Design of Network Codes for Multiple-User Multiple-Relay Wireless Networks

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***Abstract*—We investigate the design of network codes for multiple-user multiple-relay (MUMR) wireless networks with slow fading (quasi-static) channels. In these networks, M users have independent information to be transmitted to a common base station (BS) with the help of N relays, where $M \geq 2$ and $N \geq 1$ are arbitrary integers. We investigate such networks in terms of diversity order to measure asymptotic performance. For networks with orthogonal channels, we show that network codes based on maximum distance separable (MDS) codes can achieve the maximum diversity order of $N+1$. We further show that the MDS coding construction of network codes is also necessary to obtain full diversity for linear finite field network coding (FFNC). Then, we compare the performance of the FFNC approach with superposition coding (SC) at the relays. The results show that the FFNC based on MDS codes has better performance than SC in both the high rate and the high SNR regime. Further, we discuss networks without direct source-to-BS channels for $N \geq M$. We show that the proposed FFNC can obtain the diversity order $N-M+1$, which is equivalent to achieving the Singleton bound for network error-correction codes. Finally, we study the network with nonorthogonal channels and show our codes can still achieve a diversity order of $N+1$, which cannot be achieved by a scheme based on SC.**

***Index Terms*—Network coding, relay, finite field, MDS codes, diversity order.**

I. Introduction

WIRELESS relay channels and networks have recently attracted substantial research efforts with the goal of improving performance in terms of energy efficiency, throughput or coverage [1]–[3]. In relay networks, one or more intermediate nodes are employed to help the sources to transmit the information. Various transmission protocols with different complexity and performance have been proposed for these networks, e.g., amplify-and-forward and decode-and-forward [1]–[3].

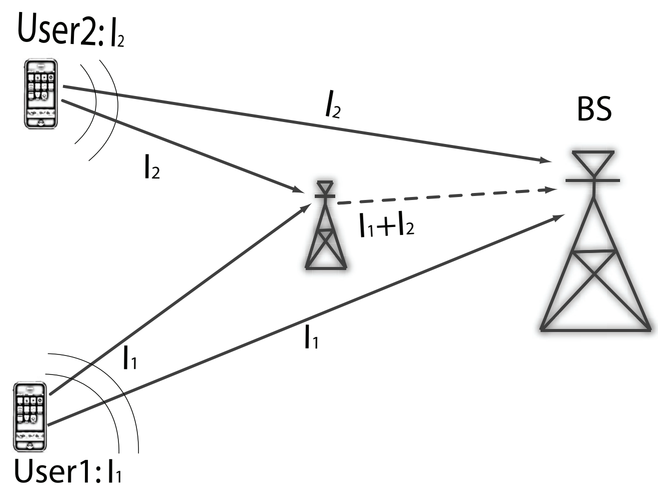

Fig. 1. Two-user one-relay network with binary network coding. User 1 and user 2 have source messages I_1 and I_2, respectively.

By allowing information processing at the intermediate nodes, network coding [4], [5] has been originally proposed for networks where the links consist of noiseless bit pipes. Here, network coding is able to achieve the min-cut for multicast transmission [6]. Later, physical layer network coding has been proposed for multiple-source multiple-hop networks [7]–[16] to improve the transmission performance. The results show that wireless networks, in particular wireless relay networks, exhibit a better performance in terms of energy-efficiency or end-to-end error probability by employing physical-layer network coding. One example of using network coding for relay networks is given by the multiple access relay channel (MARC) [7], [8], shown in Fig. 1. The relay combines the received information from two users, for example by component-wise addition in GF(2), if decoding is successful. At the sink, joint decoding can be used to decode codewords received directly from the user and from the relay by, e.g., maximum-likelihood decoding [11] or iterative decoding among the codewords of different channels [7].

In [15], an LDPC-based coding design was investigated for two-user one-relay networks which achieves full diversity. In [14], a joint design of network coding and media access control (MAC) was considered for wireless *ad hoc* networks and characterized in terms of delay, interference and throughput. Efficient conflict-free scheduling policies are proposed to maximize the throughput or to minimize the node costs (e.g., energy consumption) with network coding. In this

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paper, however, we will not consider the scheduling problem. Instead, we will primarily study the design of network codes characterized in terms of physical layer parameters such as diversity, fading, or outage probabilities.

Although the application of network coding to relay networks seems to be natural and beneficial, most of the previous schemes (e.g., [7], [9]–[11]) only address two-user one-relay networks. Design principles for using network coding in multiple-user multiple-relay (MUMR) networks are mostly unexplored. In [13], physical-layer network coding was studied for multiple-user single-relay networks, where the exact bit error rate (BER) was derived. However, the scheme in [13] employed superposition coding which, as we will show in Section III-C is not optimal for networks with multiple relays. More importantly, a significant amount of previous work [7], [9]–[11] considered binary network coding schemes which represented a special case of general finite field network coding. However, we will show in the following that binary network coding is generally not optimal for multiple-relay wireless networks in terms of transmission efficiency e.g., outage probability, frame error rate, and energy efficiency. In [17], a coding scheme was proposed for multiple-user and one-central-processor (as the encoder) wired networks. However, the scenario of multiple relay nodes, particularly for wireless channels, has not been studied in [17]. In [18], optimal coding schemes have been investigated for symmetrical multilevel diversity coding for independent or correlated sources without considering the impact of the channel errors. In [16], analog network coding was investigated for Gaussian channels where it is shown that such coding can approach the cut-set bound within a constant gap, if the received energy level grows without bound. However, the impact of channel fading was not considered in [16]. In [12] non-binary linear network coding was considered by exploiting temporal diversity of fast fading channels for cooperative communications.

In the present paper, we investigate the design of efficient network codes for wireless networks with arbitrary $M, (M \geq 2)$ users and $N, (N \geq 1)$[1] relays. The motivation is that in practical wireless networks, e.g., LTE and beyond [19] cellular or *ad hoc* wireless networks, multiple separate relays are beneficial to increase the system performance. These relays may be able to receive the signal of any user due to the broadcasting nature of the wireless medium. It is also reasonable to model wireless access networks as multiple-user multiple-relay networks. A study of efficient network codes for such networks is interesting. With increasing data rates, slow fading (quasi-static) channels are one of the most important channel models [19], [20]. A useful question is how to increase performance for relay networks with such channels. Furthermore, it is valuable to compare network codes with a multiple access scheme based on nonorthogonal channels and signal domain superposition coding. It is also valuable to find the optimal relaying and coding strategies for different access protocols with different channel costs.

The main contributions of the paper are as follows. We propose a non-binary network code construction based on maximum distance separable (MDS) codes to achieve full diversity for arbitrary M, N, which generally cannot be achieved for $N > 1$ by binary network codes. Furthermore, we show that the proposed MDS construction is actually a necessary condition to achieve full-diversity for finite field network codes (FFNC). As an alternative to FFNC we also consider superposition coding (SC) and show that for high rates FFNC is able to outperform SC. Then, we consider networks without direct source-BS channels and show that the proposed FFNC approach obtains a diversity order of $N-M+1$ which achieves the Singleton bound for network error-correction codes [21], [22]. Finally, we consider networks with nonorthogonal channels and successive interference cancellation (SIC) decoding. We show that for such networks, the proposed network codes can achieve full-diversity which cannot be achieved by the SC-based scheme. The organization of the paper is as follows. In Section II, we give the system description of the network model and the corresponding network code. We provide a performance analysis of our system in Section III. Section III-C compares the performance of FFNC and SC. In Section IV we extend our results to the case where a direct channel between the users and the BS is absent. Finally, in Section V we discuss networks with nonorthogonal channels.

[1]We mainly investigate networks with $N \geq 2$, but our design approach and the presented analysis is also applicable for $N = 1$ as a special case.

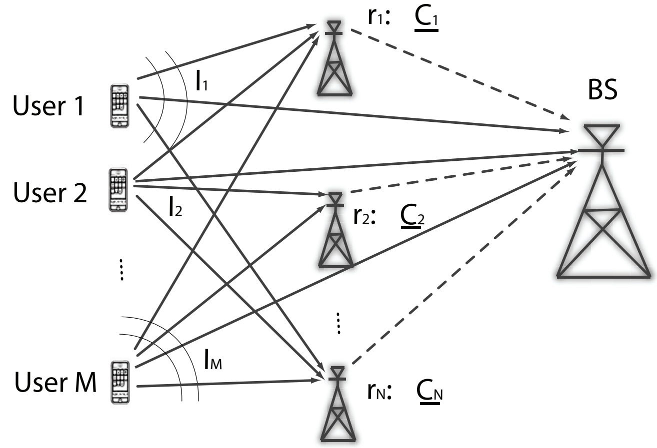

Fig. 2. Wireless network with M users and N relays.

