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NON-UNIFORM NEAR-PERFECT-RECONSTRUCTION OVERSAMPLED DFT FILTER BANKS BASED ON ALLPASS-TRANSFORMS

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ABSTRACT

In this contribution we discuss an oversampled non-uniform DFT filter bank, which is derived by allpass frequency transformations from its uniform version. Here, a perfect-reconstruction (PR) solution generally requires a non-stable synthesis filter bank. As a new result we show that by alleviating the PR conditions it is possible to construct a stable synthesis system, where the subband filters are of FIR type. The delay of the resulting near-PR system can be efficiently controlled by a factorization of the analysis and synthesis filter bank into lifting steps.

1. INTRODUCTION

Non-uniform filter banks are generally used in those applications, where the fixed time-frequency resolution provided by a uniform subband decomposition is not appropriate. A well-known example is the approximation of the critical bands in the human auditory system with non-uniform filter banks.

One simple way to obtain a non-uniform frequency resolution is based on the frequency transformation of an oversampled DFT polyphase filter bank [2, 8], where all delays are replaced by allpass elements. However, when we want to obtain perfect reconstruction (PR) for the overall analysis-synthesis system, the corresponding synthesis filter bank generally has unstable subband filters and is thus not suitable for the processing of infinite-length signals. One solution is to employ a double buffering scheme as in [1, 6], where the synthesis filtering operations are carried out alternately on time-reversed and non-time-reversed subband signal blocks. However, this method may lead to an increased system delay.

In this contribution we present a novel design for the synthesis filter bank, which leads to a near-PR solution for the allpass-based non-uniform oversampled DFT filter bank. In contrast to the method in [1, 6], the proposed synthesis filter bank only employs FIR subband filters, so that stability is always guaranteed. However, for long prototype filters or a large number of subbands this leads to an increased system delay. We show that this delay can be efficiently reduced when the design of the prototype lowpass filter is based on lifting schemes [3].

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2. ALLPASS-TRANSFORMED DFT FILTER BANKS

2.1. General Case

In the following we consider M-channel DFT filter banks, with M being an even integer number, in polyphase notation [7], where all delay elements are replaced by allpass filters $A(z)$ [2, 8]. The resulting structure is depicted in Fig. 1 for the analysis side, where $\mathbf{W}_M$ denotes the DFT matrix with $[\mathbf{W}_M]_{ki} = W_M^{ki} = e^{-j\frac{2\pi}{M}ki}$, $k, i = 0, \ldots, M-1$, and N_k the subsampling factor for the k-th subband. For the sake of

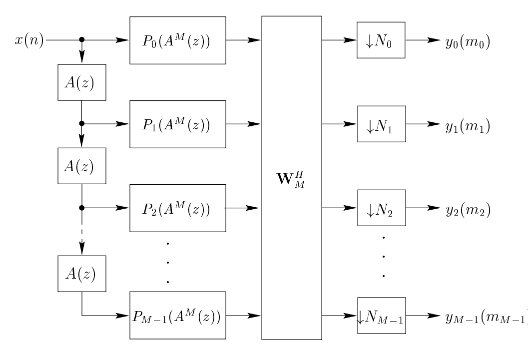

Figure 1: Generalized polyphase DFT filter bank, where all delay elements are replaced by allpass filters $A(z)$

simplicity we assume that the length L_p of the prototype impulse response $p(n)$ is restricted to integer multiples of M, i.e. $L_p = m \cdot M$, $m \in \mathbb{N}$. The (type 1) allpass-transformed polyphase components $P_\rho(A^M(z))$, $\rho = 0, \ldots, M-1$, are then defined as

$$P_\rho(A^M(z)) = \sum_{\lambda=0}^{m-1} p(\lambda M + \rho)\, A^{\lambda M}(z). \qquad (1)$$

As we can see from Fig. 1, the transfer functions of the M analysis subband filters $H_k(z)$ can generally be written according to

$$H_k(z) = \sum_{\rho=0}^{M-1} P_\rho(A^M(z))\, A^\rho(z)\, W_M^{-k\rho}. \qquad (2)$$

