Error Performance Analysis of Signal Superposition Coded Cooperative Diversity

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***Abstract*—This paper analyzes the error performance of a coded cooperative diversity system employing the Euclidean superposition of two BPSK-modulated signals. For an example using a convolutional code on block fading channels, the results show excellent agreement with computer simulations. The analysis makes it possible to optimize the power allocation between the local and relay signals numerically, circumventing the need for time consuming Monte Carlo simulations. Similarly, the analysis demonstrates how the power allocation can be "tuned" to compensate for unbalanced uplink channels and/or to provide unequal error protection to the data from the two cooperating nodes.**

***Index Terms*—Diversity methods, relay channels, fading channels, cooperative communications, signal superposition.**

I. Introduction

SIGNAL diversity - i.e., the transmission and/or reception of multiple versions of an information-bearing signal - is an effective countermeasure against the instantaneous signal-to-noise ratio (SNR) fluctuations caused by multipath fading in a wireless environment. A spatial diversity system employs two or more antennas separated in space, and it exploits the different channel gains that are realized between different transmit/receive antenna pairs. Cooperative diversity strategies employ relaying to share antennas among users, making spatial diversity possible even with single antenna transceivers.

A typical cooperative diversity system is illustrated in Figure 1. During each time slot, Node A and Node B transmit in turn to deliver their packets to a common destination D. To exploit spatial diversity and thereby enhance reliability on fading channels, each source node transmits both its own "local" packet as well as a "relay" packet that originated at its partner [1]–[10]. Methods to combine the local and relay packets in each node's transmission result in different cooperative diversity designs: separating the local and relay signals in orthogonal time slots leads to time multiplexing schemes [2]–[4]; adding the modulated local and relay signals in Euclidean space creates signal superposition schemes [1], [6], [7]; and adding the local and relay codewords over a finite field gives rise to algebraic superposition schemes [10].

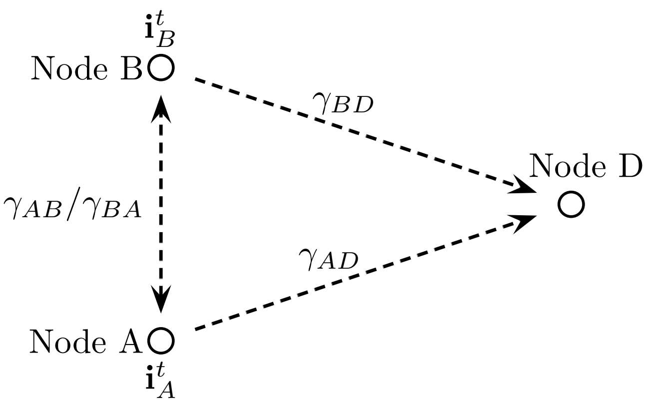

Fig. 1. The system model for cooperative diversity.

In the particular signal superposition scheme under consideration, each source node encodes and modulates its local packet and the relay packet separately and then transmits the Euclidean superposition of the two [6]. A key design parameter in such a system is the fraction of power dedicated to transmitting the relay signal, referred to as the *superposition factor*. While too little relay power would result in diminished diversity gain, allocating *too much* power to relaying reduces the cooperation success rate; to illustrate, the power that Node A dedicates to relaying Node B's information is wasted with regard to the task of successfully conveying Node A's information to Node B. In [6], the power allocation was optimized empirically via simulation, and the resulting signal superposition scheme outperformed designs based on time multiplexing for all error rates of interest.

The relayed message in a decode-and-forward system may be *repetition* based or it may provide *coded* cooperation. In the former case, the relayed message is the regenerated original codeword, and diversity is achieved by combining the repeated codewords. In coded cooperative diversity [11], the relayed message is a set of parity bits different from those used in the original transmission, and diversity is achieved by decoding

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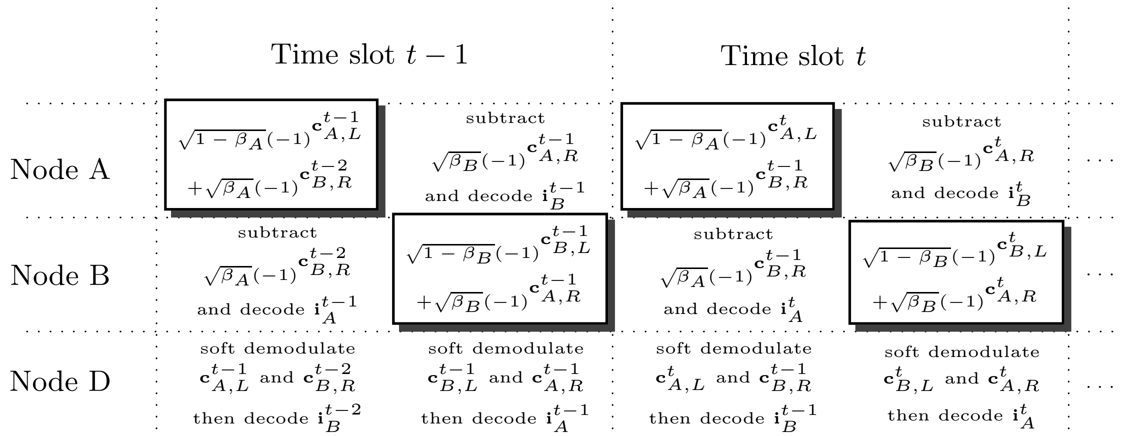

Fig. 2. The operations in the signal superposition cooperative diversity scheme. The emboldened boxes depict the transmitted signals.

with both sets of parities. Coded cooperative diversity avoids the inefficiency of repetition by using the structure of an error control code to gain diversity.

The performance of time multiplexed coded cooperative diversity was analyzed in [8] with respect to error probability and in [11] with respect to outage probability. For signal superposition schemes such as the one in [6], the two signals are added in Euclidean space and so are not orthogonal; consequently, the analysis in [8], [11] is not applicable. While an outage probability analysis for signal superposition cooperative diversity was presented in [12], there is no error probability analysis in the literature.