II. System Description

A. Network Model

As shown in Fig. 2, we consider wireless networks with an arbitrary number of users and relays, i.e., $M \geq 2$, $N \geq 1$. We assume that for each user there is a direct channel to the BS and there are in addition N paths via shared relays. We assume that all users transmit information messages with the same rates R on each channel. If the source nodes communicate to the BS, all relay nodes will also receive the corresponding codewords due to the broadcast property of the wireless medium. Then, the relay nodes try to decode, and if the decoding is successful, they can forward the information to the BS with suitable processing, e.g., by applying network coding. In Fig. 2, the output network codewords are denoted as $\underline{C}_i$ for relay i. A detailed calculation of $\underline{C}_i$ will be discussed later. Further, similar to [2], [23], we assume narrow-band transmission suffering the effects of path-loss, shadowing, frequency nonselective fading, and additive white Gaussian noise (AWGN). We also assume independent quasi-static fading in all channels and measure the performance by the outage probability to analyze the effects of spatial diversity. Fading coefficients are independent, identically distributed (i.i.d.) random variables for different channels but constant for all symbols of one or more codewords on the same channel [2], [20]. Thus, there is no temporary diversity to be exploited. For slow fading channels a received codeword (baseband) is given as

$$Y_{i,j} = a_{i,j} X_{i,j} + n_{i,j}, \tag{1}$$

where $X_{i,j}$ and $Y_{i,j}$ are the transmitted and received channel codewords, respectively. Here $n_{i,j}$ is an additive white Gaussian noise sample with double-sided power spectral density $N_0/2$, and $a_{i,j}$ denotes the channel gain due to frequency nonselective multipath fading. The indices i denote the transmitting nodes, namely the users and the relays, whereas the j's denote the receiving nodes, namely the BS and relays. The $a_{i,j}$s have zero-mean and are i.i.d. random variables. We assume that $|a_{i,j}|$ is Rayleigh distributed and has unit variance. For the sake of a tractable analysis we assume that all channels have the same average SNR. An extension to the unequal case is straightforward and generally leads to different outage probabilities for each link. We further assume that for all links channel state information (CSI) is available at the corresponding receivers (BS or relays), whereas CSI is not exploited at the transmitters. We first consider networks with orthogonal channels, assuming each node transmits in a different time slot (the analysis is similar if they are instead using different frequency bands). There are in total $M+N$ time slots used by all users and relays. For the nonorthogonal channel cases discussed in Section V, two time slots and half-duplex relays are employed.

We also define $\text{SNR} = \frac{E_s}{N_0}$, where E_s denotes the transmission energy per symbol. To measure the performance in the medium-to-high signal-to-noise ratio (SNR) regime for quasi-static fading channels the diversity order is calculated by [20]

$$D \triangleq \lim_{\text{SNR}\to\infty} \frac{-\log P_e}{\log \text{SNR}}, \tag{2}$$

where P_e is the outage probability. We can see from Fig. 2 that the corresponding codewords for each user are transmitted to the BS through $N+1$ independent fading paths: one direct path and N paths via the shared relay nodes. Thus, the maximum diversity for each user corresponds to $N+1$. To concentrate our analysis on diversity order, we assume that a perfect channel code is employed on the physical layer and that network coding is implemented on top of channel coding. This means that a relay node first forms a network codeword, and then regards the network codeword as a sequence of information symbols and produces a channel codeword which is transmitted to the BS. An outage event occurs on the channel when the transmission rate is higher than the instantaneous mutual information between channel input and output. Here the instantaneous mutual information is evaluated as $\mathcal{I} = \frac{1}{2(M+N)} \log(1 + |a_{i,j}|^2 \text{SNR})$, where the factor $\frac{1}{(M+N)}$ is due to the fact that the total transmission time is shared equally between the $M+N$ transmitting nodes, and the factor $1/2$ is because we only consider real-valued signals. Thus, an outage event occurs when the fading coefficients are smaller than a certain threshold, i.e., $|a_{i,j}|^2 < Z$, where $Z = \frac{2^{2(M+N)R}-1}{\text{SNR}}$ for Rayleigh fading channels. Then we can evaluate the outage probability of an individual channel as [2], [20]

$$P_e = \Pr\{|a_{i,j}|^2 < Z\} = 1 - e^{-Z}. \tag{3}$$

As $\text{SNR} \to \infty, P_e \approx \frac{2^{2(M+N)R}-1}{\text{SNR}} = C_I \text{SNR}^{-1}$, where $C_I = 2^{2(M+N)R} - 1$ is constant with SNR.

B. Network Coding Scheme

To facilitate analysis, we use a transfer matrix [5] to describe the network codes (linear finite field codes) for our networks. The transfer matrix $\mathcal{K}$ is given as follows. From the left to the right, we provide the global encoding kernels (GEKs) for direct transmission, corresponding to users $1, 2, \cdots, M$, and then the ones for the codewords associated with the relays $1, 2, \cdots, N$. Here, a GEK denotes the linear relation between an outgoing network codeword and source information, i.e., at relay j, $\underline{C}_j = \underline{I}\ \underline{G}_j$, where $\underline{C}_j$ denotes the outgoing codeword at relay j, and $\underline{G}_j = [\gamma_{1,j}, \gamma_{2,j}, \cdots, \gamma_{M,j}]^T$ is the GEK of relay j, and $\underline{I} = [I_1, I_2, \cdots, I_M]$ represent the source messages originating at user 1, user 2, $\cdots$, user M, respectively. Thus,

$$\begin{aligned} \mathcal{K} &= (\underline{U}, \underline{G}_1, \underline{G}_2, \cdots, \underline{G}_M) \\ &= \begin{pmatrix} 1 & 0 & \cdot\cdot & 0 & \gamma_{1,1} & \gamma_{1,2} & \cdot\cdot & \gamma_{1,N} \\ 0 & 1 & \cdot\cdot & 0 & \gamma_{2,1} & \gamma_{2,2} & \cdot\cdot & \gamma_{2,N} \\ & & \cdot\cdot & & & \cdots & & \\ 0 & 0 & \cdot\cdot & 1 & \gamma_{M,1} & \gamma_{M,2} & \cdot\cdot & \gamma_{M,N} \end{pmatrix}, \end{aligned} \tag{4}$$

where $\underline{U}$ is the $M \times M$ identity matrix. We note that $\gamma_{i,j}$ is normally in a finite field (Galois field) GF($|A|$), where $|A|$ is the alphabet size. Then, if $|A| = 2^m$, it is convenient to convert m bits into a coding symbol.

To achieve a maximum diversity order of $N+1$ for all users in the network, we now propose a network coding scheme based on maximum distance separable (MDS) codes as follows.

Definition 1 (Maximum-Diversity Network Codes): MDNCs are defined having the property that $\mathcal{K}$ is the generator matrix of a systematic MDS coding matrix in row echelon form. Clearly, each nonsystematic column of $\mathcal{K}$ represents the network coding coefficients at a specific relay node, which forms a parity-check symbol of an MDNC. Further, if any source-relay (SR) channel is in outage the corresponding relay will not send any codeword.

Here, the corresponding transfer matrix of an MDS code has the property that a submatrix formed by any M out of $M+N$ columns of $\mathcal{K}$ is nonsingular [24]. Clearly, if one transmitting block uses one time slot, totally $M+N$ time slots are needed for our scheme. Then, in general, every user has a delay of $M+N$ time slots where we assume negligible coding delay. If no network coding is used, each user equally has $\frac{N}{M}$ time slots. The first transmitter U_1 has a delay of $1 + \frac{N}{M}$, and the second transmitter U_2 has a delay of $2(1 + \frac{N}{M})$ and so on. Thus, in general, network coding increases the delay compared to the scheme without coding. In what follows we will show that MDNCs achieve full-diversity for arbitrary $M \geq 2$ and $N \geq 1$.

III. Performance Analysis

A. Diversity Order

We now analyze the performance of the MDNCs, and present the main result in the following proposition.

Proposition 1: Consider an M-user N-relay network with linear finite field network codes. The full diversity order $N+1$ is achieved if and only if $\mathcal{K}$ is an MDS coding matrix (MDNCs).

Proof: The proof is given in Appendix A. ■

MDNCs can be regarded as a networked version of MDS codes [21], [22]. In practice, Reed-Solomon (RS) codes are the most widely used variant of MDS codes, where we can easily find constructions which are suitable for any given network topology, i.e., any choice of M and N. The construction of MDNCs is thus identical to the one for RS codes with block length $M+N$ and input symbol length M [2]. Note that the alphabet size impacts the complexity and delay of the codes. In particular, MDNCs have a block length of $M+N$, i.e., M systematic blocks and N parity check blocks which lead to a required alphabet size of $|A| \geq M+N-1$ (Chapter 11, [25]). For the example in Fig. 1, the binary network code is sufficient ($|A| >= 2$). However, for a two-user two-relay network a minimum alphabet size of three is needed.

