-1)}]^{\mathrm{T}}$. Combining (46)–(48) leads to

$$\boldsymbol{a}(z) = \frac{1}{N} \cdot z^{-(N-1)}\boldsymbol{W}^{\mathrm{H}}\boldsymbol{D}(z)[\boldsymbol{R}^{(L)}(z^N)\boldsymbol{E}^{(L)}(z^N)]^{\mathrm{T}}\boldsymbol{e}(z). \tag{49}$$

Now, we use the fact that $\boldsymbol{R}^{(L)}(z)\boldsymbol{E}^{(L)}(z)$ is equal to the left-hand side of (25) if (29) is fulfilled (e.g., because of the choice $q(n) = p(n)$). By inserting this relation into (49) and by expressing the resulting equation with the polyphase components $P_j(-z^{2M})$ and $Q_j(-z^{2M})$, we get

$$\boldsymbol{a}(z) = 2(-1)^{D_1} z^{-(2M-1)} \boldsymbol{W}^{\mathrm{H}} \cdot \begin{bmatrix} \sum_{\ell=0}^{2L-1} P_{\ell N}(-z^{2M}) Q_{2M-1-\ell N}(-z^{2M}) \\ \vdots \\ \sum_{\ell=0}^{2L-1} P_{N-1+\ell N}(-z^{2M}) Q_{2M-N-\ell N}(-z^{2M}) \end{bmatrix}. \quad (50)$$

The comparison of (50) with (31) shows that for the PR case, each sum over ℓ in (50) amounts to a delay. Then, it can be shown easily that $\boldsymbol{a}(z) = [z^{-D}, 0, \cdots, 0]^{\mathrm{T}}$, where D denotes the overall system delay. This means that in the PR case, all aliasing components are canceled at the summation point in the synthesis filter bank.

In order to describe the aliasing-cancelation effects in the pseudo QMF case, where we use PR prototypes designed for $L_o = (c+1)L, c \in \mathbb{N}$ in an L-times oversampled filter bank, we write (50) with an additional scaling factor L/L_o as

$$\begin{aligned} A_k(z) &= 2\frac{L}{L_o} \cdot (-1)^{D_1} z^{-(2M-1)} \sum_{i=0}^{N-1} W_N^{-ki} \sum_{\ell=0}^{2L-1} P_{i+\ell N}(-z^{2M}) \\ &\quad \cdot Q_{2M-1-i-\ell N}(-z^{2M}) \\ &= 2\frac{L}{L_o} \cdot (-1)^{D_1} z^{-(2M-1)} \sum_{m=0}^{N_o-1} \sum_{\lambda=0}^{N/N_o-1} W_N^{-k(m+\lambda N_o)} \\ &\quad \cdot \sum_{\ell=0}^{2L-1} P_{m+\lambda N_o+\ell N}(-z^{2M}) \\ &\quad \cdot Q_{2M-1-m-\lambda N_o-\ell N}(-z^{2M}) \end{aligned}$$

for $k = 0, \cdots, N-1$. In the last step, the sum over i was split into two sums by the substitution $i = m + \lambda N_o$, where $N_o = M/L_o$ is the subband decimation factor for which the prototype filters are designed. We now substitute again $\nu = \lambda + N/N_o \ell$, which yields

$$\begin{aligned} A_k(z) = 2\frac{L}{L_o} \cdot (-1)^{D_1} z^{-(2M-1)} \sum_{m=0}^{N_o-1} \sum_{\nu=0}^{2L_o-1} W_N^{-k(m+\nu N_o)} \\ \cdot P_{m+\nu N_o}(-z^{2M}) Q_{2M-1-m-\nu N_o}(-z^{2M}). \end{aligned} \quad (51)$$

For those k where $k\nu N_o$ is a multiple of N, the term $W_N^{-k(m+\nu N_o)}$ in (51) becomes independent of ν, and then, the inner sum corresponds to the PR constraint (31) for an oversampling ratio L_o. When we use prototypes $p(n)$ and $q(n)$, which give PR for L_o [i.e., satisfy (31)], the inner sum can be replaced by the delay term $z^{-2MD_1}(-1)^{D_1}/(2N_o)$, and we have

$$A_{\hat{k}}(z) = \frac{L}{M} \cdot z^{-(2M-1)} z^{-2MD_1} \sum_{m=0}^{N_o-1} W_N^{-msN/N_o} = 0$$
$$\text{for} \quad \hat{k} = s\frac{N}{N_o} = s\frac{L_o}{L}, \quad s \in \mathbb{Z}.$$

Thus, every L_o/Lth aliasing component is canceled out.