This paper derives a union bound on the error performance of a signal superposition cooperative diversity scheme; in particular, it considers a system in which both local and relayed information is channel-encoded (using potentially different codes to achieve a cooperative coding gain) and then modulated using binary phase-shift keying (BPSK). The analytical results facilitate numerical optimization of the superposition factor without resorting to computationally intensive simulations.

The paper is organized as follows. We present the system model and establish the notation in Section II. The end-to-end packet error rate performance is bounded in Section III. Numerical results and optimization of the superposition factor are given in Section IV, and Section V concludes the paper.

II. System Model and Notation

Denote the information packets generated at Node A and B during time slot t as $\mathbf{i}_A^t$ and $\mathbf{i}_B^t$, respectively. In coded cooperation, each packet is encoded (at most) twice – once locally at the originating node and (typically) again as a relayed packet at the partner node. Denote the two resulting codewords for $\mathbf{i}_A^t$ as $\mathbf{c}_{A,L}^t$ (encoded locally at Node A) and $\mathbf{c}_{A,R}^t$ (encoded as relayed information at Node B). Define $\mathbf{c}_{B,L}^t$ and $\mathbf{c}_{B,R}^t$ similarly for $\mathbf{i}_B^t$.

One of the source nodes initiates the session by broadcasting its encoded packet to both the destination and its partner. We assume the system is in continuous operation and neglect the effect of the initial state. The signal superposition cooperative diversity scheme in [6] proceeds as follows, focusing on the operations at Node A. (Node B operations are analogous.)

We assume coherent detection with perfect channel state information at the receiver side. Node A first attempts to decode Node B's packet $\mathbf{i}_B^{t-1}$. If successful, Node A modulates the relayed codeword $\mathbf{c}_{B,R}^{t-1}$ and the local codeword $\mathbf{c}_{A,L}^t$ separately to generate the relayed and local signals and transmits the Euclidean superposition of the two. The relayed signal and the local signal are allocated different powers prior to superposition. Assume Nodes A and B have superposition factors β_A and β_B, respectively – *i.e.*, Node A uses a fraction β_A of its power for relaying while Node B uses a fraction β_B for relaying. The superposition factors β_A and β_B are known to all three nodes. Assuming both the local and relay signals are BPSK modulated, Node A thus transmits the signal [6]

$$\mathbf{y}_A^t = \sqrt{1-\beta_A}(-1)^{\mathbf{c}_{A,L}^t} + \sqrt{\beta_A}(-1)^{\mathbf{c}_{B,R}^{t-1}}, \qquad (1)$$

where the signal constellation is normalized and the transmit power is included in the channel gain. (See Figure 2, time slot t, Node A.) The weighted sum of two BPSK signals in (1) forms a 4-PAM constellation that is non-uniform due to the different signal powers for local and relayed signals.

Assume the signal transmitted by Node A is received at both Node B and Node D through two independent block fading channels. Denote the instantaneous and average SNR from Node X to Node Y as γ_{XY} and Γ_{XY}, respectively, where X can be A or B and Y can be A, B, or D. The probability density function (PDF) of γ_{XY} is denoted as $p_{XY}(\gamma_{XY})$. During time slot t, Node B subtracts the relayed signal $\sqrt{\beta_A}(-1)^{\mathbf{c}_{B,R}^{t-1}}$, which it knows, from the received signal and decodes $\mathbf{i}_A^t$ from the noisy, faded version of $\sqrt{1-\beta_A}(-1)^{\mathbf{c}_{A,L}^t}$. Node D observes a non-uniform 4-PAM signal and employs a soft demodulator to calculate the log-likelihood ratios (LLRs) of $\mathbf{c}_{A,L}^t$ and $\mathbf{c}_{B,R}^{t-1}$. It estimates $\mathbf{i}_B^{t-1}$ using the LLRs for $\mathbf{c}_{B,R}^{t-1}$ and the previously received $\mathbf{c}_{B,L}^{t-1}$ using a decoder for the low rate codeword $[\mathbf{c}_{B,L}^{t-1}\ \mathbf{c}_{B,R}^{t-1}]$, whereas the LLRs for $\mathbf{c}_{A,L}^t$ are buffered.

Figure 2 depicts the operations at Nodes A, B, and D during time slots $t-1$ and t for the system under consideration. For simplicity, we assume that the receivers know the channel and that coherent demodulation is available. If a source node fails to decode its partner's packet – or if it is initiating the session and thus has nothing to relay – it modulates only its own packet and transmits the result using full power.

This model follows [6] closely with two exceptions. First, we consider coded cooperative diversity instead of the repetition based cooperation adopted in [6]. Second, in this model the two source nodes may have different superposition factors β_A and β_B, whereas only one global superposition factor was assumed in [6].

III. Performance Analysis

Invoking the symmetry between Nodes A and B, we focus, without loss of generality, on the error probability of the packets originating at Node A. Following the analysis in [8], there are four possible combinations of local and relay signals for which the destination node must make a decision:

- I Node A transmits the superposition of local and relayed signals; Node B is able to decode and relay A's packet.
- II Node A fails to decode B's packet and therefore transmits only a local signal. Node B is able to decode and relay A's packet.
- III Node A transmits the superposition of local and relayed signals; Node B fails to decode A and offers no help.
- IV Node A fails to decode B's packet and therefore transmits only a local signal. Node B fails to decode A and offers no help.

This section analyzes the probability of occurrence $\mathbb{P}_Z$ and the conditional packet error probability $\mathbb{P}_P(e|Z)$ for each of these four cases - i.e., for $Z \in \{I, II, III, IV\}$. The end-to-end packet error probability $\mathbb{P}_P(e)$ can then be expressed as [8]

$$\mathbb{P}_P(e) = \sum_{Z \in \{I,II,III,IV\}} \mathbb{P}_Z \mathbb{P}_P(e|Z), \quad (2)$$

where e denotes the error event. We calculate $\mathbb{P}_Z$ for $Z \in \{I, II, III, IV\}$ in Section III-A and derive bounds for $\mathbb{P}_P(e|Z)$ in Section III-C.