One specific example for network codes which cannot achieve full-diversity is given as follows. Consider the binary network coding scheme (namely, $\gamma_{i,j} = 1, i = 1, \cdots, M; j = 1, \cdots, N$ in $\mathcal{K}$) [7], [9]–[11], which can achieve diversity order $D = 2$ if $N = 1$. Yet, for $N > 1$, binary network codes cannot achieve the diversity order $N+1$. The analysis is as follows.

We assume that each of $N \geq 2$ relays XORs all source messages if it can decode all of them (otherwise, the relay stays silent). With probability $P_{SRP} = (1-P_e)^{MN}$, all relays can decode all source messages and transmit the network codeword $\underline{C}_b = I_1 \oplus I_2 \oplus \cdots \oplus I_M$. Clearly, if any one of the direct source-BS channels are in outage, then the BS can decode all source messages with the other $M-1$ source messages and $\underline{C}_b$. Yet, if two source-BS channels are in outage with probability $P_{SB,2} = \binom{M}{2} P_e^2 (1-P_e)^{M-2}$, then the BS cannot decode all messages and an outage event occurs. The outage probability is lower bounded by $P_{o,bnc} \geq P_{SRP} P_{SB,2} = \binom{M}{2} P_e^2$ in high SNR, where the lower bound is obtained for the assumption of perfect source-relay channels. Then, the diversity order is upper-bounded by $D = 2$ which is smaller than $N+1$. Thus, the binary network coding scheme cannot achieve the full-diversity $N+1$ for $N > 1$.

Remark 1: For $N = 1$ the binary network codes are MDS codes, as, for example, constructed based on single parity check (SPC) codes. In the single relay case, the network codewords are formed as the binary XOR of the incoming packets from the M sources, namely, $\underline{C} = I_1 \oplus I_2 \oplus \cdots I_M$. Clearly, if only one channel is in outage in the whole network, the BS can still recover all M sources. Thus, binary network codes are sufficient to achieve a maximum diversity order of two. MDNCs include this scenario as a special case. For $N > 1$, the binary codes cannot be MDS if the binary symbols $\gamma_{i,j}$ are treated as scalars in the code. However, if binary vectors are used as coding symbols, e.g., as in array codes [26], there exist binary MDS codes with $N > 1$. The array codes are actually the image of RS codes where the finite field symbols are interpreted as binary vectors. In such scenario $\gamma_{i,j}$ is a binary vector with a certain length. There are requirements on N and the structure $\gamma_{i,j}$, and the codes are fundamentally binary images of nonbinary MDS codes. Following the same analysis as in the proof of Proposition 1, a diversity order of $N+1$ can be achieved.

[2]There are many references on the topic, see, e.g., [24] and [25, Chapter 5] for the construction of RS codes with systematic generator matrices.

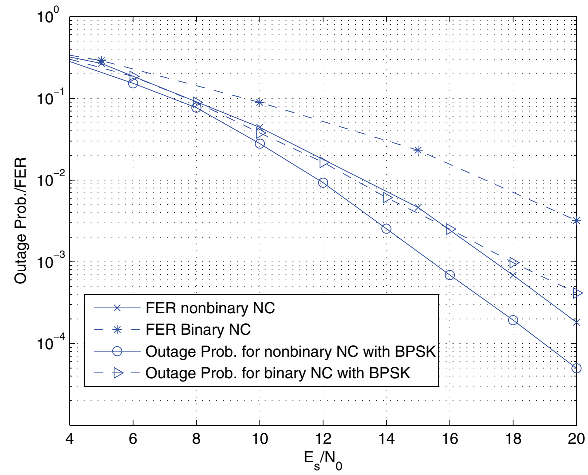

Fig. 3. Outage probability and frame error rate simulations for the two-user two-relay network with both binary and nonbinary network coding. BPSK is used on all channels in the network. The rate of each user is chosen as $1/2$ bits/second/Hz.

Remark 2: From Proposition 1, it is clear that the diversity order stays the same with an increasing number of users M, but the delay of $M+N$ time slots increases with M. Compared to a scheme in which relays only network encode subsets of all sources, our scheme, in which each relay possibly can encode all sources, exhibits a higher diversity order. Clearly, if a relay only encodes a subset of all source messages it cannot help the remaining sources. Thus, the diversity order for these sources decreases. For example, if in the two-user two-relay network each relay transmits only one source message, we observe that the diversity order is two since the resulting network can be seen as two one-source one-relay networks operated in parallel.

Remark 3: Our analysis also includes the special scenario when $M = 1$ where $\mathcal{K}$ is a row vector. Then any $\gamma_{1,j} \neq 0$, $j = 1, \cdots, N$, can meet the requirement of MDS codes. Clearly, an outage even occurs only when all $N+1$ paths are in outage, which leads to a full diversity order of $N+1$.

Above we have considered only finite field network codes. In following sections, we will discuss the achieved diversity order of a network coding scheme based on superposition coding.

B. Numerical Results

As an illustration we present a specific example with two users, two relays, and with orthogonal channels among users and relays. In Fig. 3 we compare binary network coding with MDNCs in terms of simulation results for frame error rates (FER) and outage probability versus the SNR, respectively. For MDNCs, we use $[\gamma_{1,1}, \gamma_{2,1}]^T = [1,1]^T$ and $[\gamma_{2,1}, \gamma_{2,2}]^T = [1,2]^T$ as the GEKs of relay 1 and relay 2, respectively. Here we assume the coding coefficients are in the finite field GF(4), which is constructed based on the minimal polynomial $p(X) = X^2 + X + 1$. Hence, the four elements are the polynomials $0, 1, X$ and $X+1$. For simplicity, we also use integer notation for the field elements, i.e., $0, 1, 2$ and 3, respectively. It is easy to see that the transfer matrix has an MDS-type construction. As transmission format, we use binary phase shift keying (BPSK) modulation. Both outage probabilities and FERs are obtained through simulations. For the FER results we use regular $(3, 6)$ Gallager low-density parity-check (LDPC) codes and a block length of 400 code bits. From Fig. 3 we observe gaps between the outage probabilities and the FER results. One of the reasons is that the employed channel codes have not been optimized for the considered networks. However, as we can see, the diversity orders are predicted correctly by the analytical results of outage probabilities. From the results, we can clearly see the advantage of MDNCs.

Above, we have assumed that a relay encodes and forwards source messages only if all messages are successfully decoded at the relay. If we relax this constraint and let the relay encode and transmit whatever it decodes, the BS may receive more mutual information, and the error probability will decrease. We call this protocol opportunistic relaying. However, the diversity order will not change with opportunistic relaying. A dominant outage event leading to a diversity order of $N+1$ happens when all relays are able to decode all source messages (perfect source-relay channels). The analysis is as follows. The probability of perfect source-relay channels is $P_{SRP} = (1-P_e)^{MN}$. Clearly, we have $\lim_{\text{SNR}\to\infty} P_{SRP} = 1$. Among $N+M$ blocks received by the BS, the probability that all N blocks from the relay are in outage is given as P_e^N. Then, no network codewords are received by the BS. Obviously, if in addition one of M direct source-BS channels is in outage, which occurs with probability P_e, the system is in outage. In this case the resulting diversity order is $N+1$. As we can see, the outage probability and therefore the diversity order for perfect source-relay channels is the same with or without opportunistic relaying. Thus, opportunistic relaying cannot increase the diversity order, but it can improve the outage probability and therefore the coding gain.

Similarly, for binary network codes a diversity order $N+1$ cannot be achieved. The reason is that in the above analysis the error event for which binary network codes loose the diversity order of $N+1$ is given when all relays are able to decode all messages with probability P_{SRP}. Then, opportunistic relaying does not affect the analysis and the diversity gain, although an increase in coding gain may be observed. One example is shown in Fig. 4 for the two-user two-relay network. We can see that compared to the schemes without opportunistic relaying, this strategy can improve the outage probabilities but provides the same diversity order for both binary and nonbinary network codes.

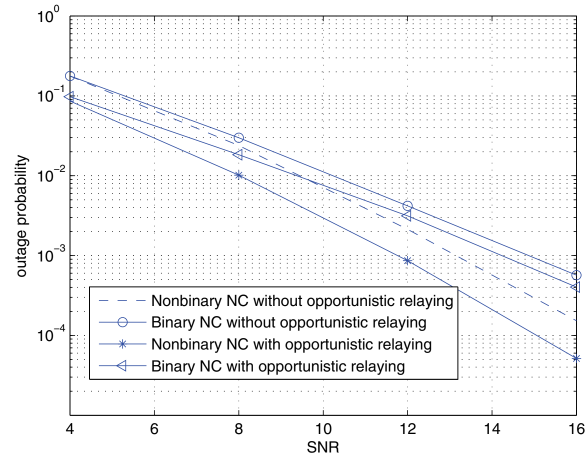

Fig. 4. Outage probability comparison for two-user two-relay networks with and without opportunistic relaying. The rate of each user is chosen as $1/3$ bits/second/Hz.