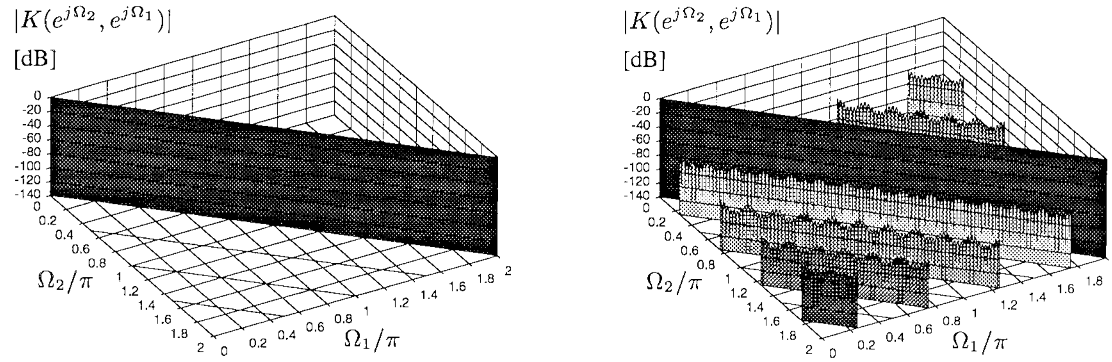
The overall transfer function $A_0(z)$ amounts to $A_0(z) = L/L_o \cdot z^{-(2M-1+2MD_1)}$, where the overall system delay (again) can be found as $D = 2M(D_1+1) - 1$. Furthermore, we have to scale the analysis and synthesis prototype with factor $\sqrt{L_o/L}$ in order to obtain the original amplitude. With such scaling, we have no linear distortion at all, which is not the case when using prototype filters designed as approximate Nyquist $(2M)$ filters for almost PR filter banks [17]–[19]. The only output distortion is due to those aliasing components that are not canceled.

2) Examples for Pseudo QMF Solutions: Generally, the filter bank in Fig. 1 can be regarded as a periodically time-varying system with the periodic system response $k(n_2, n_1) = k(n_2 + \ell N, n_1 + \ell N), \ell \in \mathbb{Z}$, where $k(n_2, n_1)$ denotes the response of the system at discrete time n_2 to a unit sample applied at discrete time n_1. The aliasing distortions in such systems are best visualized via bifrequency system functions [20]–[22]. The bifrequency system function is defined as [22]

$$K(e^{j\Omega_2}, e^{j\Omega_1}) = \frac{1}{2\pi} \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} k(n_2, n_1) e^{j(\Omega_1 n_1 - \Omega_2 n_2)}.$$

Since $k(n_2, n_1)$ is periodic in n_1 and $n_2, K(e^{j\Omega_2}, e^{j\Omega_1})$ contains only discrete lines, which refer to the aliasing components and the overall frequency response. Fig. 5(a) shows the magnitude bifrequency system function for the PR oversampled case with $M = 8$ subbands, an oversampling ratio of $L = 2$, and a prototype of length $L_p = 128$ designed for $L_o = 2$. The diagonal lines indicate the principal location of all $N-1$ possible aliasing components, which are completely canceled due to the PR property, and only the line for the transfer function of the system remains. Since we have a one-to-one mapping between the input frequency Ω_1 and the output frequency Ω_2, this system can be regarded as time invariant. The PR prototype for $L_o = 2$ is now scaled by $\sqrt{2}$ and applied to a critically subsampled filter bank, where the resulting system bifrequency function is shown in Fig. 5(b). We can see that every second possible aliasing spectrum is zero, which confirms the result of the proof above. Since the remaining aliasing components are significantly suppressed by the high stopband attenuation of the subband filters and by the property of the cosine-transform in (13), respectively, this case can be regarded as the pseudo QMF case. However, since the input-output mapping is not one-to-one, the system response of this filter bank is not time invariant.

In a second example, we apply a prototype of length $L_p = 128$ designed for $L_o = 8$ to a critically subsampled filter bank with $M = 8$. This case can be regarded as the classical pseudo QMF case, where all $M-1$ aliasing components are present with a magnitude that corresponds to the prototype filter's stopband attenuation. The resulting bifrequency system function is shown in Fig. 6. Note that although we have chosen $L_o = M$, the aliasing spectrum for $\Omega_2 = \Omega_1 \pm \pi$ is not present. This is due to the even length of the prototype filter $p(n) = q(n)$, which leads to a zero at $z = -1$ in the transfer function $P(z)$ and, thus, to a suppression of all signal components (including aliasing) at the frequency $\Omega = \pi$.