A. Cooperation Success Rate

Node A transmits either a superimposed signal (cooperative mode) or a signal modulated with only local bits (noncooperative mode). Since Node B can subtract the relayed portion from the received signal prior to decoding in the former case, it can decode A's message with fractional power $(1-\beta_A)$ when signal superposition is used or with full power when A is not relaying. Let η_A and η_B be the stationary probabilities that Nodes A and B, respectively, transmit superimposed signals. Because superposition is used at Node A if and only if Node B's packet was successfully decoded at Node A (and vice versa), these relations hold:

$$1 - \eta_A = \eta_B \cdot P^S_{B\to A} + (1 - \eta_B) \cdot P^N_{B\to A} \quad (3a)$$
$$1 - \eta_B = \eta_A \cdot P^S_{A\to B} + (1 - \eta_A) \cdot P^N_{A\to B}, \quad (3b)$$

where $P^S_{B\to A}$ and $P^N_{B\to A}$ are the packet (block) error probabilities of Node B's packets at Node A with superposition (fractional power $(1 - \beta_B)$) and without superposition (full power), respectively. Similarly $P^S_{A\to B}$ and $P^N_{A\to B}$ are the packet error probabilities of Node A's packets at Node B with fractional power $(1-\beta_A)$ and full power, respectively. Solving these equations for η_A and η_B, we obtain

$$\eta_A = \frac{\left(1 - P^N_{B\to A}\right) - \left(1 - P^N_{A\to B}\right)\left(P^S_{B\to A} - P^N_{B\to A}\right)}{1 - \left(P^S_{B\to A} - P^N_{B\to A}\right)\left(P^S_{A\to B} - P^N_{A\to B}\right)} \quad (4a)$$
$$\eta_B = \frac{\left(1 - P^N_{A\to B}\right) - \left(1 - P^N_{B\to A}\right)\left(P^S_{A\to B} - P^N_{A\to B}\right)}{1 - \left(P^S_{B\to A} - P^N_{B\to A}\right)\left(P^S_{A\to B} - P^N_{A\to B}\right)}. \quad (4b)$$

As a special case, when the channels $A \to B$ and $B \to A$ have the same statistics and the superposition factors at A and B are identical, then $P^N_{A\to B} = P^N_{B\to A}$, $P^S_{A\to B} = P^S_{B\to A}$, and (4) can be simplified as

$$\eta_A = \eta_B = \frac{1 - P^N_{A\to B}}{1 + P^S_{A\to B} - P^N_{A\to B}}. \quad (5)$$

The probability of the four cases described above can now be expressed as

$$\mathbb{P}_I = \eta_A \cdot \left(1 - P^S_{A\to B}\right) \quad (6a)$$
$$\mathbb{P}_{II} = (1 - \eta_A) \cdot \left(1 - P^N_{A\to B}\right) \quad (6b)$$
$$\mathbb{P}_{III} = \eta_A \cdot P^S_{A\to B} \quad (6c)$$
$$\mathbb{P}_{IV} = (1 - \eta_A) \cdot P^N_{A\to B}. \quad (6d)$$

If the inter-user channels are independent, then techniques such as those in [13] may be used to evaluate the block error probabilities $P^S_{B\to A}$, $P^N_{B\to A}$, $P^S_{A\to B}$, and $P^N_{A\to B}$. Conversely, if the instantaneous channel gain from Node A to Node B is identical to that from Node B to Node A – *i.e.*, $\gamma_{AB} = \gamma_{BA}$, a scenario we refer to by saying that the two inter-user channels form *reciprocal channels* – then the decoding failures at Nodes A and B are correlated. However, conditioned on the value of the instantaneous SNR γ_{AB}, the errors are independent. In this case, the probabilities $P^S_{B\to A}$, $P^N_{B\to A}$, $P^S_{A\to B}$, and $P^N_{A\to B}$ are evaluated assuming AWGN channel transmission with SNRs $\gamma_{AB}(1-\beta_B)$, γ_{AB}, $\gamma_{AB}(1-\beta_A)$, and γ_{AB}, respectively, and the expressions in (6) are averaged over the distribution of γ_{AB} to take the effect of fading into account.

More generally, a pair of correlated complex Gaussian channel gains can be decomposed into two independent parts and one common part [14, Eqn. (1)-(3)]. Hence general correlated Rayleigh/Ricean inter-user channels can be analyzed by first conditioning on the common part and calculating the probabilities in (6) as if the channels were independent and then averaging the results over the distribution of the common part.

B. The Soft Demodulator

When the two source nodes cooperate, the destination node must calculate the LLR of each bit from a received superimposed signal of the form

$$r = a(-1)^s + b(-1)^i + n, \quad (7)$$

where s is the desired bit (either relayed or local), i is the superimposed interfering bit, n is zero mean Gaussian noise with variance $N_0/2$, and a and b are the amplitudes of the desired signal and the interfering signal, respectively, whose

$$m_1(0|0) = -m_1(1|1) = \mathbb{E}\big[\tilde{\ell}(r)|s=0, i=0\big]$$
$$= \frac{2b}{\sqrt{\pi N_0}}\left(e^{-\frac{b^2}{N_0}} - e^{-\frac{(2a+b)^2}{N_0}}\right) + \frac{4}{N_0}\left[a^2 - b^2 Q\left(\sqrt{\frac{2b^2}{N_0}}\right) + b(2a+b)Q\left(\sqrt{\frac{2}{N_0}}(2a+b)\right)\right] \tag{10a}$$

$$m_1(0|1) = -m_1(1|0) = \mathbb{E}\big[\tilde{\ell}(r)|s=0, i=1\big]$$
$$= \frac{2b}{\sqrt{\pi N_0}}\left(e^{-\frac{b^2}{N_0}} - e^{-\frac{(2a-b)^2}{N_0}}\right) + \frac{4}{N_0}\left[a^2 + b(b-2a)Q\left(\sqrt{\frac{2}{N_0}}(b-2a)\right) - b^2 Q\left(\sqrt{\frac{2b^2}{N_0}}\right)\right] \tag{10b}$$