In our analysis, we have assumed that the relay nodes do not overhear the transmissions of all other relays. However, if we relax this assumption the diversity order does not change. More formally, consider an error event such that all SR channels and all relay-relay channels are not in outage. The probability for this to happen is $P_{SR} = (1-P_e)^{MN+N(N-1)}$. Then, if both all relays to the BS channels and one of the source-BS channels is in outage, the overall system is in outage as one user cannot be decoded. The probability for this to happen is $P_{RB} = CP_e^{N+1}$, where C is a constant depending on SNR. Thus, the overall probability (denoted by $P_{o,ov}$) is evaluated by $P_{o,ov} = P_{SR}P_{RB}$ which leads to a diversity order of $N+1$. Note that we have assumed slow fading channels and that no temporal diversity can be exploited for a channel.

C. Comparison to Superposition Coding

In addition to FFNC, superposition coding (SC) at the relays is another way to share a relay among the users. Since SC is employed at the relays it can be seen as real-domain or analog network coding [16], [27], [28]. Thus, it is valuable to investigate the performance of SC for multi-user multi-relay networks.

We first provide an SC strategy at the relays, where we consider a selective decode-and-forward scheme [2]. That is, if the relay can decode a received channel codeword, it decodes the information message. Otherwise, the relay drops the codeword. An SC scheme based on an amplify-and-forward approach will be discussed in Section V. After receiving signals from M sources via M independent channels the i-th relay, $i = 1, 2, \ldots, N$, tries to decode. If it can decode $Z_{r_i} \le M$ messages, then the relay re-encodes these Z_{r_i} messages separately by using the same channel codewords as the sources. Also, the relay equally divides the transmission energy among these Z_{r_i} codewords and transmits the sum of them. At the BS a minimum mean-squared error receiver with successive interference cancellation (MMSE-SIC) is employed to decode the received signals. As shown in [20, Chapter 8], such a receiver is optimal in the sense of providing a sufficient statistic based on the channel outputs. Specifically, X_j denotes the channel codeword of user j, and $\underline{X}$ denotes a set of the codewords from all sources, respectively. We assume that Z_{r_i} channel codewords are re-encoded by the i-th relay as $X_{1,r_i}, X_{2,r_i}, \cdots, X_{Z_{r_i},r_i} \in \underline{X}$. Hence, the transmitted codeword of relay i is given as $X_{r_i,0} = \frac{1}{\sqrt{Z_{r_i}}}(X_{1,r_i} + X_{2,r_i} + \cdots + X_{Z_{r_i},r_i})$, where the factor $\frac{1}{\sqrt{Z_{r_i}}}$ is used to normalize transmission energy[3]. We assume that K relays (denoted as $r_1, r_2, \cdots, r_K$) are able to decode the codeword X_j. The BS now receives $K+1$ codewords, including X_j which is obtained directly from the source. Thus, for user j, the received vector is given as

$$\underline{Y}_j = \underline{a}_j X_j + \underline{n}_j, \tag{5}$$

where $\underline{a}_j$ is the vector of channel gains, defined as

$$\underline{a}_j = \left[a_{j,0}, \frac{1}{\sqrt{Z_{r_1}}} a_{r_1,0}, \cdots, \frac{1}{\sqrt{Z_{r_K}}} a_{r_K,0} \right]^T, \tag{6}$$

and where $\underline{n}_j$ denotes noise including the interference from other superimposed codewords. Here $a_{r_\ell,0}$, $\ell = 1, \ldots, K$, is the channel gain from relay r_ℓ to the BS (see (1)), and Z_{r_ℓ} represents the number of decoded codewords at the relay r_ℓ. We note that the noise samples in $\underline{n}_j$ are correlated since at the BS the codewords of other users of the same relay are considered as interference. If, without loss of generality, we assume that $X_j = X_{1,r_m}$ for $m = 1, \ldots, K$, then

$$\underline{n}_j = \Bigg[n_{j,0}, n_{r_1,0} + \frac{a_{r_1,0}}{\sqrt{Z_{r_1}}}(X_{2,r_1} + \cdots + X_{Z_{r_1},r_1}), \ \ldots \ , \\ n_{r_K,0} + \frac{a_{r_K,0}}{\sqrt{Z_{r_K}}}(X_{2,r_K} + \cdots + X_{Z_{r_K},r_K}) \Bigg], \tag{7}$$

where $n_{r_i,0}, (i = 1, \cdots, K)$ is the noise variable for the channel between relay r_i and the BS, and $X_{m,r_i}, m > 1$ denotes the transmitted signals at relay r_i. Note that the elements $a_{r_i,0}$ of $\underline{a}_j$ in (5) have zero mean and unit variance and are independent random variables for different i. The MMSE receiver first whitens the colored noise before subsequent matched filtering [20]. Then, the mutual information (MI) of user j at the BS can be obtained as

$$\mathcal{I}_j = \frac{1}{2(M+N)} \log_2 \Bigg(1 + |a_{j,0}|^2 \mathrm{SNR} + \frac{\frac{1}{Z_{r_1}}|a_{r_1,0}|^2 E_s}{N_0 + \frac{Z_{r_1}-1}{Z_{r_1}}|a_{r_1,0}|^2 E_s} \\ + \cdots + \frac{\frac{1}{Z_{r_K}}|a_{r_K,0}|^2 E_s}{N_0 + \frac{Z_{r_K}-1}{Z_{r_K}}|a_{r_K,0}|^2 E_s} \Bigg). \tag{8}$$

That is, $\mathcal{I}_j$ represents the MI between the (Gaussian) codeword X_j of user j and the received signal at the BS after filtering, treating the remaining interference resulting from codewords X_k, $k \neq j$, as being part of the noise. The first decoding round starts for the user with the strongest MI, $\mathcal{I}_j$, then the corresponding decoded codeword of that user is subtracted. Then, the second decoding round considers the user with the strongest MI among all remaining users, and so on, until the decoding fails in a certain stage, if the effective MI received by the BS for a user is smaller than its rate, or if all users are decoded successfully.

Clearly, in (8) the impact of the signals from the relays on the performance of the SC scheme is determined by the following SNR term, which denotes the received signal energy for the k-th path to the BS:

$$\mathrm{SNR}_{SC,k} = \frac{\frac{1}{Z_{r_k}}|a_{r_k,0}|^2 E_s}{N_0 + \frac{Z_{r_k}-1}{Z_{r_k}}|a_{r_k,0}|^2 E_s} = \frac{\frac{1}{Z_{r_k}}|a_{r_k,0}|^2}{\frac{1}{\mathrm{SNR}} + \frac{Z_{r_k}-1}{Z_{r_k}}|a_{r_k,0}|^2}, \tag{9}$$

where $k = 1, \ldots, K$. If $Z_{r_k} > 1$, we obtain $\lim_{\mathrm{SNR}\to\infty} \mathrm{SNR}_{SC,k} = \frac{1}{Z_{r_k}-1}$ (for $k = 1, \ldots, K$), which cannot increase with SNR. From (8) and (9), it is extremely difficult to derive the exact diversity order of the SC-based scheme since the fading coefficients appear in both the numerator and the denominator of the SNR term, and the SIC decoding process makes analysis even more involved. However, when the SNR goes to infinity and $\mathrm{SNR}_{SC,k}$ approaches a constant, we obtain $\lim_{\mathrm{SNR}\to\infty} \mathcal{I}_j = \frac{1}{2(M+N)} \log_2(|a_{j,0}|^2 \mathrm{SNR})$, which leads to a diversity order close to $D = 1$. Thus, in the high SNR regime SC nearly loses the multipath diversity.

Note that the SC decoder uses opportunistic decoding [20], i.e., first decodes the strongest user, then second strongest and so on. From (9) we can see that multipath diversity from the relays can only be obtained when the relays are able to help only one user, i.e., $Z_{r_k} = 1$. This means that the source-relay channels of all other $M-1$ users are in outage, but the probability of this event is negligible. For general scenarios, the users with strong signals can have higher opportunistic decoding gains, but they mostly have no multipath diversity for above reason ($Z_{r_k} > 1$). Since the users with weaker signals do not experience any opportunistic decoding gain, the benefit from opportunistic decoding is insignificant for this class of users.

As shown above, the outage probability for the FFNC-based scheme with MDNCs is given as (14) with a diversity order of $N+1$. Hence, in the high SNR regime, employing FFNC at the relays still provides multipath diversity and thus yields better performance than the SC-based relaying strategy.