(a) (b)

Fig. 5. Normalized magnitude system bifrequency functions. (a) Oversampled perfect reconstruction case for $M = 8, L = 2$ and $L_o = 2$. (b) Critically sampled case with $M = 8$ and the same prototype as in (a). The prototype filter has a length of $L_p = 128$.

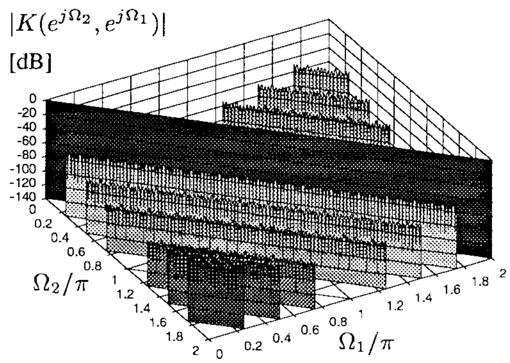


Fig. 6. Normalized magnitude system bifrequency function for a critically sampled filter bank with $M = 8$ and a prototype of length 128 designed for $L_o = M$.

V. Design Procedure and Examples

A. Prototype Optimization

1) General Case: We have seen in Section III-C that in the case of identical prototypes for the analysis and synthesis bank, (29) is satisfied, and the prototype, according to (31), remains to be designed. Expressing the latter equation in the time domain with $p(n) = q(n)$ yields

$$\sum_{\ell=0}^{2L-1} \underbrace{\sum_{i=0}^{m-1} p_{k+\ell N}(i) \cdot p_{2M-1-k-\ell N}(n-i)}_{r_{k+\ell N}(n)} = \frac{1}{2N} \cdot \delta(n - D_1) \tag{52}$$

for $k = 0, \cdots, \lceil N/2 \rceil - 1$ and $n = 0, \cdots, 2m-2$, where $\delta(n)$ denotes the unit sample sequence. The sequences $r_{k+\ell N}(n)$ stand for the result of the convolution of the polyphase impulse responses $p_{k+\ell N}(n)$ and $p_{2M-1-k-\ell N}(n)$. Since we consider an analysis prototype filter of length $L_p = 2mM, m \in \mathbb{N}$, each polyphase component $p_\ell(n)$ has the length m, which results in a total convolution length of $2m - 1$. The index k can be restricted to the first $\lceil N/2 \rceil$ convolution results $r_{k+\ell N}(n)$ because the first $\lfloor N/2 \rfloor$ results lead to exactly the same expressions as the $\lfloor N/2 \rfloor$ last ones. The case of odd N is formally taken into account by use of the ceiling operator. Expressing the sums $\Sigma_\ell \, r_{k+\ell N}(n)$ as impulse responses $s_k(n)$ results in the requirements

$$\sum_{\ell=0}^{2L-1} r_{k+\ell N}(n) = \sum_{i=0}^{2m-2} s_k(i)\delta(n-i) \tag{53}$$

and this yields the PR constraints in the time domain

$$s_k(n) = \begin{cases} 1/(2N) & \text{for } n = D_1 \\ 0 & \text{otherwise} \end{cases} \tag{54}$$

for $k = 0, \cdots, \lceil N/2 \rceil - 1, n = 0, \cdots, 2m-2$. The delay parameter D_1 can be selected in the range $D_1 \in [0, \cdots, 2m-2]$. The number of PR constraints in (54) is $\lceil N/2 \rceil (2m-1)$. For M being a power of two and $L \leq M/2$, this means a reduction by factor L compared with the critically subsampled case.

For the design of the prototype, we use a quadratic-constrained least-squares approach [23]. Each $s_k(n)$ from (54) can be written as a quadratic constraint of the form $s_k(n) = \boldsymbol{p}^{\mathrm{T}} \boldsymbol{Q}_{k,n} \boldsymbol{p}$ with $\boldsymbol{p} = [p(0), \cdots, p(L_p - 1)]^{\mathrm{T}}$ and some matrix $\boldsymbol{Q}_{k,n}$. For details on the optimization approach, see [23].