2.2. Special Cases

(a) Uniform filter bank. By choosing $A(z) = z^{-1}$ we obtain the classical DFT analysis filter bank with uniform frequency resolution. Here, the filter $A(z)$ can be regarded as an allpass of zeroth order with the frequency response $A(e^{j\omega}) = e^{-j\omega}$ and the linear phase response $\phi_u(\omega) = -\omega$. With (2) the analysis filters $H_k^{(u)}(z)$ in the uniform case are then given as

$$H_k^{(u)}(z) = \sum_{\rho=0}^{M-1} P_\rho(z^M)\, z^{-\rho}\, W_M^{-k\rho}. \qquad (3)$$

(b) Allpass transform of first order. Here, we choose the transfer functions $A(z)$ as stable and causal allpass filters of first order according to

$$A(z) = A_p(z) = \frac{z^{-1} - a}{1 - az^{-1}}, \quad -1 < a < 1, \qquad (4)$$

where we restrict ourselves to a real-valued parameter a. From (4) we obtain the frequency response

$$A_p(e^{j\omega}) = e^{j\phi_p(\omega)} \qquad (5)$$

with

$$\phi_p(\omega) = -\omega + 2\arctan\left(\frac{a\sin\omega}{a\cos\omega - 1}\right). \qquad (6)$$

Thus, replacing all terms z^{-1} by first order allpass elements leads to a transformation $\omega \to \phi_p(\omega)$ of the frequency scale. By inserting (5) and (6) in (2) the frequency responses of the allpass-transformed subband filters $H_k^{(n)}(z)$ can be written as

$$H_k^{(n)}(e^{j\omega}) = \sum_{\rho=0}^{M-1} P_\rho(e^{jM\phi_p(\omega)})\, e^{j\rho\phi_p(\omega)}\, W_M^{-k\rho},$$
$$= H_k^{(u)}(e^{-j\phi_p(\omega)}) = P(e^{-j(\phi_p(\omega)+k\,\frac{2\pi}{M})}).$$

These filters generally have different bandwidths, which results in a non-uniform frequency resolution of the corresponding analysis filter bank. Examples are depicted in Fig. 2 for different allpass parameters a.

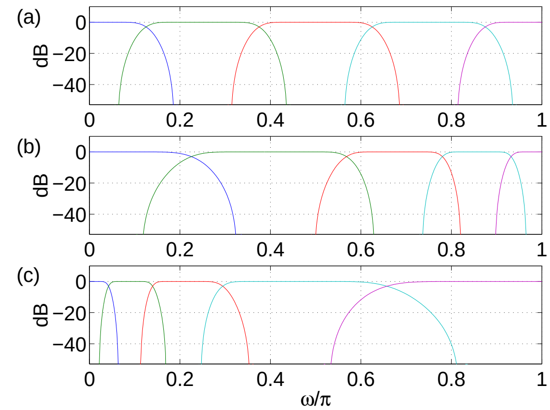

Figure 2: First order allpass-transformed analysis filter bank: Frequency resolution for $M = 8$ and (a) $a = 0$, (b) $a = -0.3$, (c) $a = 0.5$

3. NEAR-PR SYNTHESIS FILTER BANK

When designing an overall PR or near-PR analysis-synthesis system with the allpass-transformed DFT bank from section 2, we require both the analysis and synthesis subband filters to be stable and causal. Therefore, it is not possible to use a simple inverse of all allpass transfer functions (of non-zero order) in the synthesis filter bank. In this section we present a novel synthesis structure, which is only based on FIR subband filters and thus avoids all stability problems. Furthermore, it leads to near-PR for the corresponding non-critically subsampled analysis-synthesis system. For simplicity reasons we restrict ourselves in the following to allpass transforms of first order as discussed in section 2.2(b).