In Fig. 5, we compare the outage probability for both the FFNC- and SC-based schemes for a three-user two-relay network. The outage probabilities for both schemes are evaluated based on simulations, where we assume Gaussian channels and model the fading coefficients as Rayleigh distributed. An outage event occurs if the rate R is higher than the mutual information in any stage of the MMSE-SIC decoder for any user. From Fig. 5 we can see that FFNC has better performance in the region of high SNR, while SC outperforms FFNC at low SNRs. Also, as already shown above, the simulations verify that SC cannot achieve full diversity, which in this example corresponds to a diversity order of $D = 3$. More generally, we have the following proposition.

Proposition 2: For sufficiently high SNR, MUMR schemes based on MDNCs have a larger outage capacity than SC-based schemes.

[3]For clarity, we here use 0 in $X_{r_i,0}$ to denote that the receiving node is the BS.

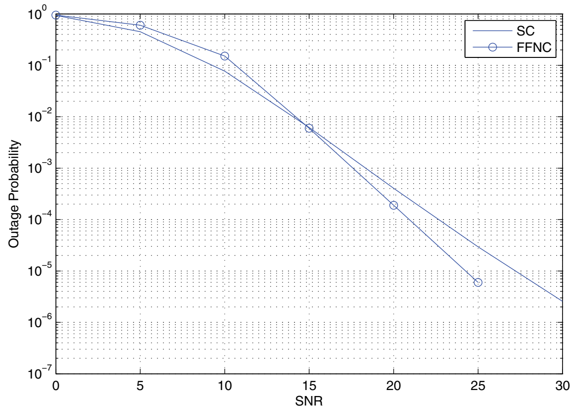

Fig. 5. Comparison of outage probabilities for FFNC- and SC-based schemes for three-user two-relay networks. The rate of each user is chosen as $2/3$ bits/second/Hz.

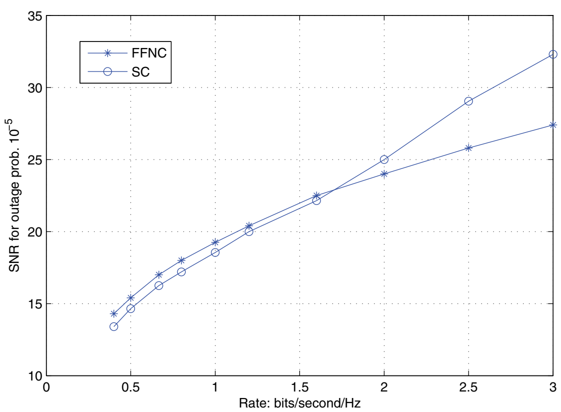

Fig. 6. Comparison of SNR versus achievable rates for FFNC and SC-based schemes for a fixed outage probability of 10^{-5} and the two-user two-relay network.

Proof: The mutual information between the codeword X_j of user j and the received signal at the base station for SC-based schemes is given as in (8). Clearly, $\lim_{\text{SNR}\to\infty} \mathcal{I}_j = \frac{1}{2(M+N)} \log_2(|a_{j,0}|^2\text{SNR})$. Thus, the outage probability for SC schemes can be obtained as $P_{e,SC} = \frac{2^{2(M+N)R}}{\text{SNR}}$ in high SNR. On the other hand, the outage probability for MDNC-based schemes is $P_{e,MDNC} = C_{e,MDNC}\text{SNR}^{-(N+1)}$ for high SNR, where $C_{e,MDNC}$ is constant with SNR.

Let ϵ denote a given outage probability. By the definition in [20, Chapter 5], the ϵ-outage capacity (in high SNR) for SC-based schemes is evaluated as $\mathcal{C}_{SC,\epsilon} = \frac{1}{2(M+N)} \log((1-\epsilon)\text{SNR})$, and the one for FFNC based schemes is $\mathcal{C}_{FFNC,\epsilon} = \frac{1}{2(M+N)} \log((1-\epsilon)^{\frac{1}{(N+1)}}\text{SNR})$. Clearly, since $0 < 1-\epsilon < 1$, for given the SNR and ϵ, $\mathcal{C}_{FFNC,\epsilon} > \mathcal{C}_{SC,\epsilon}$. ■

In Fig. 6, we compare the achievable rates for the two-user two-relay network for both FFNC- and SC-based strategies and a fixed outage probability of 10^{-5}. We can see that in the high rate/throughput regime FFNC is significantly better than SC. However, for lower rates SC slightly outperforms FFNC. Further, compared to the SC-based scheme, the FFNC scheme avoids the use of complexity-increasing whitening matched filters. These filters are required due to the colored noise resulting from the superposition (see [20, Chapter 8]).

IV. Networks without Direct Source-BS Channels

Up to now we have assumed that all channels have the same average SNR. However, in certain situations the channels between the users and the BS correspond to much longer distances leading to considerably lower received SNRs. Hence, we can practically disregard any direct channels between the users and the BS. In this section, we consider the design of network codes, assuming the absence of a direct source-BS channel. Specifically, we can construct the network codes in such a way that each output symbol from a relay represents an output symbol of a nonsystematic MDS code. We consider two different scenarios. First, we assume that each relay is only allowed to transmit one coded block of data, for example to avoid congestion at the BS, which may receive data also from other networks besides the one under consideration. In the second scenario we relax the restriction on the number of coded blocks to be transmitted for each relay.

A. Each Relay Only Transmits One Coded Block

For such scenario, we assume $N \geq M$. Note that this is only for the case of networks without direct channels, since no information can be decoded if $N < M$ and each relay transmits only one coded block. For a single-user network without direct source-BS channels it is clear that the full diversity order is N. However, it is not directly clear what the full diversity order is in the multi-user scenario. Although the codewords for each user are transmitted through N paths we cannot use N as the full diversity order since these N channels are shared among different users. This is addressed in the following proposition.

Proposition 3: The maximum diversity order for M-user N-relay wireless networks employing MDNCs based on MDS codes is given as $N-M+1$ if direct source-BS channels are absent.

Proof: We first consider the scenario for perfect SR channels. In high SNR the probability of no overall outage event is given as $P_{SR,NoO} = (1-P_e)^{MN}$. Clearly, $\lim_{\text{SNR}\to\infty} P_{SR,NoO} = 1$. Then, each of N network codewords from the relays is encoded based on M source messages. Clearly, to recover M source messages the BS should successfully decode at least M codewords from the relays. Since our network codes are MDS, any M out of N network codewords can be used to recover the M source messages. Outage occurs only when $N-M+1$ or more relay-BS channels are in outage. The probability for this event is $P_{o,NoD} = P_e^{N-M+1}(1-P_e)^{M-1} + P_e^{N-M+2}(1-P_e)^{M-2} + \cdots + P_e^N$. By (2), the diversity order is $N-M+1$.

If there are SR channels in outage, we can still achieve a diversity order of $D_{\max} = N-M+1$, where the proof is similar to the one of Proposition 1. For instance, consider that Z out of NM SR channels are in outage and that these Z channels connect to $K \leq Z$ relays. These K relays do not send any codeword and keep silent. By a similar analysis as

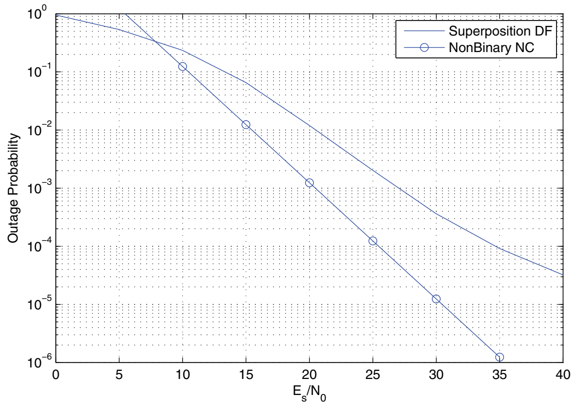

Fig. 7. Outage probabilities for SC and MDNCs for a two-user three-relay network without direct source-BS channels. The source rates are 0.8 bits/second/Hz for both users.

in Proposition 1 we can show that an outage event occurs only when M or more out of these $N-K$ symbols are in outage. Thus, the outage probability is

$$\begin{aligned} P_{o,NoD} &= P_e^Z P_e^{N-K-M+1}(1-P_e)^{MN-Z+M-1} + \\ &+ P_e^Z P_e^{N-K-M+2}(1-P_e)^{MN-Z+M-2} + \\ &\cdots + P_e^Z P_e^{N-K}(1-P_e)^{MN-Z}. \end{aligned} \quad (10)$$

Clearly, we have $\lim_{\text{SNR}\to\infty} \frac{\log P_{o,NoD}}{\log P_e} = N - M + 1$. In summary, the diversity order is $N - M + 1$. ∎

Remark 4: In Section III we showed that opportunistic relaying cannot increase the diversity order for the networks with direct source-BS channels. The conclusion also holds for networks without direct source-BS channels. From the proof of Proposition 3 we can see that for the event of perfect source-relay channels N relays transmit network codewords in N relay-BS channels whether or not they perform opportunistic relaying. Then, an outage event occurs if $N - M + 1$ or more relay-BS channels are in outage since fewer than M codewords are received in this case. Thus, networks with opportunistic relaying still have a diversity order of $N-M+1$ if direct source-BS channels are absent, but they again can increase the coding gain.