The constraints (54) are fulfilled numerically under additional minimization of the weighted prototype's stopband energy, which is independent of L and can be written as

$$E_s = \frac{1}{\pi} \cdot \sum_{k=0}^{Q-1} w_k \cdot \int_{\Omega_{s_k}}^{\Omega_{s_{k+1}}} |P(e^{j\Omega})|^2 \, d\Omega \stackrel{!}{=} \min. \tag{55}$$

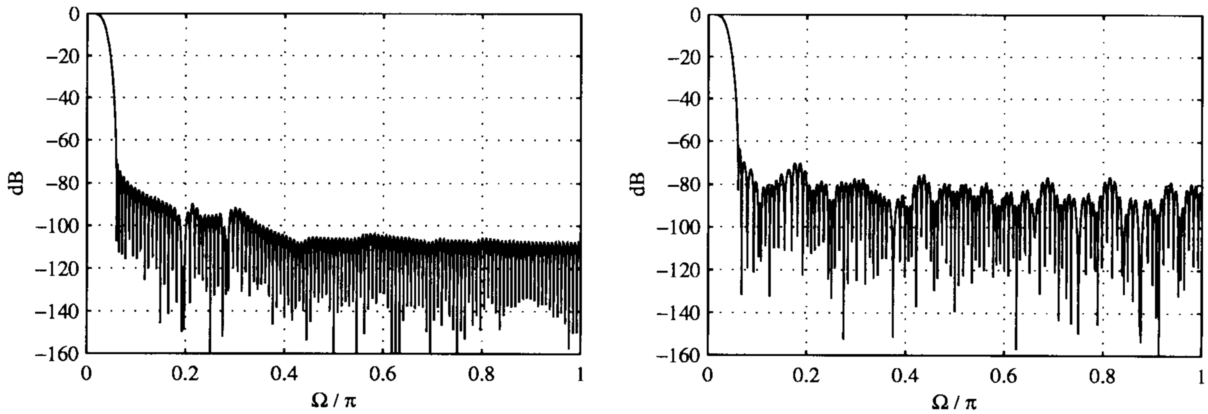
Here, $[w_0, \cdots, w_{Q-1}]$ denote the weighting factors for the different frequency regions, which are characterized by the edge frequencies $[\Omega_{s_0}, \cdots, \Omega_{s_{Q-1}}, \Omega_{s_Q} = \pi]$. Equation (55) can be expressed as a quadratic form $E_s = \boldsymbol{p}^{\mathrm{T}} \boldsymbol{S} \boldsymbol{p}$ according to

$$E_s = \frac{1}{\pi} \cdot \sum_{k=0}^{Q-1} w_k \cdot \boldsymbol{p}^{\mathrm{T}} \boldsymbol{S}_k \boldsymbol{p} = \boldsymbol{p}^{\mathrm{T}} \underbrace{\frac{1}{\pi} \cdot \sum_{k=0}^{Q-1} w_k \cdot \boldsymbol{S}_k}_{\boldsymbol{S}} \boldsymbol{p}$$

and

$$\boldsymbol{S}_k = \int_{\Omega_{s_k}}^{\Omega_{s_{k+1}}} \mathrm{Re}\left\{\boldsymbol{c}(\Omega)\boldsymbol{c}^{\mathrm{H}}(\Omega)\right\} d\Omega, \quad \text{with}$$

$$\boldsymbol{c}(\Omega) = [1, e^{-j\Omega}, \cdots, e^{-j\Omega(L_p-1)}]^{\mathrm{T}}.$$



(a) (b)

Fig. 7. Design Example 1. Magnitude frequency responses for linear-phase prototypes. Design parameters: $L_p = 256, M = 16, D_1 = 7, Q = 2$, edge frequencies $[\Omega_{s_0}, \Omega_{s_1}, \Omega_{s_2}] = [0.06\pi, 0.3\pi, \pi]$, weights $w_0 = 1, w_1 = 2, \epsilon = 10^{-11}$. (a) Oversampled case with $L = 2$. (b) Critically sampled case.

Note that since $\boldsymbol{p}$ is a real vector, we only need to take the real part of the outer product $\boldsymbol{c}(\Omega)\boldsymbol{c}^{\mathrm{H}}(\Omega)$. The elements of the matrix $\boldsymbol{S}_k$ can be given analytically as

$$\begin{aligned}[\boldsymbol{S}_k]_{m,n} &= \int_{\Omega_{s_k}}^{\Omega_{s_{k+1}}} \mathrm{Re}\,\{e^{-jm\Omega}e^{jn\Omega}\}\,d\Omega \\ &= \frac{1}{m-n}\cdot[\sin((n-m)\Omega_{s_{k+1}}) \\ &\quad - \sin((m-n)\Omega_{s_k})].\end{aligned}$$