3.1. Basic Idea

In order to explain the basic idea of our approach we begin with a simple non-subsampled three-channel system, which is given in Fig. 3. Herein, $A(z)$ is chosen as the first order

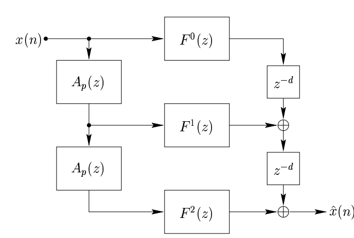

Figure 3: Simple three-channel allpass-based analysis-synthesis system

allpass $A_p(z)$ (4) with parameter a, and $F(z)$ is defined as

$$F(z) = -a\,z^{-d} + (1-a^2)\,z^{-(d-1)} + a\,(1-a^2)\,z^{-(d-2)} +$$
$$+ \cdots + a^{d-2}\,(1-a^2)\,z^{-1} + a^{d-1} \qquad (7)$$

with $d \in \mathbb{N}$. The overall transfer function of the system in Fig. 3 can thus be stated with $F^0(z) = 1$ as

$$T(z) = z^{-2d} + z^{-d} A_p(z)\, F(z) + A_p^2(z)\, F^2(z). \quad (8)$$

For the product $A_p(z)\, F(z)$ we now obtain with (4) and (7) the expression

$$\begin{aligned} A_p(z)\, F(z) &= \frac{z^{-1} - a}{1 - az^{-1}}\, (1 - a\, z^{-1})\, (z^{-(d-1)} + \\ &\quad + z^{-(d-2)}\, a + \cdots + z^{-1}\, a^{d-2} + a^{d-1}), \\ &= z^{-d} - a^d = z^{-d} + \epsilon(a, d). \end{aligned} \quad (9)$$

Since $|a| < 1$, we can select the order $d \in \mathbb{N}$ of the FIR-filter $F(z)$ such that we have an approximate compensation of the phase distortions introduced by the allpass filter up to a certain error $\epsilon(a, d)$. This error can be made arbitrarily small at the expense of additional delay. Thus, (9) can be approximated as

$$A_p(z)\, F(z) \approx z^{-d}, \quad (10)$$

and the overall transfer function (8) now becomes $T(z) \approx 3\, z^{-2d}$.

Note that the analysis-synthesis system in Fig. 3 can be regarded as a generalized (allpass-based) "delay" chain with a near-PR property, where in order to obtain a sufficient compensation of the allpass phase distortions, the system delay must be increased by a factor d.

3.2. General Synthesis Structure

Based on the phase compensation approach discussed above, we are now able to construct a near-PR synthesis filter bank, when the prototypes are designed appropriately (see below). We restrict ourselves to prototypes with identical lengths on the analysis and the synthesis side.

(a) Case $L_p = M$. The resulting modified synthesis structure for this special case is depicted in Fig. 4, where the synthesis prototype of length $L_q = M$ is denoted with $q(n)$. When we now consider the non-subsampled case with $N_k \equiv 1$ for $k = 0, \ldots, M-1$ and choose the synthesis prototype according to $q(L_p - 1 - n) = 1/p(n)$, $p(n) \neq 0$ for all $n = 0, \ldots, L_p - 1$, we can see that the only distortion in the reconstructed signal is due to the non-ideal behavior of the compensation filter $F(z)$. Note that for $M = 3$ after some structural simplifications we again obtain the system in Fig. 3.

(b) Case $L_p > M$. For longer prototypes with $L_p = L_q = m\, M$ and $m = 2, 3, \ldots$ the resulting synthesis structure is depicted in Fig. 5. Again, if the prototype is designed appropriately, the overall analysis-synthesis system yields near-PR, which will be shown in the following for the non-subsampled case. In a first step we redraw the