$$m_2(0|0) = m_2(1|1) = \mathbb{E}\big[\tilde{\ell}^2(r)|s=0, i=0\big]$$
$$= \frac{8b}{\sqrt{\pi N_0^3}}\left[(b^2 - 2a^2 - 2ab)e^{-\frac{(2a+b)^2}{N_0}} + (2a^2 - b^2)e^{-\frac{b^2}{N_0}}\right] - \frac{8b}{N_0^2}(4a^2 b + 2aN_0 - 2b^3 - bN_0)Q\left(\sqrt{\frac{2b^2}{N_0}}\right)$$
$$+ \frac{8b}{N_0^2}(8a^3 + 12a^2 b - 2b^3 + 2aN_0 - bN_0)Q\left(\sqrt{\frac{2}{N_0}}(2a+b)\right) + \frac{8a^2}{N_0^2}\left(2a^2 + N_0\right) \tag{10c}$$

$$m_2(0|1) = m_2(1|0) = \mathbb{E}[\tilde{\ell}^2(r)|s=0, i=1]$$
$$= \frac{8b}{\sqrt{\pi N_0^3}}\left[(2a^2 - 4ab + b^2)e^{-\frac{b^2}{N_0}} - (2a^2 - 2ab + b^2)e^{-\frac{(2a-b)^2}{N_0}}\right] - \frac{8b}{N_0^2}(4a^2 b + 2b^3 - 8ab^2 + bN_0 - 2aN_0)Q\left(\sqrt{\frac{2b^2}{N_0}}\right)$$
$$- \frac{8b}{N_0^2}(8a^3 - 12a^2 b + 8ab^2 - 2b^3 + 2aN_0 - bN_0)Q\left(\sqrt{\frac{2}{N_0}}(b-2a)\right) + \frac{8a^2}{N_0^2}(2a^2 + N_0) \tag{10d}$$

values are determined by the appropriate channel gains and the superposition factors β_A and β_B.

To provide a meaningful analysis for such a system, a model for the soft demodulator is required. Since both signals $a+b$ and $a-b$ correspond to $s=0$ while both signals $-a+b$ and $-a-b$ correspond to $s=1$, the ideal soft demodulator output for bit s is [15]

$$\ell(r) = \log\frac{\exp(-\frac{(r-a+b)^2}{N_0}) + \exp(-\frac{(r-a-b)^2}{N_0})}{\exp(-\frac{(r+a+b)^2}{N_0}) + \exp(-\frac{(r+a-b)^2}{N_0})}. \tag{8}$$

A good approximation $\tilde{\ell}(r) \approx \ell(r)$ that also simplifies the analysis can be obtained by taking only the dominant terms in the numerator and denominator:

$$\begin{aligned}\tilde{\ell}(r) &= \frac{1}{N_0}\Big\{\min[(r+a+b)^2, (r+a-b)^2] \\ &\quad - \min[(r-a+b)^2, (r-a-b)^2]\Big\} \\ &= \begin{cases} \frac{4a}{N_0}(r+b) & r < -a \\ \frac{4(a-b)}{N_0}r & -a \le r \le a \\ \frac{4a}{N_0}(r-b) & r > a. \end{cases}\end{aligned} \tag{9}$$

The means $m_1(s|i)$ and second moments $m_2(s|i)$ of $\tilde{\ell}(r)$ are given by (10) at the top of this page, where the channel observation r in (7) is a Gaussian random variable for any particular combination of s and i.

In a Gaussian channel, the mean of the output is the signal amplitude and the variance of the output is the noise variance. The channel SNR is proportional to the ratio between the square of the signal amplitude (signal energy) and the variance of the output. Since the LLRs at the output of the demodulator represent the soft inputs of the channel decoder, we now define two equivalent SNRs (given as $2m_1^2/(m_2 - m_1^2)$) by[1] [16]

$$\gamma_{eq,1} = \frac{2m_1^2(0|0)}{m_2(0|0) - m_1^2(0|0)} \triangleq \rho_1\left(\frac{a^2}{N_0}, \frac{b^2}{N_0}\right) \tag{11a}$$
$$\gamma_{eq,2} = \frac{2m_1^2(0|1)}{m_2(0|1) - m_1^2(0|1)} \triangleq \rho_2\left(\frac{a^2}{N_0}, \frac{b^2}{N_0}\right), \tag{11b}$$

where the quantities $\frac{a^2}{N_0}$ and $\frac{b^2}{N_0}$ can be interpreted as the SNRs of the desired bit and the superimposed interfering bit, respectively. The functions $\rho_1(\cdot,\cdot)$ and $\rho_2(\cdot,\cdot)$ characterize how the equivalent SNR depends on the SNRs of the desired bit and the superimposed bit, given that the two bits have the same or opposite values.

C. Conditional Error Probabilities

In the absence of cooperation, each source node transmits its own BPSK-modulated symbols, and thus the LLRs at the demodulator output have a Gaussian distribution. When superposition is applied, however, both the exact and simplified approximate LLRs in (8) and (9) are no longer Gaussian; under this scenario, the exact pairwise error probabilities can be computed via the moment generating function (MGF) [17, Appendix A]; however, the MGF method requires inverse Laplace transforms, which can be computationally burdensome. As a result, we instead adopt a Gaussian approximation for these LLRs. This can be justified for two reasons: first, the LLRs are piecewise linear functions of Gaussian random variables, as seen in (9), and hence have Gaussian-like tails; second, an error event in a coded system involves several LLRs, and so a central limit theorem argument can be invoked.

