On the Relationship between Pseudo-QMF Designs and Perfect-Reconstruction Solutions for Modulated Filter Banks

Jörg Kliewer
University of Kiel
Chair for Circuits and Systems Theory
Kaiserstr. 2, D-24143 Kiel, Germany
Phone: +49 (431) 77572-406, Fax: +49 (431) 77572-403
E-Mail: jkl@techfak.uni-kiel.de

Abstract

In this paper the classical pseudo-QMF approach for modulated filter banks is related to a recently published general perfect reconstruction (PR) description for integer oversampling ratios. It is shown that a pseudo-QMF prototype approximately satisfies the PR conditions, where the error depends on the remaining linear distortions and on the stopband attenuation of the prototype. For the non-subsampled case the pseudo-QMF and the PR condition are equivalent. Furthermore, an upper bound for the PR-approximation error can be given, when the maximum allowable deviation from the ideal flat frequency response of the analysis-synthesis system is specified.

1. Introduction

When designing modulated filter banks one generally has the choice between a perfect reconstruction (PR) design (e.g. [3, 8]) or the classical pseudo-QMF approach [6, 10], where the latter is giving rise to near-perfect-reconstruction (NPR) solutions. The PR approach has the advantage that it leads to a simple and compact matrix description of the filter bank. However, due to less severe design constraints, the pseudo-QMF approach often leads to more selective prototypes compared to PR designs. Furthermore, since in many applications small linear and aliasing distortions are tolerated at the output of the analysis-synthesis system, an NPR solution is often sufficient. Subject of this paper is the relation between the pseudo-QMF design and a general PR approach for arbitrary integer oversampling ratios, where the latter has recently been published in [7] for DFT and cosine-modulated filter banks. In contrast to this approach the filter bank literature always treats both design methods as separate cases. It turns out that the pseudo-QMF design can be regarded as a special case of the general PR description. As a consequence it is also possible to design pseudo-QMF prototypes with methods developed for the PR case.

2. Pseudo-QMF and PR Design Conditions

In K-channel modulated filter banks the analysis and synthesis filters $h_k(n)$ and $f_k(n)$ of the general analysis-synthesis system in Figure 1 are given according to

$$h_k(n) = p(n)\, w_k^{(a)}(n) \quad \text{and} \quad f_k(n) = q(n)\, w_k^{(s)}(n) \quad (1)$$

for $n = 0, \ldots, L_p - 1$ and $k = 0, \ldots, K-1$, where $w_k^{(a/s)}(n)$ denote the modulation sequences and $p(n)$, $q(n)$ the prototype filters. For the sake of simplicity we restrict ourselves in this paper to identical prototypes $p(n) = q(n)$ of length L_p and to modulation sequences, which lead to DFT or cosine-modulated filter banks [11]. However, a generalization to arbitrary modulation sequences is possible.

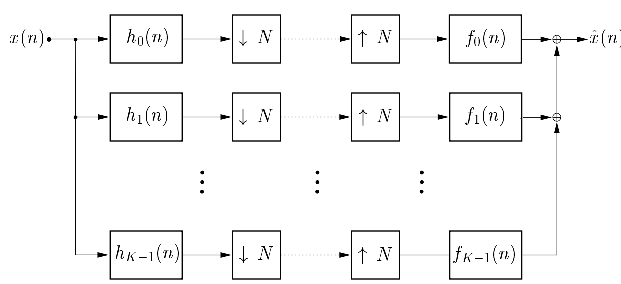

Figure 1. General analysis and synthesis filter bank with subsampling factor N. For (G)DFT filter banks we have $K = 2M$ subbands, for DCT-IV-based filter banks $K = M$ subbands.

In order to ensure equal prototype passband widths for all systems, the following design conditions will be derived for $2L$-times oversampled DFT filter banks [1, 7] and DCT-II/DST-II-based filter banks [9] with $K = 2M$ subbands and L-times oversampled DCT-IV-based filter banks [7, 8] with $K = M$ subbands, respectively. The oversampling ratio L is defined as $L = M/N$, where N denotes the subband subsampling factor.

2.1. Classical Pseudo-QMF Approach

A pseudo-QMF design requires the overall transfer function $T(z)$ of the analysis-synthesis-system in Figure 1 to be approximately a simple delay. This can be expressed in the time domain as

$$t(n) = \mathcal{Z}^{-1}\{T(z)\} = \frac{1}{N}\sum_{k=0}^{K-1}\sum_{\ell=0}^{L_p-1} h_k(\ell) f_k(n-\ell) \stackrel{!}{\approx} \delta(n-D), \quad (2)$$

where $\delta(n)$ denotes the unit sample sequence and D the overall system delay. Exemplarily, the following derivation is carried out for the Generalized DFT (GDFT) filter bank [2] with $K = 2M$. The result also holds for the other above mentioned types of modulated filter banks, since these systems use the GDFT filter bank as a "building block".