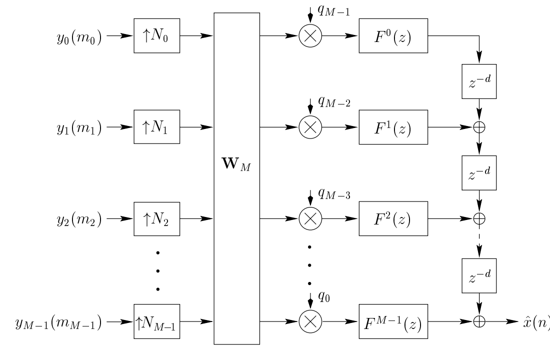

Figure 4: Modified DFT synthesis filter bank, $L_q = M$

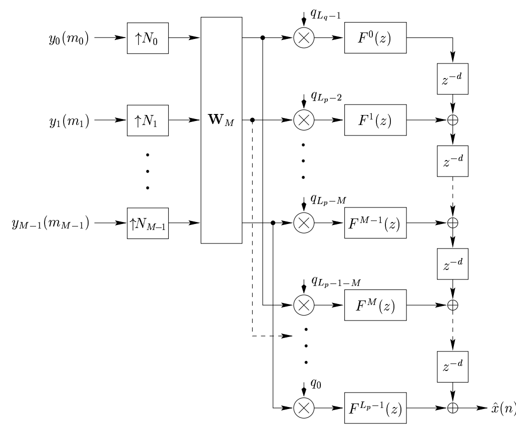

Figure 5: Modified DFT synthesis filter bank for $L_q > M$

synthesis filter bank in Fig. 5 as depicted in Fig. 6, where the modified (type 1) "polyphase components" $\hat{Q}_\rho(z)$ are defined as

$$\hat{Q}_\rho(z) = \sum_{\lambda=0}^{m-1} q(\lambda M + \rho)\, F^{(M(m-\lambda)-\rho-1)}(z)\, z^{-Md\lambda} \quad (11)$$

with $\rho = 0, \ldots, M-1$. Furthermore, the polyphase matrices of the modified DFT filter bank for the non-subsampled case may be written as

$$\mathbf{E}_n(z) = \mathbf{W}_M^H\, \mathbf{P}(A_p(z)), \quad \mathbf{R}_n(z) = \mathbf{Q}(z)\, \mathbf{W}_M.$$

Herein, $\mathbf{P}(A_p(z))$ is defined as

$$\mathbf{P}(A_p(z)) = \begin{bmatrix} \mathbf{p}_0(A_p^M(z)) \\ A_p^{\frac{M}{2}}(z)\, \mathbf{p}_1(A_p^M(z)) \end{bmatrix}$$

with the diagonal matrices

$$\begin{aligned} \mathbf{p}_0(A_p^M(z)) &= \text{diag}\,[P_0(A_p^M(z)), \ldots, P_{\frac{M}{2}-1}(A_p^M(z))], \\ \mathbf{p}_1(A_p^M(z)) &= \text{diag}\,[P_{\frac{M}{2}}(A_p^M(z)), \ldots, P_{M-1}(A_p^M(z))]. \end{aligned}$$

Likewise, we have

$$\mathbf{Q}(z) = \begin{bmatrix} z^{-\frac{Md}{2}}\, \mathbf{q}_1(z) & \mathbf{q}_0(z) \end{bmatrix}$$

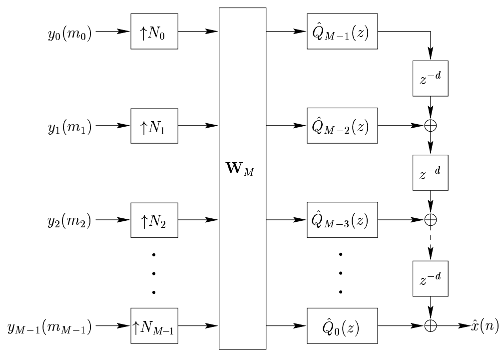

Figure 6: Alternative structure for the synthesis bank in Fig. 5 (see text).

with

$$\mathbf{q}_0(z) = \text{diag}\,[\hat{Q}_{\frac{M}{2}-1}(z), \hat{Q}_{\frac{M}{2}-2}(z), \ldots, \hat{Q}_0(z)],$$
$$\mathbf{q}_1(z) = \text{diag}\,[\hat{Q}_{M-1}(z), \hat{Q}_{M-2}(z), \ldots, \hat{Q}_{\frac{M}{2}}(z)].$$