[1] The factor of two comes from the fact that the SNR is defined as E_s/N_0 and the noise variance is $N_0/2$.

$$P_I(d|\gamma_{AD},\gamma_{BD}) = \sum_{j=0}^{d_1}\sum_{k=0}^{d_2} 2^{-(d_1+d_2)}\binom{d_1}{j}\binom{d_2}{k} Q\left(\sqrt{2\Delta_1+2\Delta_2}\right) \tag{13a}$$

$$\Delta_1 \triangleq j\rho_1\big(\gamma_{AD}(1-\beta_A),\gamma_{AD}\beta_A\big) + (d_1-j)\rho_2\big(\gamma_{AD}(1-\beta_A),\gamma_{AD}\beta_A\big) \tag{13b}$$

$$\Delta_2 \triangleq k\rho_1\big(\gamma_{BD}\beta_B,\gamma_{BD}(1-\beta_B)\big) + (d_2-k)\rho_2\big(\gamma_{BD}\beta_B,\gamma_{BD}(1-\beta_B)\big) \tag{13c}$$

$$P_{II}(d|\gamma_{AD},\gamma_{BD}) = \sum_{k=0}^{d_2} 2^{-d_2}\binom{d_2}{k} Q\left(\sqrt{2d_1\gamma_{AD} + 2k\rho_1\big(\gamma_{BD}\beta_B,\gamma_{BD}(1-\beta_B)\big) + 2(d_2-k)\rho_2\big(\gamma_{BD}\beta_B,\gamma_{BD}(1-\beta_B)\big)}\right) \tag{13d}$$

$$P_{III}(d_1|\gamma_{AD}) = \sum_{j=0}^{d_1} 2^{-d_1}\binom{d_1}{j} Q\left(\sqrt{2j\rho_1\big(\gamma_{AD}(1-\beta_A),\gamma_{AD}\beta_A\big) + 2(d_1-j)\rho_2\big(\gamma_{AD}(1-\beta_A),\gamma_{AD}\beta_A\big)}\right) \tag{13e}$$

$$P_{IV}(d_1|\gamma_{AD}) = Q\left(\sqrt{2d_1\gamma_{AD}}\right) \tag{13f}$$

$$\mathbb{P}_P(e|Z) \le \begin{cases} 1-\int_0^\infty\int_0^\infty \left(1-\min\left[1,\sum_{d=d_f}^{\infty} a(d)P_Z(d|\gamma_{AD},\gamma_{BD})\right]\right)^K p_{AD}(\gamma_{AD})p_{BD}(\gamma_{BD})\mathrm{d}\gamma_{AD}\mathrm{d}\gamma_{BD}, & Z=I,II \\ 1-\int_0^\infty \left(1-\min\left[1,\sum_{d_1=d'_f}^{\infty} a'(d_1)P_Z(d_1|\gamma_{AD})\right]\right)^K p_{AD}(\gamma_{AD})\mathrm{d}\gamma_{AD}, & Z=III,IV \end{cases} \tag{14}$$

Assuming that linear codes are used, we only need to consider the pairwise error probability between the all-zero codeword and an erroneous codeword with Hamming weight d to employ the union bound [18]. In a Gaussian channel, the pairwise error probability can be written as [19, eqn. (12.13)]

$$Q\left(\sqrt{2\sum_{i=1}^{d}\gamma_i}\right), \tag{12}$$

where γ_i is the channel SNR of the i-th nonzero bit in the erroneous codeword. Note that, for the LLR of a superimposed bit, the Gaussian approximation is conditioned on the value of the interfering bit; thus, the equivalent SNR in (11) depends on whether the two bits are equal or not. Hence, for an erroneous codeword with Hamming weight d', there are $2^{d'}$ possible combinations of the two equivalent SNRs $\rho_1(\cdot,\cdot)$ and $\rho_2(\cdot,\cdot)$, and the multiplicity of combinations where k out of d' bits have value one is given by $\binom{d'}{k}$. The pairwise error in each combination can be calculated using (12), and it follows that we can formulate the pairwise error probabilities for each superposition case as the weighted sum of the pairwise error probabilities in each combination, shown in (13), where d_1 is the Hamming weight contributed to the erroneous codeword by the local codeword $\mathbf{c}^t_{A,L}$ generated at Node A, d_2 is the Hamming weight contributed to the same erroneous codeword by the relay codeword $\mathbf{c}^t_{A,R}$ regenerated at Node B, and $d_1+d_2=d$ is the Hamming weight of the erroneous codeword in the overall low rate code that combines $\mathbf{c}^t_{A,L}$ and $\mathbf{c}^t_{A,R}$.

For each case except case IV, the pairwise probability is obtained as a weighted sum of Q-functions over all possible combinations of the equivalent SNRs, where the weights correspond to the total probability of each combination. The arguments of the Q-functions depend on the sum of the SNRs associated with those bits that contribute Hamming weight to the erroneous codeword. For the LLRs corresponding to signal superposition, the equivalent SNR for the transmission of local information from Node A can be either $\rho_1\left(\gamma_{AD}(1-\beta_A),\gamma_{AD}\beta_A\right)$ or $\rho_2\left(\gamma_{AD}(1-\beta_A),\gamma_{AD}\beta_A\right)$. Similarly, the equivalent SNR for the transmission of relay information from Node B is either $\rho_1\left(\gamma_{BD}\beta_B,\gamma_{BD}(1-\beta_B)\right)$ or $\rho_2\left(\gamma_{BD}\beta_B,\gamma_{BD}(1-\beta_B)\right)$. The "limit before averaging" technique [13] can provide tight and numerically useful upper bounds in fading channels by limiting the union bound to be less than one before averaging over the fading distribution. Thus, we adopt this technique to evaluate the conditional packet error probability as shown in (14) at the top of this page, where d_f is the free distance of the overall low rate convolutional code combining $\mathbf{c}^t_{A,L}$ and $\mathbf{c}^t_{A,R}$ and d'_f is the free distance of the high rate local convolutional code. Additionally, $a(d)$ represents the multiplicity of combined codewords with Hamming weight $d=d_1+d_2$, and $a'(d_1)$ is the multiplicity of local codewords with Hamming weight d_1. Finally, K is the number of information bits in one packet. (The conditional bit error probabilities can also be bounded using the pairwise error probabilities given in (13) in a straightforward manner [8], [13].)