As a specific example, we employ MDNCs based on nonsystematic RS codes. In Fig. 7 the outage probabilities of MDNCs for a two-user three-relay network (without direct source-BS channels) are shown and compared to an SC-based strategy. We can observe that a diversity order of $D = 2$ is achieved for MDNCs. From the figure, we also find that for the two-user three-relay network MDNCs have better performance than an SC-based strategy in the high SNR regime which seems to suffer from an error floor in this regime. This can also be seen from the fact that (8) approaches a constant with increasing SNR. For a more general case, we have the following proposition.

Proposition 4: Consider MUMR relaying networks without direct source-BS channels. For sufficiently high SNR, the schemes based on FFNCs have a larger outage capacity than the schemes based on SC.

Proof: The proof is similar to that of Proposition 2. For the schemes based on SC, clearly, the mutual information between the (Gaussian) codeword X_j of user j and the received signal at the BS (K signals from K relays) is

$$\begin{aligned} \mathcal{I}_{SC,NoD} &= \frac{1}{2(M+N)} \log_2\Bigg(1 + \frac{\frac{1}{Z_{r_1}}|a_{r_1,0}|^2 E_s}{N_0 + \frac{Z_{r_1}-1}{Z_{r_1}}|a_{r_1,0}|^2 E_s} + \\ &\ldots + \frac{\frac{1}{Z_{r_K}}|a_{r_K,0}|^2 E_s}{N_0 + \frac{Z_{r_K}-1}{Z_{r_K}}|a_{r_K,0}|^2 E_s}\Bigg) = \log_2(C_S) \end{aligned}$$

for sufficiently high SNR. Here C_S is constant with SNR. In this case, the outage capacity is close to zero for SC-based schemes. On the other hand, a scheme based on FFNC can always have a diversity order of $N - M + 1$ as stated in Proposition 3. In high SNR, following a similar approach as in the proof of Proposition 2, we can show that the outage capacity is $\mathcal{C}_{FFNC,\epsilon} = \frac{1}{2(M+N)} \log((1-\epsilon)^{-(N-M+1)}\text{SNR})$ which is larger than for SC-based schemes. ∎

Since we can regard an outage in the channel as a single erasure, MDNCs can also be seen as network error correction codes [21], [22]. In particular, it has been shown in [21], [22] that the network Singleton bound for t-error correction codes in a single-source acyclic network is

$$\log|A| \leq (N - 2t)\log q, \quad (11)$$

where $|A|$ is the source alphabet size with the basis q, and N is the min-cut. Similar results are also presented in [29] for coherent and noncoherent multi-source networks. For channel erasures one can easily obtain the Singleton bound of an E-erasure correction network code as $E \leq (N - \log_q|A|)$. For MDNCs we can state the following proposition.

Proposition 5: MDNCs achieve equality in the Singleton bound for M-user N-relay networks without direct source-BS channels and E erasures, i.e, $E = N - M$.

Proof: Since we assume all sources have the same rates, the M source nodes with the alphabet size $|A|$ can be easily reduced to a network with one super source and alphabet size $|A|^M$. This super source is connected to M sources via M error-free channels, and each channel transmits an independent source message with the alphabet size $|A|$. Since all sources have the same rates, we can easily assume $|A| = q$ to simplify the analysis. For the single super source network, the min-cut between the super source and the BS is N, i.e., equivalent to the number of channels between the relays and the BS. Hence, the network Singleton bound for M-user N-relay networks is given as $E \leq N-M$, i.e., all erasures or channel outages with $E < N - M + 1$ can be corrected. By comparing this result with Proposition 3, we can see that the proposed MDNCs achieve equality in the network Singleton bound. ∎

Remark 5: Although Proposition 5 is obtained for networks without direct source-BS channels, a similar result also holds for networks with direct source-BS channels. Clearly, for such networks the BS receives $M + N$ network coded blocks. Then, the Singleton bound is $E \leq N$. As MDNCs achieve the diversity order $N + 1$, they also achieve equality in the Singleton bound by a similar argument as in Proposition 5.

Remark 6: By comparing Propositions 1 and 3 we observe that the diversity orders are different in terms of N and M for the networks with or without direct source-BS channels. We can unify both scenarios by assuming in total L different paths from all M users to the BS. Clearly, we have $L = N$ for networks without direct source-BS channels and $L = N + M$ for the ones with direct source-BS channels. Then, in average, each user has L/M different paths to the BS. Thus, the diversity order is

$$D = L - M + 1, \tag{12}$$

irrespective whether there are direct source-BS channels or not. In this sense, MDNCs achieve the Singleton bound for both scenarios.

B. Each Relay Transmits Multiple Blocks

Up to now, we have assumed that each relay only transmits one block. If we relax this assumption and let a relay node transmit multiple blocks, the analysis will be different. We assume that each relay transmits M^* coded blocks. Then N relays transmit NM^* coded blocks in total. We use a single joint MDNCs based on MDS codes over all NM^* coded blocks. Then, any M out of these NM^* coded blocks can be used to rebuild the M sources. Clearly, if $M^* < M$, the diversity order N cannot be achieved, since an outage of $N-1$ relay-BS channels may lead to an overall outage event. In general, we can easily calculate that the diversity order is $N - \lceil \frac{M}{M^*} \rceil + 1$, where $\lceil \frac{M}{M^*} \rceil$ denotes the smallest integer larger or equal to $\frac{M}{M^*}$. For $M^* = M$, the BS may be able to decode all sources even if only one relay-BS channel is not in outage, since the BS can obtain M coded blocks. Thus, a diversity order of N is achieved. For the scenario with $M^* > M$, the analysis is similar to the one for $M^* = M$.

Clearly, fixing the number of transmitting blocks for each relay may lead to a loss of transmission resources, since for slow fading channels multiple transmitted blocks of the same channel will all be in outage. To efficiently use the available resources, the relay with a high transmit SNR should transmit more blocks, and the one with a low transmit SNR should transmit fewer blocks, respectively. Thus, the number of transmitting blocks for each relay may be determined dynamically by the instantaneous channel gains. This will require a feedback mechanism to provide channel state information to the sources.

V. Nonorthogonal Channels

For the above analysis we have assumed a network with orthogonal channels, thus eliminating inter-user interference but limiting the achievable rates [20], since each transmitter uses their own time slot. To efficiently exploit channel resources, it is valuable to consider transmission schemes with nonorthogonal channels, in particular, multiple access channels (MACs). In the following, we will investigate efficient network coding schemes and their performance for multi-user multi-relay networks with nonorthogonal channels and half-duplex relays.

The transmission scheme for the network with nonorthogonal channels uses two time slots. In the first slot, all M users *concurrently* transmit their own information. Then, the received signal at node j (relay or BS) is

$$Y_j = \sum_i a_{i,j} X_{i,j} + n_j. \tag{13}$$

On receiving Y_j, node j tries to decode all source messages using a SIC decoder. If it can decode *all* source messages and if it is a relay, the node computes a network codeword specified by the MDNC, otherwise, it stays silent. If node j is the BS, it successfully decodes as many source messages as possible. These messages will be used to decode all available source messages jointly with the network codewords received in the second time slot. In this time slot, those relays, which are able to decode all source messages, concurrently transmit their network codewords to the BS. The BS first tries to decode these network codewords with a SIC decoder. Then, by combining the decoded messages in the first time slot and the received network codewords in the second time slot, the BS will try to decode the source messages. In the following, for networks with nonorthogonal channels we will first discuss the performance of MDNCs, and then compare with SC-based schemes.

Proposition 6: Consider an M-user N-relay network with nonorthogonal channels and half-duplex relays, MDNCs achieve a maximal diversity order of $N+1$ with SIC decoding at both the BS and the relays.