2) Paraunitary Case: In this special case with $q(n) = p(n)$ and linear-phase prototypes satisfying the symmetry condition (40), we have only $\lceil N/2 \rceil m$ time-domain constraints to be satisfied in order to achieve PR. It was already stated in Section III-D that for linear-phase filters, the delay parameter D_1 is fixed to $D_1 = m - 1$ so that we can write (52) as

$$\sum_{\ell=0}^{2L-1} \underbrace{\sum_{i=0}^{m-1} p_{k+\ell N}(i)\cdot p_{k+\ell N}(m-1-n'+i)}_{r_{k+\ell N}(n'-m+1)} = \frac{1}{2N}\cdot\delta(n'-m+1)$$

for $k = 0, \cdots, \lceil N/2 \rceil - 1$ and $n' = 0, \cdots, 2m-2$. The expressions $r_{k+\ell N}(n'-m+1)$ can now be regarded as the autocorrelation sequences of the polyphase impulse responses $p_{k+\ell N}(n)$ shifted by $m-1$ samples. Due to the even symmetry of the autocorrelation $(r_{k+\ell N}(n') = r_{k+\ell N}(2m-n'-2))$, we can restrict ourselves to $n' = 0, 1, \cdots, m-1$. Hence, in the paraunitary case, we have $\lceil N/2 \rceil m$ time-domain PR constraints according to

$$\begin{aligned}s_k(0) = s_k(1) = \cdots = s_k(m-2) = 0 \\ s_k(m-1) = \frac{1}{2N}, \qquad k = 0, \cdots, \lceil N/2 \rceil - 1\end{aligned}$$

with $s_k(n)$ defined as in (53). This is a further reduction of the required conditions compared with the general case above. When comparing the number of PR constraints for different oversampling ratios L, here, we obtain the same results as in the general case.

B. Design Examples

In the following, we present design examples, which show that by relaxing the number of constraints in the oversampled case, we can design filters with better properties. In all examples, the initial filters for the optimization process were constructed via the pseudo QMF method from [19]. As an objective function, we used the stopband energy (55). The optimization was carried out with the NAG Fortran library. The PR constraints (54) were fulfilled up to some accuracy ϵ, which will be specified in the examples. For displaying frequency responses, the prototypes were normalized to 0 dB DC amplification.

Example 1: In this example, we address the paraunitary case and design a linear-phase prototype filter of length $L_p = 256$ for $M = 16$ subbands and an overall system delay of $D = 255$ samples. Fig. 7(a) shows the magnitude frequency response of the prototype designed for the oversampled case with $L = 2$, and Fig. 7(b) corresponds to the critically subsampled case. The oversampled case yields much higher stopband attenuation, which shows that the additional design freedom in the oversampled case can indeed be used to improve the prototype.

Example 2: Here, we refer to the biorthogonal case and consider an example with reduced delay, where a prototype of length $L_{p_0} = 128$ is designed for $M = 8$, an overall system delay of $D = 47$ samples and an oversampling ratio of $L = 2$. The prototype is then compared with a linear-phase design with the same reconstruction delay and $L_{p_1} = 48$. The solid line in Fig. 8(a) shows the magnitude frequency response of the low-delay prototype with $L_{p_0} = 128$, and the dotted line corresponds to the linear-phase case with $L_{p_1} = 48$. As we can see, the longer prototype yields higher stopband attenuation while having the same reconstruction delay.

Fig. 8(b) shows the magnitude frequency response for the critically subsampled low-delay case $(L = 1)$ with the same parameters as above. We see again that in the biorthogonal case as well, where the prototypes do not have linear phase, we gain advantages by designing prototypes especially for a given oversampling ratio L.