IV. Numerical Results and Optimization

We use the cooperative convolutional code design derived from a block fading channel perspective in [4] for our numerical examples. The local codewords $\mathbf{c}^t_{A,L}$ and $\mathbf{c}^t_{B,L}$ are generated by the rate 1/2 code (in octal notation) $[15,\ 17]_8$ and the relay codewords $\mathbf{c}^t_{A,R}$ and $\mathbf{c}^t_{B,R}$ are generated by the rate 1/2 code $[13,\ 15]_8$. The destination node uses the decoder corresponding to the rate $1/4$ code $[15,\ 17,\ 13,\ 15]_8$ to exploit diversity when a relayed signal is available. In evaluating

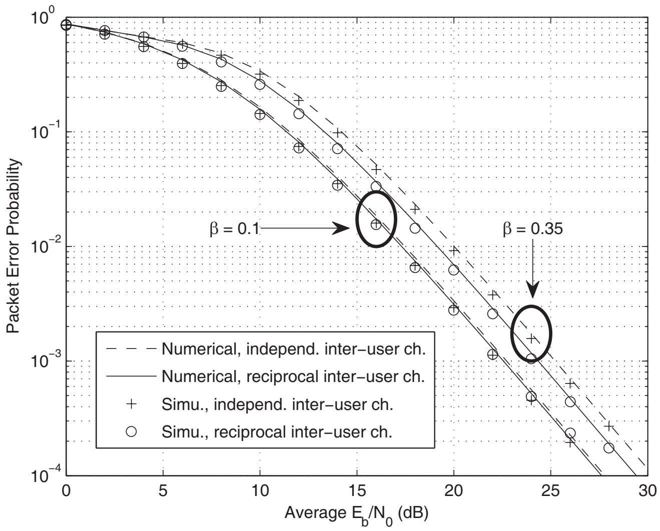

Fig. 3. Packet error probability for Rayleigh fading channels with $\beta = \beta_A = \beta_B$, $\Gamma_{AD} = \Gamma_{BD} = \Gamma_{AB}$.

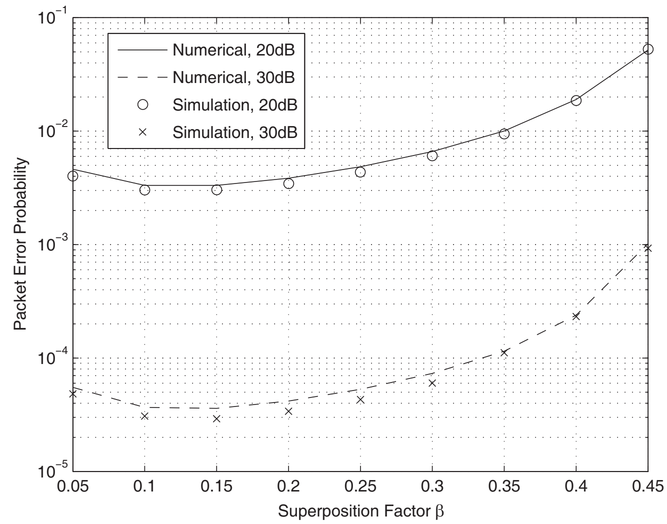

Fig. 4. Packet error probability as a function of the superposition factor $\beta = \beta_A = \beta_B$ at an average E_b/N_0 of 20 dB and 30 dB, assuming $\Gamma_{AD} = \Gamma_{BD} = \Gamma_{AB}$ and independent inter-user channels.

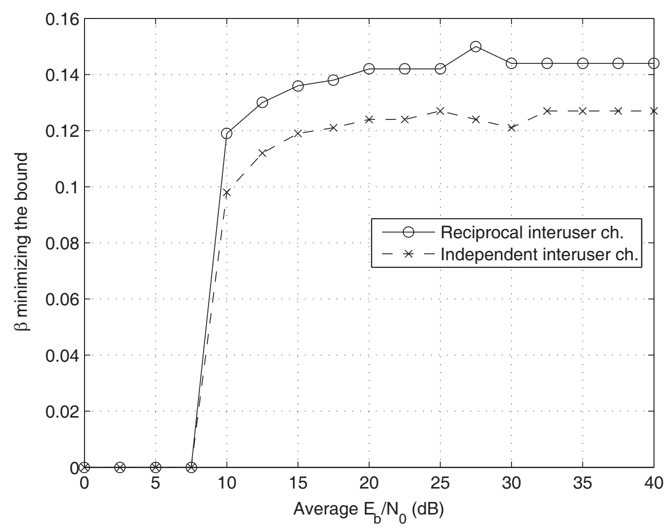

Fig. 5. Values of $\beta = \beta_A = \beta_B$ that minimize the analytical bound at different SNRs $\Gamma_{AD} = \Gamma_{BD} = \Gamma_{AB}$, for both independent inter-user channels and reciprocal inter-user channels.

the union bound, terms corresponding to distance greater than $d_f + 15$ are dropped. The Hamming distances and their corresponding multiplicities can be efficiently calculated using a slightly modified BEAST algorithm [20]. All channels are subject to independent block Rayleigh fading. The information packets contain 500 bits each, and 12 additional CRC bits are appended prior to convolutional encoding to detect decoding failures at the partner nodes. (Since the CRC is neglected in our analysis, we do not distinguish between information bits and CRC bits and use $500 + 12 = 512$ bit packets for the numerical calculation.)

First consider the case in which the fading observed on all the channels has the same distribution and the power allocations at Nodes A and B are identical – *i.e.* $\beta_A = \beta_B = \beta$. In Figure 3, the calculated packet error probability union bound approximations are compared with Monte Carlo simulations, with superposition factors $\beta = 10\%$ and $\beta = 35\%$, for two different scenarios - one in which the inter-user channels have independent fading and another in which the fading observed in both directions in a given time slot is the same (i.e., the "reciprocal channel" assumption). In all four cases, the analytical results track the simulation results closely. For example, at a packet error rate of 10^{-2}, the bound is only about 0.3 dB from the simulation curve, and this level of agreement is consistent over the entire range of interest. (This consistency is due to the averaging effect of the fading distribution.) Comparing the independent inter-user channels case with the reciprocal inter-user channels case, the latter results in a slightly lower error probability. This is best explained by the fact that $\gamma_{AB} = \gamma_{BA}$ reduces the probability of case III – the worst possible scenario, in which one node is helping the other node but is not being helped itself. The more pronounced disparity with increased β is due to the error performance in case III being made worse by large β's.