The modulation sequences of the GDFT filter bank are defined as

$$w_k^{(a/s)}(n) = W_{2M}^{-(k+k_0)(n-n_{a/s})}, \;\; W_{2M} = e^{-j\pi/M}, \quad (3)$$

where the parameter k_0 specifies the frequency displacement of the subband filter frequency responses, and n_a and n_s denote some phase parameters. Here $k_0 = 1/2$ has to be chosen for DCT-IV-based filter banks and $k_0 = 0$ in the DFT- or DCT-II/DST-II-based case.

Combining (1), (2) and (3) yields after some intermediate steps

$$t(n) = \frac{1}{N}\sum_{k=0}^{2M-1} W_{2M}^{-(k+k_0)(n-n_a-n_s)}\, g(n) \stackrel{!}{\approx} \delta(n-D), \quad (4)$$

where $g(n)$ denotes the result of the linear convolution $g(n) = p(n) * p(n)$. Equation (4) can be written in the z-domain as

$$T(z) = \frac{1}{N}\sum_{k=0}^{2M-1} W_{2M}^{(n_a+n_s)(k+k_0)}\, G(zW_{2M}^{k+k_0}) \stackrel{!}{\approx} z^{-D}, \quad (5)$$

which will be subsequently needed in Section 3. A closer evaluation of (4) reveals that, when we further identify the overall delay as $D = n_a + n_s$, the sequence $g(n)$ has to approximately satisfy the following Nyquist($2M$) condition:

$$g(n) = \sum_{\ell=0}^{L_p-1} p(\ell)\, p(n-\ell) \stackrel{!}{\approx} \begin{cases} \dfrac{e^{-j2\pi k_0\lambda_0}}{2L} & \text{for } n = D + \lambda_0\, 2M,\ \lambda_0 \in \mathbb{Z}, \\ 0 & \text{for } n = D + \lambda\, 2M,\ \lambda \neq \lambda_0,\ \lambda \in \mathbb{Z}, \\ \text{arbitrary} & \text{elsewhere.} \end{cases} \quad (6)$$

If the approximation error in (6) is zero, we have no linear distortions in the reconstructed signal at all. Without any loss of generality we restrict ourselves in the following to $k_0 = 0$.

The main aliasing components due to the subsampling in the subbands should be eliminated by an appropriate choice of the factor N and/or proper selection of the modulation sequences.

2.2. PR Approach for Integer Subsampling Ratios and Partial Aliasing Cancelation

In order to simplify the PR formulation we only regard special system delays $D = (d_0+1)2M-1$, $d_0 \in \mathbb{N}_0$, in the sequel of this paper. With the type-1 polyphase components of the prototype filter

$$P_\ell(z) = \sum_{\lambda=-\infty}^{\infty} p(\lambda N + \ell)\, z^{-\lambda} \quad (7)$$

it is shown in [7] that the PR conditions for all the above mentioned modulated filter banks can be derived as

$$\sum_{\lambda=0}^{2L-1} P_{2M-1-\lambda N-\ell}(z)\, P_{\lambda N+\ell}(z) \stackrel{!}{=} \frac{z^{-d_0}}{2M} \quad (8)$$

for $\ell = 0, 1, \ldots, N-1$. It is interesting to note that (8) is independent of k_0.

3. Relationship between Near-PR and PR Solutions

3.1. Partial Aliasing Compensation in the Oversampled PR Case

It has already been observed in [7] that when a PR prototype is designed such that it satisfies the PR conditions for a given oversampling ratio L_o, we obtain an NPR solution when the prototype is applied to a filter bank with $L < L_o$, $L_o = cL$, $c \in \mathbb{N}$. These solutions have a partial aliasing cancelation property which eliminates every L_o/L-th aliasing component. As an example, a PR prototype designed for

$L_o = 4$ and $M = 16$ is applied to a critically subsampled (DCT-IV-based) filter bank with $L = 1$. The magnitude bifrequency system function [2] in Figure 2 shows that every fourth aliasing component is canceled.

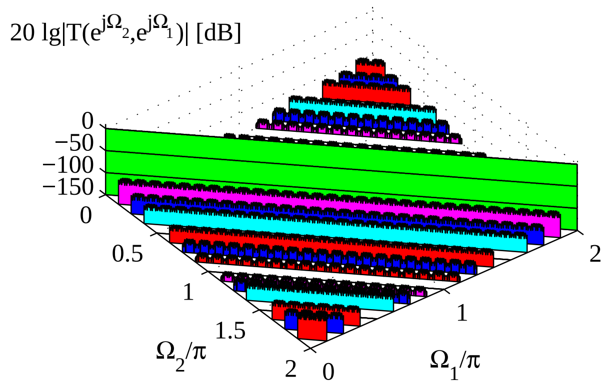

Figure 2. Magnitude bifrequency system function for $L = 1$, $L_o = 4$, $M = 16$

3.2. Approximation of the PR Conditions in the Pseudo-QMF Case

As an extension to the relations in Section 3.1 we show in this subsection that the type-1 polyphase components of a given pseudo-QMF prototype $p(n)$ satisfy the PR conditions approximately.