Based on this notation, the requirement for the overall transfer function of the near-PR analysis-synthesis system $T(z) \overset{!}{\approx} z^{-D}$ can be stated as

$$T(z) = \mathbf{e}_d(z)\,\mathbf{R}_n(z)\,\mathbf{E}_n(z)\,\mathbf{a}(z) \overset{!}{\approx} z^{-D}, \qquad (12)$$

where $\mathbf{a}(z) = [1, A_p(z), \ldots, A_p^{\frac{M}{2}-1}(z)]^T$ and $\mathbf{e}_d(z) = [z^{-(\frac{M}{2}-1)d}, z^{-(\frac{M}{2}-2)d}, \ldots, 1]$. When we furthermore define the matrix $\mathbf{F}(z) = \text{diag}\,[F^0(z), F^1(z), \ldots, F^{\frac{M}{2}-1}(z)]$ and choose the order d of the compensation filter $F(z)$ appropriately, it is obvious from the discussion in section 3.1 that

$$\mathbf{e}_d(z)\,\mathbf{F}(z)\,\mathbf{a}(z) \approx \frac{M}{2} z^{-(\frac{M}{2}-1)d}. \qquad (13)$$

Comparing this with the right-hand side of (12) we can see that the condition

$$\mathbf{Q}(z)\,\mathbf{P}(A_p(z)) \overset{!}{=} \frac{2}{M}\, z^{-s}\,\mathbf{F}(z) \quad \text{with} \quad s \in \mathbb{N} \qquad (14)$$

must be satisfied, where we assume that the phase compensation error $\epsilon(a, d)$ represents the only distortion in the reconstructed signal. From (14) we can obtain the following conditions for the polyphase components:

$$z^{-\frac{M\,d}{2}}\,\hat{Q}_{M-1-k}(z)\,P_k(A_p^M(z)) + A_p^{\frac{M}{2}}(z)\,\hat{Q}_{\frac{M}{2}-1-k}(z)\cdot$$
$$\cdot\, P_{\frac{M}{2}+k}(A_p^M(z)) \overset{!}{=} \frac{2}{M}\, z^{-s}\, F^k(z) \qquad (15)$$

for $k = 0, 1, \ldots, \frac{M}{2}-1$. In order to fulfill (15) the parameter s has to be chosen as $s = q\,M\,d + \frac{M}{2}\,d,\ q \in \mathbb{N}$, which leads to an overall delay of $D = q\,M\,d + (M-1)\,d$ samples.

When we have a linear-phase prototype $p(n) = q(n) = p(L_p - 1 - n)$ satisfying (15) it can be shown that its type 1 polyphase components also satisfy the PR-conditions for the twofold oversampled DFT filter bank [5] given as

$$P_k(z)\,P_{M-1-k}(z) + P_{k+\frac{M}{2}}(z)\,P_{\frac{M}{2}-1-k}(z) \overset{!}{=} \frac{2}{M}\, z^{-(m-1)}. \qquad (16)$$

Clearly, since the allpass-transformed system has been obtained by a frequency scale transformation from a uniform DFT bank, the same design criteria hold for linear-phase prototypes, which do not introduce any additional non-linear phase into the system. Thus, it is possible to use a linear-phase pseudo-QMF design [4] as well. As we have seen above, in the non-subsampled case a design via the PR conditions for the uniform case only leads to distortions due to the phase compensation error $\epsilon(a, d)$, whereas a pseudo-QMF approach also introduces additional linear distortions into the reconstructed signal. In the more important subsampled case the aliasing components should be suppressed by a sufficient stopband attenuation of the prototype filters. However, critical subsampling without strong aliasing distortions in the reconstructed signal is not possible here (exactly as for the uniform DFT filter bank with $L_p > M$ [9]).

Note that for long prototype filters with high stopband attenuation the length of the allpass chain increases, which also leads to an increased overall system delay. This problem will be addressed in the next section.

4. IMPROVED DESIGN WITH LIFTING SCHEMES

Lifting is a technique for the construction of biorthogonal wavelet bases. However, it has been shown in [3] that lifting can also be successfully applied to the design of PR cosine-modulated filter banks. By applying lifting and dual lifting steps with different amounts of additional delay to the polyphase components of the prototype filter, the length of the filters can be increased while constraining the overall delay to a desired value.