Proof: The proof is given in Appendix B. ■

Similar to the analysis the system with orthogonal channels in Section IV, we now compare the performance of schemes based on FFNC and superposition coding in this nonorthogonal channel setup. For nonorthogonal channels, all transmitted signals arrive at a receiving node simultaneously. These superposed signals naturally perform network coding in the real (or complex) domain. Thus, it is often termed as *analog network coding* [16]. Let X_j denote the network codeword output at relay j, then $X_j = \beta_j Y_j$, where Y_j is the received signal (13) at the first time slot, and the factor $\beta_j = \sqrt{\frac{E_s}{\sum_i |a_{i,j}|^2 E_s + N_0}}$ is employed to normalize the transmit energy at relay j. For the purpose of comparison, we now present a diversity order analysis for SC-based schemes.

Proposition 7: Consider an M-user N-relay network with nonorthogonal channels, direct source-BS channels, and half-duplex relays. SC-based schemes cannot achieve the full diversity order of $N + 1$.

Proof: The proof is given in Appendix C. ■

By combining Proposition 6 and Proposition 7 we can see that for sufficiently high SNR the scheme based on FFNC outperforms the ones based on SC for nonorthogonal channels and half-duplex relays. In Fig. 8, we compare the outage probabilities of a two-user two-relay network. The numerical results are evaluated from simulations, where we assume Gaussian channels with Rayleigh distributed channel gains. We can see that for nonorthogonal channels FFNC exhibits a lower outage probability in the high SNR region, but the scheme based on SC has better performance in the low SNR regime. This observation is similar to the case with orthogonal channels.

For the case without direct source-BS channels, we have similar results as stated in the following proposition.

Proposition 8: Consider an M-user N-relay network, $N \geq M$, with nonorthogonal channels and half-duplex relays. In such a network MDNCs achieve a diversity order of $N - M + 1$ with SIC decoding at both the BS and the relays if direct source-BS channels are absent.

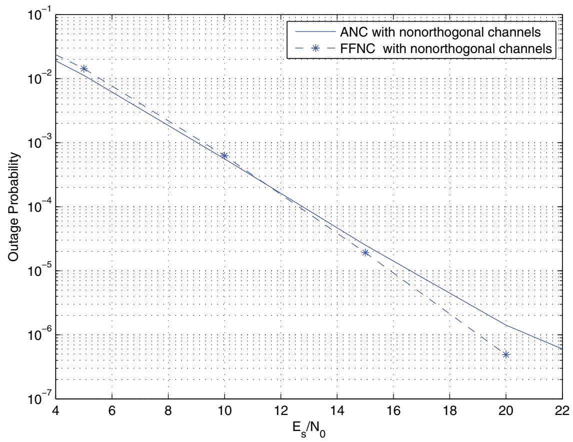

Fig. 8. Outage probabilities for the two-user two-relay network with nonorthogonal channels. The source rates are 0.25 bits/second/Hz.

Proof: The proposition can be proved by combining the proofs of Propositions 3 and 6. ■

Furthermore, for the SC-based scheme we can show that the diversity order $N-M+1$ cannot be achieved if the direct source-BS channels are absent. Since this proof is similar to the one of Proposition 7 we skip the detailed analysis.

VI. Conclusion

We have investigated the design of efficient network coding strategies for MUMR wireless networks with slow fading channels. For FFNC-based schemes we propose maximum diversity network codes based on MDS codes to achieve a full diversity order of $N+1$. We also show that the MDS codes are necessary and sufficient to achieve full-diversity for linear FFNCs. As an alternative, we have also addressed superposition coding at the relay and compared its performance with FFNC-based schemes where the results show that in the high SNR regime FFNC has significantly better performance in terms of outage capacity. Then, we have considered networks without direct source-BS channels and show that a diversity order of $N-M+1$ can be achieved by using the proposed FFNC-based strategies. This construction also achieves equality in the network Singleton bound for the presented class of MUMR networks. Finally, we study networks with nonorthogonal channels and half-duplex relays. We show that FFNCs based on an MDS code construction can also achieve a full diversity order of $N+1$, which is not achievable for an SC-based scheme.

Appendix A
Proof of Proposition 1

We first prove that Proposition 1 is sufficient. Clearly, if the SR channels are error-free each user can achieve a diversity order of $N+1$. This is because by using MDS codes we have the property that from any M out of $N+M$ received codewords M source messages can be recovered. Hence, an outage event for any of the users occurs only if $N+1$ or more codewords are in outage. Since the channels are independent, and each channel outage occurs with probability $P_e = C_I \mathrm{SNR}^{-1}$, the diversity order $N+1$ is achieved for perfect SR channels.

Now we assume that Z out of MN SR channels are in outage and none of the remaining channels, but that these Z channels connect to K relays, $K \leq Z$. These K relays do not send any codeword and stay silent. The probability of the event is $P_{SR,ZOut} = P_e^Z(1-P_e)^{MN-Z}$. Then $\lim_{\mathrm{SNR}\to\infty} \frac{\log P_{SR,ZOut}}{\log P_e} = Z$. Further, there are $X = N-K$ relays which receive all M source messages, where $0 \leq X \leq N$. In combination with M direct transmission codewords, any M of these $M+X$ codewords are able to recover M source messages. Hence, an outage event occurs only when $X+1$ or more codewords are in outage, which has a probability of $P_{o,XR} = P_e^{X+1}(1-P_e)^{M-1} + P_e^{X+2}(1-P_e)^{M-2} + \cdots + P_e^{X+M}$ and $\lim_{\mathrm{SNR}\to\infty} \frac{\log P_{o,XR}}{\log P_e} = N-K+1$. Combining this result with $P_{SR,ZOut}$, an outage event for any of the M users occurs with probability (see (14) at the top of the next page) in high SNR. Since $Z \geq K$, it follows $P_{o,T2} \leq P_e^{N+1}$. Hence, by (2), the diversity order is $N+1$ for any of the users.

The necessary part of the proof can be shown as follows. Assume a submatrix $\mathcal{K}_s$ of M columns has a rank smaller than M. With probability $P_{SRP} = (1-P_e)^{MN}$ all source-relay channels are perfect and all relays transmit a network codeword. Then the BS totally receives $M+N$ codewords specified by $\mathcal{K}$ through $M+N$ channels. Consider the event in which information can be perfectly conveyed over those M channels associated with the transfer matrix $\mathcal{K}_s$, but the remaining N channels are in outage. This event happens with probability $P_{Ms} = P_e^N(1-P_e)^M$. Since the transfer matrix $\mathcal{K}_s$ is not full-rank, some source messages cannot be decoded, which leads to an error event. The overall probability for this error event is $P_{o,Ms} = P_{Ms}P_{SRP} = P_e^N$ in high SNR. Clearly, the diversity order is upper-bounded by N, and full diversity cannot be achieved.

Appendix B
Proof of Proposition 6

After the first time slot, the BS tries to decode all source messages directly received from the users. We assume that $K \leq M$ source messages cannot be decoded; the remaining $M-K$ messages can be perfectly decoded. We denote the set of users which can be decoded as $\mathcal{D}$, and the set of users which cannot be decoded as $\mathcal{B} = \mathcal{D}^{\mathcal{C}}$, respectively. Then, for MAC fading channels with a SIC decoder, we have [30]

$$\mathcal{C}\left(\frac{\sum_{i\in\mathcal{S}} \mathrm{SNR}|a_{i,0}|^2}{1+\sum_{j\in\mathcal{B}} \mathrm{SNR}|a_{j,0}|^2}\right) > |\mathcal{S}|R, \forall \mathcal{S} \subseteq \mathcal{D},$$
$$\mathcal{C}\left(\frac{\sum_{i\in\mathcal{T}} \mathrm{SNR}|a_{i,0}|^2}{1+\sum_{j\in\mathcal{T}^c\cap\mathcal{B}} \mathrm{SNR}|a_{j,0}|^2}\right) < |\mathcal{T}|R, \forall \mathcal{T} \subseteq \mathcal{B}, \quad (15)$$

where $\mathcal{C}(x) = \frac{1}{2}\log(1+x)$, $a_{i,0}(a_{j,0})$ is the fading coefficient of the channel from user i (user j) to the BS. Then the probability that exactly K users can be decoded is

$$P_{MAC,K} = \binom{M}{K} \Pr\left\{\mathcal{C}\left(\frac{\sum_{i\in\mathcal{S}} \mathrm{SNR}|a_{i,0}|^2}{1+\sum_{j\in\mathcal{B}} \mathrm{SNR}|a_{j,0}|^2}\right) > |\mathcal{S}|R,\right.$$
$$\mathcal{C}\left(\frac{\sum_{i\in\mathcal{T}} \mathrm{SNR}|a_{i,0}|^2}{1+\sum_{j\in\mathcal{T}^c\cap\mathcal{B}} \mathrm{SNR}|a_{j,0}|^2}\right)$$

$$P_{o,T2} = P_{SR,ZOut}P_{o,XR} = P_e^Z P_e^{N-K+1}(1-P_e)^{MN-Z+M-1} + + P_e^Z P_e^{N-K+2}(1-P_e)^{MN-Z+M-2} + \cdots + P_e^Z P_e^{N-K+M}(1-P_e)^{MN-Z} = P_e^{N+1+Z-K} \quad (14)$$

$$\begin{aligned} &< |\mathcal{T}|R, \forall \mathcal{S} \subseteq \mathcal{D}, \forall \mathcal{T} \subseteq \mathcal{B}\Big\} \\ &\le \binom{M}{K} \Pr\left\{\mathcal{C}\left(\frac{\sum_{i\in\mathcal{T}} \mathrm{SNR}|a_{i,0}|^2}{1+\sum_{j\in\mathcal{T}^c\cap\mathcal{B}} \mathrm{SNR}|a_{j,0}|^2}\right) \right. \\ &< |\mathcal{T}|R, \forall \mathcal{S} \subseteq \mathcal{D}, \forall \mathcal{T} = \mathcal{B}\Big\} \\ &= \Pr\left\{\frac{1}{2}\log(1+\sum_{i,i\in\mathcal{B}} |a_{i,0}|^2\mathrm{SNR}) < KR)\right\}. \quad (16) \end{aligned}$$