Clearly the superposition factor β plays an important role in error performance. The effect of β on the packet error probability is shown in Figure 4 for independent inter-user channels and average SNR's of 20 dB and 30 dB. Once again, the approximation provides good agreement with the simulation results for different values of $\beta = \beta_A = \beta_B$. It is worth noting that the simulation results tend to be closer to the bound for larger values of β. We conjecture the reason is the error in the Gaussian approximation, which potentially offsets some looseness in the union bound.

Figure 5 shows the optimal value of $\beta = \beta_A = \beta_B$ as a function of the average SNR, for both the reciprocal inter-user channels and the independent inter-user channels. Here, the "optimal" value of β is the one that minimizes the union bound, obtained by searching over the interval $(0, 0.5)$. The optimum β is shown to be zero in the low SNR region; setting $\beta = 0$ degenerates the signal superposition system into a noncooperative system, which performs better than signal superposition cooperation at low SNRs. The optimum β for SNRs above 10 dB falls into the 10-15% range found empirically in [6] for a repetition based configuration; more

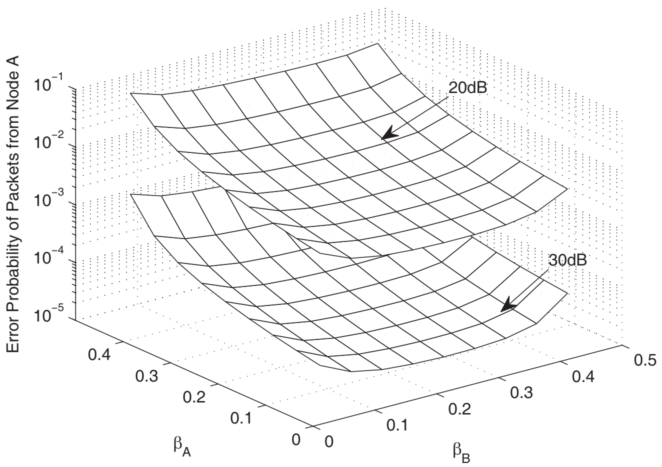

Fig. 6. The error probabilities of the packets originating at Node A as a function of the superposition factors β_A and β_B. The average E_b/N_0 is 20 dB and 30 dB, respectively.

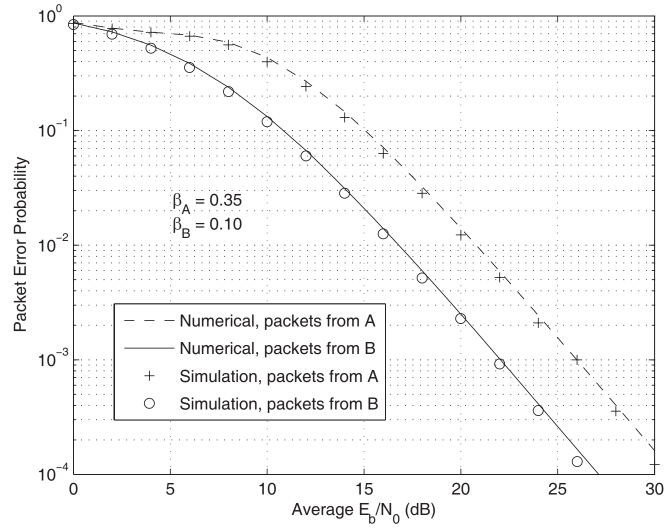

Fig. 7. Setting $\beta_A \neq \beta_B$ can create error performance disparities, even with equivalent uplink channels. Independent inter-user channels and identical average channel SNRs are assumed.

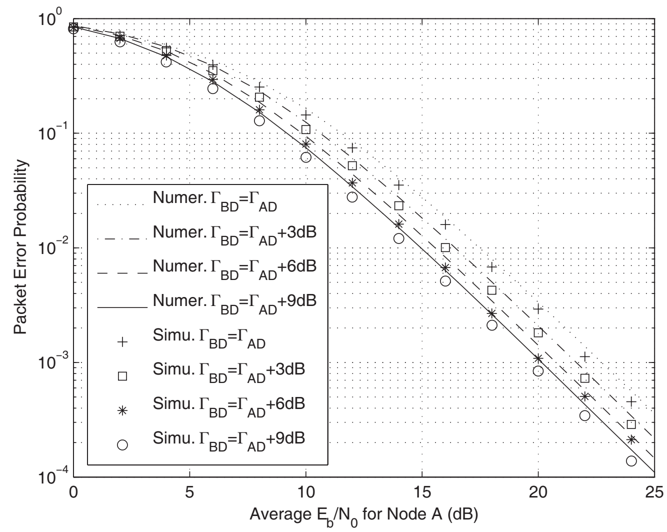

Fig. 8. The error probabilities of the packets originating at Node A, when its partner, Node B, enjoys a better channel, *i.e.* a larger Γ_{BD}. Independent inter-user channels and $\beta_A = \beta_B = 0.1$ are assumed.

specifically, Figure 5 indicates that the optimum β approaches a constant at high SNRs – $\beta = 0.127$ for independent inter-user channels and $\beta = 0.144$ for reciprocal inter-user channels. (At all SNR values, the optimum value of β for reciprocal channels is higher than for independent channels.) More generally, the optimal value of β depends on the particular channel model and the convolutional code being used.

Now we consider the more general case where $\beta_A \neq \beta_B$. Figure 6 depicts the union bound on the error probability of the packets originating at Node A as a function of β_A and β_B. The interuser channels are independent and all channels have the same average SNR of 20 or 30 dB (the error probability of the packets originating at Node B can be obtained by exchanging β_A and β_B). Figure 6 shows that the packet error rate for Node A is more sensitive to changes in its own superposition factor (β_A) than to changes in its partner's (β_B). Comparing the 20 dB case to the 30 dB case, it is apparent that the superposition factors have a more significant impact at higher SNRs.