In the following we present an adaptation of this lattice-like structure to the allpass-transformed DFT filter bank, where in a first step the PR-conditions in (15) for every k can be written as

$$\begin{bmatrix} P_k(A_p^M(z)) & A_p^{\frac{M}{2}}(z) P_{k+\frac{M}{2}}(A_p^M(z)) \end{bmatrix} \mathbf{I} \begin{bmatrix} z^{-\frac{M\,d}{2}}\hat{Q}_{M-1-k}(z) \\ \hat{Q}_{\frac{M}{2}-1-k}(z) \end{bmatrix}$$
$$\overset{!}{=} \frac{2}{M}\, z^{-s}\, F^k(z).$$

The identity matrix is then replaced by $\mathbf{I} = \mathbf{C}\,\mathbf{B}\,(\mathbf{C}\,\mathbf{B})^{-1}$, where for example in a zero delay lifting step the matrices

$\mathbf{B}$ and $\mathbf{C}$ are defined according to

$$\mathbf{C}\mathbf{C}^{-1} = \begin{bmatrix} 1 & 0 \\ c\,A_p^{\frac{M}{2}}(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -c\,A_p^{\frac{M}{2}}(z) & 1 \end{bmatrix} = \mathbf{I},$$

$$\mathbf{B}\mathbf{B}^{-1} = \begin{bmatrix} 1 & b\,A_p^{\frac{M}{2}}(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -b\,A_p^{\frac{M}{2}}(z) \\ 0 & 1 \end{bmatrix} = \mathbf{I}.$$

Herein, the parameters c and b denote free parameters, which can be utilized to improve the stopband attenuation of the prototype filter. The resulting allpass-transformed zero delay lifting structure for the analysis filter bank is depicted in Fig. 7. This approach can also be used to design a single lifting step for the allpass-transformed filter bank, which increases the delay by $2\,M\,d$ samples. The resulting non-uniform resolution analysis filter bank based on allpass-transformed lifting steps is shown in Fig. 8. The synthesis filter bank may be derived by the inverse lifting operations. Furthermore, the overall system delay can be given as $D = q_2\,M\,d + (M-1)\,d$ with $q_2 = (m_i - 1) + r_s$, where $L_{p_i} = m_i\,M$ denotes the length of the initial prototype and r_s the number of the single delay lifting steps.

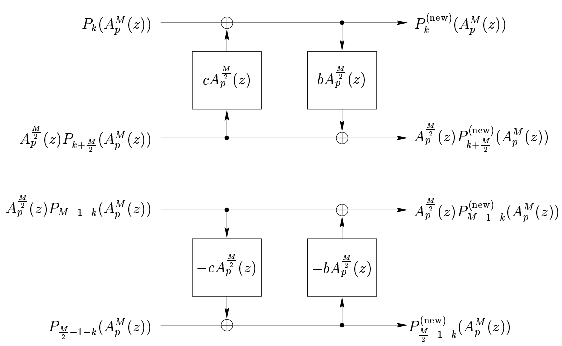

Figure 7: Allpass-transformed zero delay lifting and dual lifting step for the analysis filter bank

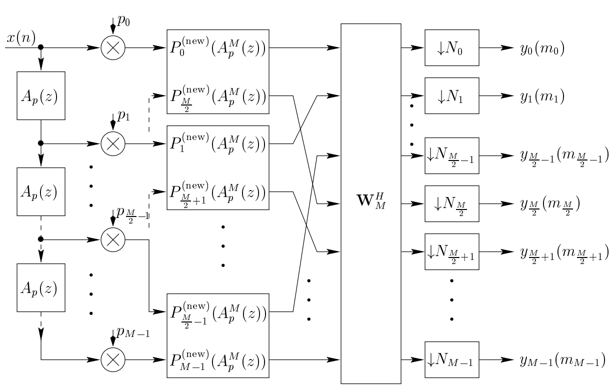

Figure 8: Generalized polyphase DFT analysis filter bank using allpass-transformed lifting steps

5. DESIGN PROCEDURE AND EXAMPLES

In the following design examples we assume identical prototypes $p(n) = q(n)$ on the analysis and synthesis side for simplicity reasons.