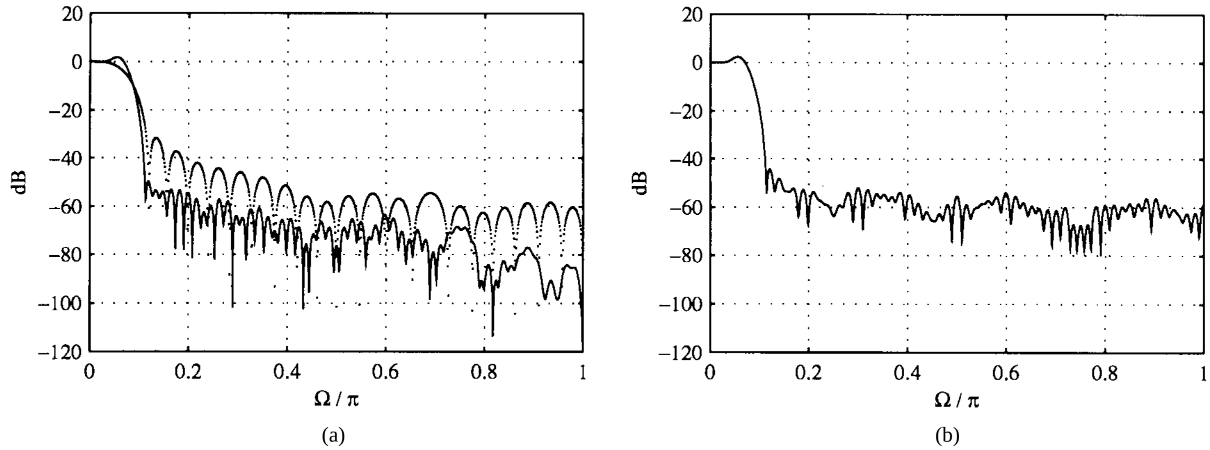


Fig. 8. Design Example 2: Magnitude frequency responses for $M = 8$ and an overall system delay of $D = 47$; one frequency band in the stopband region with $\Omega_{s_0} = 0.1\pi, \epsilon = 10^{-9}$. (a) Oversampled case for $L = 2$ with $L_{p_0} = 128$ (solid line) and $L_{p_1} = 48$ (linear-phase case, dotted line). (b) Critically sampled case with $L_{p_0} = 128$.

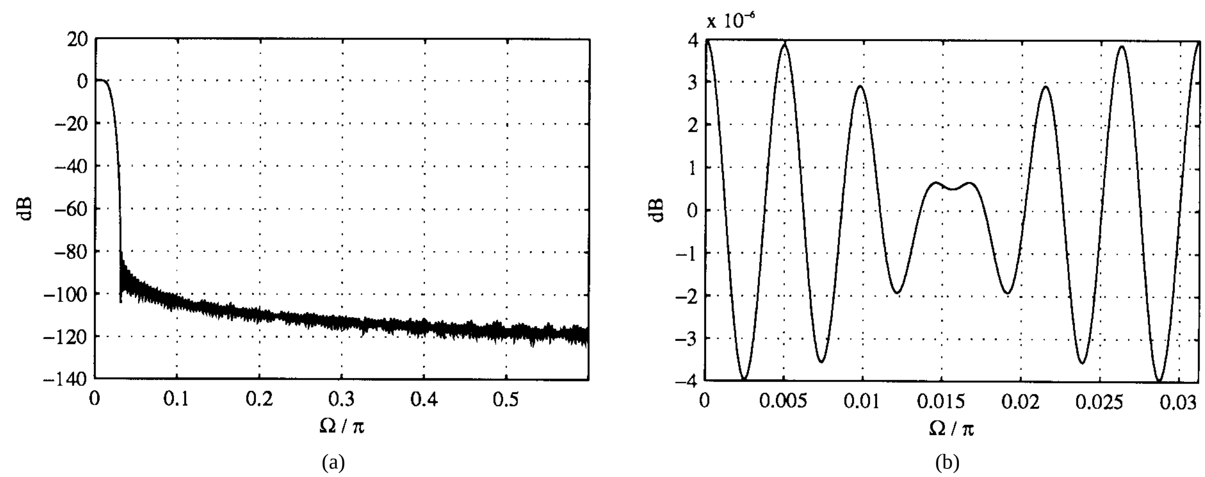


Fig. 9. Design Example 3: Prototype length $L_p = 512$ for $M = 32$ subbands, oversampling ratio $L = M$, and an overall system delay of $D = 447$ samples ($D_1 = 6, Q = 1, \Omega_{s_0} = 0.03\pi, \epsilon = 10^{-7}$). (a) Magnitude frequency response. (b) One period of the filter bank's overall magnitude frequency response $|A_0(e^{j\Omega})|$.

Example 3: In this example, we design a prototype for the nonsubsampled case $(L = M)$ of length $L_p = 512, M = 32$, and an overall system delay of $D = 447$ samples. Fig. 9 shows the filter bank's magnitude frequency response and one period of the overall magnitude frequency response $|A_0(e^{j\Omega})|$. Due to the high stopband attenuation, this prototype can be used for a critically subsampled pseudo QMF filter bank, where the aliasing components in the stopband region of the prototype are suppressed. The main aliasing components, however, are canceled out by the properties of the transforms (13) and (16).