Since $|\mathcal{B}| = K$, $\Pr\{\frac{1}{2}\log(1+\sum_{i,i\in\mathcal{B}}|a_{i,0}|^2\mathrm{SNR}) < KR)\} = \frac{(2^{2KR}-1)^K}{K!\mathrm{SNR}^K}$ in high SNR (Chapter 5, [20]). Hence $P_{MAC,K} \le C_{MAC,K} \cdot \mathrm{SNR}^{-K}$, where $C_{MAC,K} = \frac{(2^{2KR}-1)^K}{K!}$ is independent of the SNR. By our transmission protocol, the probability that a relay is silent is given as $P_{R,S} = \Pr\{\text{one or more messages cannot be decoded}\} = C_{R,S}\mathrm{SNR}^{-1}$ in high SNR, where $C_{R,S}$ is also independent of the SNR. The result follows since the probability that exactly one message cannot be decoded will dominate for medium-to-high SNRs. Let us further assume that $Z \le N$ relays are unable to decode all source messages. Since all channels are independent the probability for this event is $P_{R,Z} = C_{R,Z}\mathrm{SNR}^{-Z}$. The other $N-Z$ relays will transmit $N-Z$ network codewords specified by the MDNCs in the second time slot. Similar to the first time slot, the probability of $L \le N-Z$ network codewords being unable to be decoded at the BS is $P_{MAC,L} \le C_{MAC,L}\mathrm{SNR}^{-L}$. Clearly, with MDNC, an outage event occurs if the number of totally decoded source messages or network codewords are smaller than M. That is, $M-K+N-Z-L \le M-1$, and thus $N+1 \le K+Z+L$. The overall outage probability is thus $P_{O,MAC} = P_{MAC,K}P_{R,Z}P_{MAC,L} \le C_{O,MAC}\mathrm{SNR}^{-(K+Z+L)} \le C_{O,MAC}\mathrm{SNR}^{-(N+1)}$, where $C_{O,MAC}$ is constant with SNR. Thus, the diversity order of M-user N-relay networks with MDNC and nonorthogonal channels is given as $N+1$. ■

APPENDIX C
PROOF OF PROPOSITION 7

In the first time slot, the BS receives $Y_1 = \sum_i a_{i,0}X_i + n_{1,0}$ directly from all users, where X_i is the (Gaussian) codeword of user i, $a_{i,0}$ is the channel gain from user i to the BS, and $n_{1,0}$ is the noise at the BS. With an MMSE-SIC decoder, the receiver treats the signal of one user (say user i) as the intended codeword and all other signals as interference. Then the signal-to-interference-noise ratio (SINR) of user i is $\mathrm{SINR}_{i,1} = \frac{|a_{i,0}|^2E_s}{\sum_{j,j\ne i}|a_{j,0}|^2E_s+N_0}$, and the MI between X_i and Y_1 in the first time slot is $\mathcal{I}_{MSC,i}(1) = \frac{1}{4}\log_2(1+\mathrm{SINR}_{i,1})$. In the second time slot, the BS receives $Y_2 = \sum_i\sum_{z=1}^N a_{i,z}\beta_z a_{z,0}X_i + \sum_{z=1}^N n_z\beta_z a_{z,0} + n_{2,0}$, where $a_{i,z}$ and $a_{z,0}$ are the channel gains from user i to the relay z, and from that relay to the BS, respectively. Also, n_z and $n_{2,0}$ are the noise variables at relay z and the BS. We assume that the BS decoder knows $\{a_{i,z}\}$ and $\{a_{z,0}\}$. The BS treats the signals other than X_i as interference, when trying to decode X_i. Then $Y_2 = X_i\sum_{z=1}^N a_{i,z}\beta_z a_{z,0} + \sum_{j,j\ne i}\sum_{z=1}^N a_{j,z}\beta_z a_{z,0}X_i + \sum_z n_z\beta_z a_{z,0} + n_{2,0}$. Thus, the SINR of user i at the second time slot is $\mathrm{SINR}_{i,2} = \sum_{z=1}^N |a_{i,z}|^2\beta_z^2|a_{z,0}|^2E_s \Big/ (\sum_{z=1}^N\sum_{j=1,j\ne i}^M |a_{j,z}|^2\beta_z^2|a_{z,0}|^2E_s + N_0 + N_0\sum_{z=1}^N|\beta_z|^2|a_{z,0}|^2)$ and the corresponding MI between X_i and Y_2 is $\mathcal{I}_{MSC,i}(2) = \frac{1}{4}\log_2(1+\mathrm{SINR}_{i,2})$. The accumulated MI over the two time slots is $\mathcal{I}_{MSC,i} \le \mathcal{I}_{MSC,i}(1) + \mathcal{I}_{MSC,i}(2)$. Note that without loss of generality, we assume that user i has the largest MI, $\mathcal{I}_{MSC,i}$, for the first round of decoding.

It is difficult to analyze the achievable diversity order based on $\mathcal{I}_{MSC,i}$. In order to make the analysis tractable, let us assume error-free source-relay channels in the network. We further assume that every relay transmits all user information with equal energy. By these modifications, we can show that the MI for such a network is larger than $\mathcal{I}_{MSC,i}$. The accumulated MI for the assumed network in two time slots is given as

$$\begin{aligned} \mathcal{I}_{ASC} &\le \frac{1}{4}\log_2\left(1+\frac{|a_{i,0}|^2E_s}{\sum_{j,j\ne i}|a_{j,0}|^2E_s+N_0}\right) \\ &\quad + \frac{1}{4}\log_2\left(1+\frac{\frac{1}{M}\sum_{z=1}^N|a_{z,0}|^2E_s}{\frac{M-1}{M}\sum_{z=1}^N|a_{z,0}|^2E_s+N_0}\right). \end{aligned} \quad (17)$$

Here the left term corresponds to the first time slot, and the right to the second. We note that $\lim_{\mathrm{SNR}\to\infty}\frac{\frac{1}{M}\sum_{z=1}^N|a_{z,0}|^2E_s}{\frac{M-1}{M}\sum_{z=1}^N|a_{z,0}|^2E_s+N_0} = \lim_{\mathrm{SNR}\to\infty}\frac{\frac{1}{M}\sum_{z=1}^N|a_{z,0}|^2}{\frac{M-1}{M}\sum_{z=1}^N|a_{z,0}|^2+\frac{1}{\mathrm{SNR}}} = \frac{1}{M-1}$. Consequently, at high SNR

$$\begin{aligned} \mathcal{I}_{ASC} &\le \frac{1}{4}\log_2\Big(1+\frac{1}{M-1}+\frac{|a_{i,0}|^2E_s}{\sum_{j,j\ne i}|a_{j,0}|^2E_s+N_0} \\ &\quad + \frac{1}{M-1}\frac{|a_{i,0}|^2E_s}{\sum_{j,j\ne i}|a_{j,0}|^2E_s+N_0}\Big) \\ &< \frac{1}{4}\log_2\left(1+\frac{1}{M-1}+|a_{i,0}|^2\mathrm{SNR}\frac{M}{M-1}\right). \end{aligned} \quad (18)$$

An outage event occurs if $R > \frac{1}{4}\log_2(1+\frac{1}{M-1}+|a_{i,0}|^2\mathrm{SNR}\frac{M}{M-1})$. Thus, since $|a_{i,0}|$ is Rayleigh distributed, for high SNR the outage probability is lower bounded by

$$\frac{2^{4R}-1-\frac{1}{M-1}}{\mathrm{SNR}}\frac{M-1}{M}.$$

It is hence easy to see that we cannot get a full diversity order of $N+1$ by SC-based coding in this networking scenario. ■

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