5.1. Prototype design

In order to design the prototype filters we can simply utilize the lifting steps for the uniform case with $a = 0$, where a rectangular window of length M is applied as an initial solution. Note that it is also possible to use a longer PR oder PQMF linear-phase prototype as an initial filter. We then apply zero lifting and single lifting steps such that the desired overall system delay of $D = (3M-1)\,d$ is achieved. The free parameters in the lifting blocks are obtained via a nonlinear optimization under additional minimization of the prototype's stopband energy. Fig. 9 shows the magnitude frequency responses for two design examples. The design parameters are chosen as

(a) $L_p = 64$, $L_{p_i} = 8$, $M = 8$, $D = 23\,d$, and

(b) $L_p = 192$, $L_{p_i} = 24$, $M = 24$, $D = 71\,d$.

5.2. Design examples

(a) Case *M = 8*. In this example we apply the prototype filter designed in section 5.1(a) to the allpass-transformed analysis-synthesis system with $a = 0.1$ and $d = 5$, leading to a delay of $D = 115$ samples. The magnitude frequency responses of the analysis subband filters are shown in Fig. 10(a) and the overall amplitude distortion for the non-subsampled case, which is only due to the phase compensation error of $|\epsilon(a,d)| = 10^{-5}$, is depicted in Fig. 10(b). The aliasing distortion in the reconstructed signal for the subsampled case with $N_k \equiv 2$, $k = 0, \ldots, M-1$, is given in Figure 11. Here the solid line corresponds to the result for $a = 0.1$ and $d = 5$, whereas the dashed line denotes the choice $a = 0.3$ with $d = 10$ ($D = 710$ samples). Note that the aliasing error corresponds to the stopband attenuation of the prototype filter.

(b) Case *M = 24*. Here, we use the prototype filter from the example in section 5.1(b), where the magnitude frequency response for the subband filters and the aliasing distortion for the parameters $a = 0.3$, $d = 10$ (resulting in $D = 710$ samples), $N_k \equiv 4$ are depicted in Fig. 12.

6. CONCLUSION

We have presented a novel design approach for a stable (FIR) synthesis filter bank, which corresponds to an allpass-transformed non-uniform oversampled DFT analysis bank. It has been shown that the overall analysis-synthesis system satisfies the near-PR property, where the nonlinear phase distortion introduced by the allpass-transform in the analysis filter bank is almost canceled by FIR compensation filters in the synthesis bank. Furthermore, the overall sys-

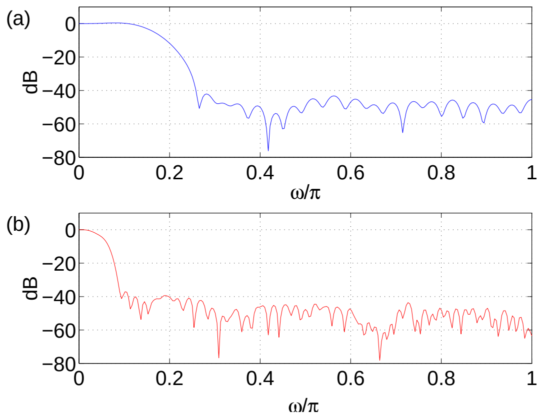

Figure 9: Magnitude frequency responses for two prototype design examples: (a) $M = 8$, $L_p = 64$, $L_{p_i} = 8$, $D = 23\,d$; (b) $M = 24$, $L_p = 192$, $L_{p_i} = 24$, $D = 71\,d$

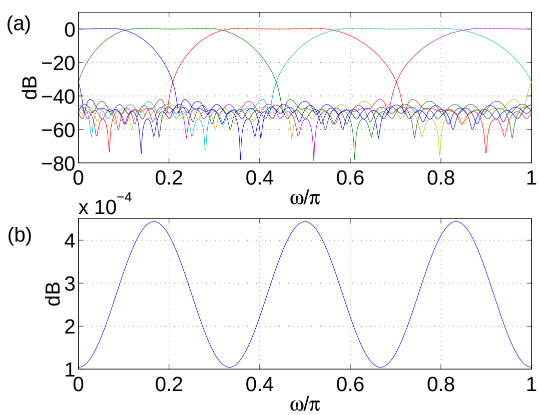

Figure 10: Overall analysis-synthesis system (parameters $M = 8$, $a = 0.1$, $d = 5$, prototype from Fig. 9(a)): (a) Magnitude frequency responses for the subband filters, (b) overall amplitude distortion for $N_k \equiv 1$.

tem delay can be efficiently reduced by applying allpass-transformed lifting steps to the polyphase components of the prototype. The resulting distortion and aliasing components in the reconstructed signal are small and correspond to the stopband attenuation of the prototype filter.