1) Relations Between Stopband Energy, Stopband Attenuation, Filter Length, and Oversampling Ratio: In the following, we show the effects on the stopband energy and the stopband attenuation in the paraunitary case when the filter length L_p and the oversampling ratio L are varied. The results for $M = 16$ and $M = 32$ are displayed in Fig. 10. All prototypes are designed with $\epsilon = 10^{-9}, Q = 1$, and $\Omega_{s_0} = 0.055$ for $M = 16$, and $\Omega_{s_0} = 0.027$ for $M = 32$, respectively. Due to the quadratic nature of the objective function (55), the fact whether or not the obtained solutions correspond to local or global minima highly depends on the size of the problem and on the optimization method used.

As expected, we can observe that the stopband energy decreases when the filter length increases; see Fig. 10. However, in the case $M = 16$ for filter lengths larger than 350, there is not much improvement, and one encounters the limits of the used optimization algorithm. There are even cases where the best filters have been found for critical subsampling. Since these prototypes also give PR in the oversampled case, we can conclude that in these cases, the solutions for $N < M$ represent local minima of the objective function.

From Fig. 10(a) and (b), we see that for fixed prototype lengths, the stopband energy decreases when the subsampling rate N becomes smaller. This confirms the observation in Example 1. A decreasing N leads to a decreasing number of PR constraints, and thus, we may expect a lower stopband energy. However, oversampling ratios $L > 2$ bring only negligibly small improvements compared with $L = 2$. For the stopband attenuation, which is the interesting measure in practice, we qualitatively have the same results as for the stopband energy. See Fig. 10(c) and (d) for average stopband attenuations.

2) Relations Between Stopband Energy, Stopband Attenuation, System Delay, and Oversampling Ratio: We here address