Figure 6 also indicates that it is possible to provide unequal error protection for the traffic from the two different nodes, even with similar uplink channels ($\Gamma_{AD} = \Gamma_{BD}$). As an example, Figure 7 shows the packet error probabilities for a cooperative diversity system with $\beta_A = 0.35$ and $\beta_B = 0.1$. By allocating more of Node A's power to cooperation, Node B's performance can be improved at the cost of deteriorated performance for Node A's packets. This property could be of value when Node A and Node B have different error rate requirements; the analysis provides a useful tool to design and tune these unequal error requirements.

The effect of "unbalanced" uplinks to the destination - i.e., $\Gamma_{AD} \neq \Gamma_{BD}$ - is considered in Figures 8 and 9. In both figures, $\Gamma_{AD} = \Gamma_{AB} = \Gamma_{BA}$ is the value on the X-axis and Γ_{BD} is varied to be 9 dB, 6 dB, 3 dB, and 0 dB higher than Γ_{AD}. It is evident that Node B enjoys an improved error performance when Γ_{BD} increases, and Node A also benefits from Node B's better channel to the destination thanks to the effect of diversity. Comparing Figures 8 and 9, it is observed - not surprisingly - that Node A has slightly worse performance than Node B when Node B's uplink channel is better than Node A's. This is because, while both nodes benefit from the improved Γ_{BD}, Node B exploits it more effectively in two ways: first, because $\beta_B = 0.1$, only 10% of the better channel is used to relay Node A's data while 90% of the better channel is used to transmit Node B's data; and second, Node B *always* gets to transmit over the better channel, while Node A's packets are conveyed over the better channel only when Node B successfully decodes Node A's transmission.

Figure 10 displays the effects of both unequal superposition factors ($\beta_A \neq \beta_B$) *and* unbalanced uplinks. Specifically, this figure assumes $\Gamma_{AB} = \Gamma_{BA} = \Gamma_{AD} = 30$ dB and $\Gamma_{BD} = 33$ dB. As in the identical uplink channel case, tuning β_A and β_B improves the performance of one source at the cost of the other. Moreover, it is interesting to observe that, despite the unbalanced uplink channels, the two error

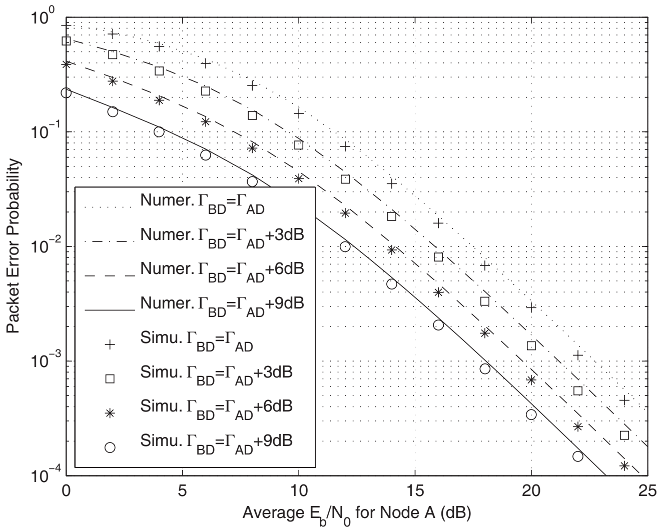

Fig. 9. The error probabilities of the packets originating at Node B, when it enjoys a better channel than its partner, Node A. Independent inter-user channels and $\beta_A = \beta_B = 0.1$ are assumed.

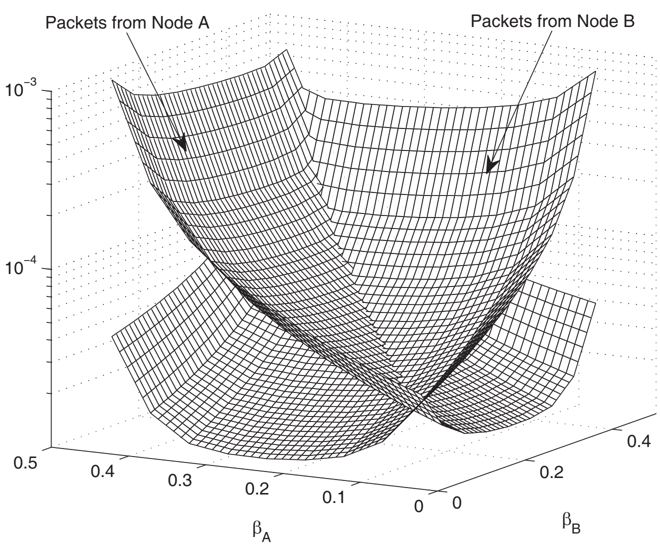

Fig. 10. Packet error probabilities for packets originating at Node A and Node B, respectively, as functions of the superposition factors β_A and β_B when $\Gamma_{AD} = \Gamma_{AB} = 30$ dB and $\Gamma_{BD} = 33$ dB.

probability mesh intersects. Thus, with proper tuning of β_A and β_B, it is possible to provide *equal* error protection even when the uplink channel conditions are unequal.

V. Conclusions

We have developed an analytical performance bound for a coded cooperative diversity signal superposition system. Numerical calculations based on the bound match computer simulation results very closely. Minimizing the bound shows the dependence of the optimum superposition factor on the operating SNR. By allowing different superposition factors at the two partner nodes, we can either provide unequal error protection for the traffic from the two nodes, or, conversely, equalize the error probabilities even when one node has a better channel to the destination. QPSK modulated signal superposition systems can also be addressed using our analysis in a straightforward manner since a QPSK signaling can be viewed as a BPSK signaling in both the I and the Q channel. The analysis provides a useful tool for performance prediction and design parameter optimization for signal superposition based cooperative diversity systems.

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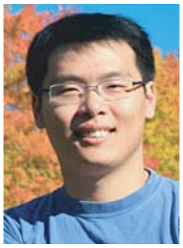

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