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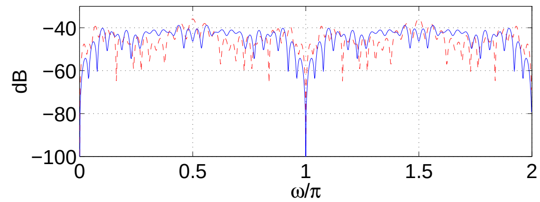

Figure 11: Aliasing distortion in the reconstructed signal for $N_k \equiv 2$, $M = 8$ and $a = 0.1$ (solid line), $a = 0.3$ (dashed line)

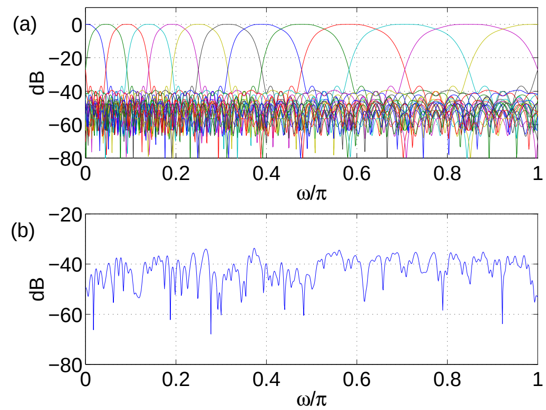

Figure 12: Overall analysis-synthesis system (parameters $M = 24$, $a = 0.3$, $d = 10$, prototype from Fig. 9(b)): (a) Magnitude frequency responses for the subband filters, (b) aliasing distortion for $N_k \equiv 4$.

REFERENCES

[1] C. D. Creusere and S. K. Mitra. Efficient audio coding using perfect reconstruction noncausal IIR filter banks. *IEEE Trans. on Speech and Audio Processing*, 4:115–123, Mar. 1996.

[2] M. Kappelan, B. Strauss, and P. Vary. Flexible nonuniform filter banks using allpass transformation of multiple order. In *Proc. European Signal Processing Conf.*, Signal Processing VIII: Theory and Applications, pages 1745–1748, Trieste, Italy, Sept. 1996.

[3] T. Karp and A. Mertins. Lifting schemes for biorthogonal modulated filter banks. In *Proc. Int. Conf. on Digital Signal Processing*, Santorini, Greece, June 1997.

[4] J. Kliewer. On the relationship between pseudo-QMF designs and perfect reconstruction solutions for modulated filter banks. In *Proc. 33th Asilomar Conf. on Signals, Systems, and Computers*, Asilomar, USA, Nov. 1999.

[5] J. Kliewer and A. Mertins. Oversampled cosine–modulated filter banks with arbitrary system delay. *IEEE Trans. on Signal Processing*, 46(4):941–955, Apr. 1998.

[6] S. K. Mitra, C. D. Creusere, and H. Babic. A novel implementations of perfect reconstruction QMF banks using IIR filters for infinite length signals. In *Proc. IEEE Int. Sympos. Circuits and Systems*, pages 2312–2315, San Diego, USA, May 1992.

[7] K. Swaminathan and P. P. Vaidyanathan. Theory and design of uniform DFT, parallel, quadrature mirror filter banks. *IEEE Trans. on Circuits and Systems*, CAS-33(12):1170–1191, Dec. 1986.

[8] P. Vary. Digital filter banks with unequal resolution. In *Proc. European Signal Processing Conf.*, pages 41–42, 1980.

[9] M. Vetterli and D. le Gall. Perfect reconstruction filter banks: Some properties and factorizations. *IEEE Trans. on Acoust., Speech, Signal Processing*, ASSP-37:1057–1071, July 1989.