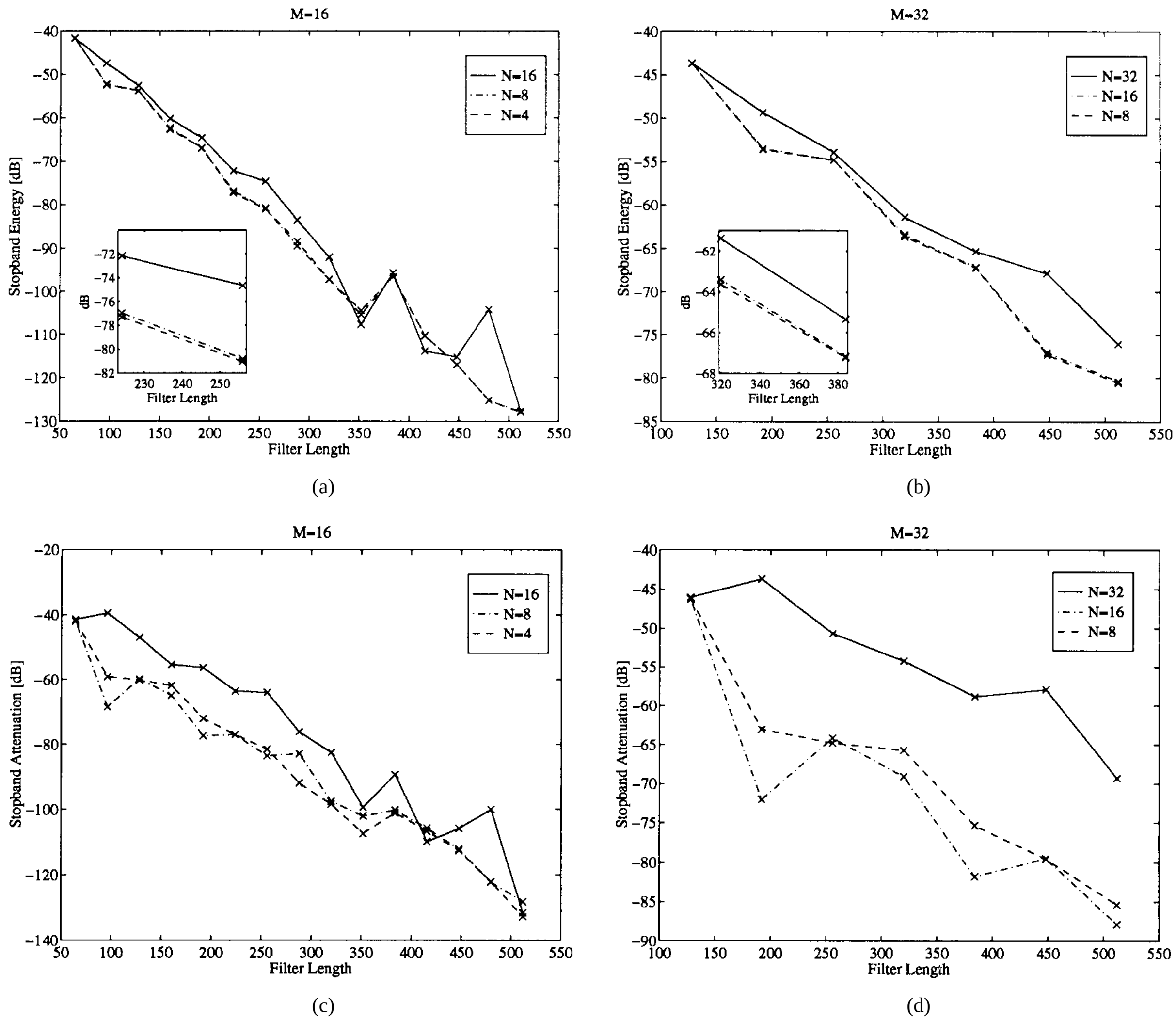


Fig. 10. Stopband energy for (a) $M = 16$. (b) $M = 32$ and average stopband attenuation for (c) $M = 16$. (d) $M = 32$ versus prototype length for different subsampling rates.

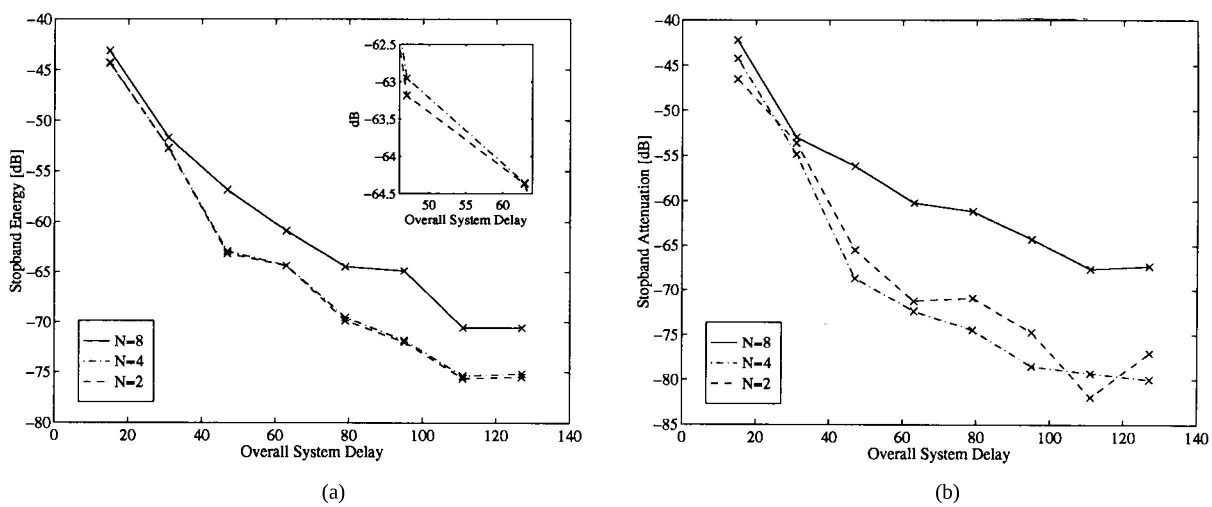


Fig. 11. Stopband energy (a) and average stopband attenuation (b) versus overall system delay for different subsampling rates ($M = 8, L_p = 128$).

the biorthogonal case and measure the effects on the stopband energy and the stopband attenuation due to a change of the parameter D_1 and, thus, due to the overall system delay D for different oversampling ratios L. The other design parameters are fixed; they are $L_p = 128, M = 8, Q = 1, \Omega_{s_0} = 0.11$, and $\epsilon = 10^{-9}$. The results are shown in Fig. 11. We observe that for increasing delay, the objective function decreases. As in the previous example, we see that an oversampling ratio of

$L > 2$ only gives a small amount of improvement compared with $L = 2$.

VI. CONCLUSION

We have derived the PR conditions for oversampled cosine-modulated filter banks with arbitrary system delay. It turned out that given an analysis prototype, the synthesis prototype is not uniquely determined. The complete set of solutions was expressed in terms of a single solution and the solutions from the nullspace of a matrix operator. Within this framework, PR filter banks with unidentical prototypes for the analysis and synthesis stage can be designed. The design examples have shown that PR prototypes for oversampled filter banks with much higher stopband attenuations than for critically subsampled filter banks can be found. All PR solutions in the cosine-modulated case can be used for oversampled DFT filter banks as well. Finally, relations between PR oversampled and critically subsampled cosine-modulated pseudo QMF banks were discussed. It was shown that pseudo QMF banks having a partial aliasing-cancelation property can be designed via the design of PR oversampled filter banks.

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