In a first step we write the polyphase components $P_\ell(z^{2M})$ as decimated versions of some filter $P(z)$ [4]:

$$P_\ell(z^{2M}) = \frac{z^\ell}{2M} \sum_{k=0}^{2M-1} P(zW_{2M}^k)\, W_{2M}^{k\ell}.$$

Using this expression in (8), where z has to be replaced by z^{2M}, we obtain

$$\sum_{\lambda=0}^{2L-1} P_{2M-1-\lambda N-\ell}(z^{2M})\, P_{\lambda N+\ell}(z^{2M}) = \frac{z^{2M-1}}{4M^2} \sum_{k=0}^{2M-1} \sum_{k'=0}^{2M-1} P(zW_{2M}^k)\, P(zW_{2M}^{k'}) \cdot \cdot W_{2M}^{k\ell-k'(\ell+1)} \sum_{\lambda=0}^{2L-1} W_{2M}^{(k-k')\lambda N}. \quad (9)$$

However, only those terms in (9) remain, where the difference $|k - k'|$ is an integer multiple of $2L$. Under the assumption that the prototype lowpass has a sufficiently high stopband attenuation and a stopband edge frequency of $\Omega_s < \pi/M$ we have no appreciable overlapping between both shifted versions of the prototype frequency response in (9). In this case, when we furthermore use the relations $G(z) = P^2(z)$ and $D = (d_0 + 1)2M - 1$, eq. (9) can be reduced to only one sum and and one error term $E_\ell^{(S)}(z^{2M})$ according to

$$\sum_{\lambda=0}^{2L-1} P_{2M-1-\lambda N-\ell}(z^{2M})\, P_{\lambda N+\ell}(z^{2M}) = \frac{L\, z^{2M-1}}{2M^2} \sum_{k=0}^{2M-1} W_{2M}^{Dk}\, G(zW_{2M}^k) + E_\ell^{(S)}(z^{2M}). \quad (10)$$

When the Nyquist condition (5) with $D = n_a + n_s$, $k_0 = 0$, is at least approximately satisfied by $G(z)$ and z^{2M} is formally replaced by z, (10) can finally be written as

$$\sum_{\lambda=0}^{2L-1} P_{2M-1-\lambda N-\ell}(z)\, P_{\lambda N+\ell}(z) = \frac{z^{-d_0}}{2M} + + E^{(N)}(z) + E_\ell^{(S)}(z), \quad \ell = 0, 1, \ldots, N-1. \quad (11)$$

The quality of the approximation depends on the error terms on the right-hand side of (11). The higher the stopband attenuation of the prototype filter, the smaller are the coefficients in $E_\ell^{(S)}(z)$, and the size of the coefficients in $E^{(N)}(z)$ depends on how well the sequence $g(n)$ satisfies the Nyquist condition, i.e. on the approximation error in (6).

4. Relationship in the Non-subsampled Case

As we know from Section 3.1 an exact PR solution for $N = 1$ completely eliminates all linear distortions. Thus the prototype has to be an ideal square-root Nyquist filter. On the other hand its polyphase components satisfy the PR condition in (8) for $N = 1$, since the conditions in (6) and (8) are equivalent. In the following we extend this similarity between both approaches also to the (approximate) pseudo-QMF case.

4.1. Equivalence of PR and Pseudo-QMF Solutions

In the non-subsampled case with $L = M$ only those terms in (9) remain, which correspond to $k' - k = 0$. This leads to $E_\ell^{(S)}(z^{2M}) = 0$ in (10), yielding

$$\sum_{\ell=0}^{2M-1} P_{2M-1-\ell}(z)\, P_\ell(z) = \frac{z^{-d_0}}{2M} + E^{(N)}(z). \quad (12)$$

Hence, the approximation quality of the PR conditions only depends on the linear distortions of the analysis-synthesis system. Note that when $g(n)$ has the Nyquist property the PR conditions are exactly satisfied and both conditions are equivalent. In the following we show that this also holds in the approximate case.

In a first step we assume that (12) is satisfied with $E^{(N)}(z) = 0$. The z-transform of the Nyquist filter $g(n)$ can be written with the type-1 polyphase components of the prototype in (7) as

$$\mathcal{Z}\{g(n)\} = G(z) = \sum_{\lambda=0}^{2M-1} z^{-\lambda} P_\lambda(z^{2M}) \sum_{\ell=0}^{2M-1} z^{-\ell} P_\ell(z^{2M}).$$

When we now construct the sequence $\hat{g}(n) = g(n-1)$ the first zero sample can be found in $\hat{g}(0)$, since we have $D = (d_0{+}1)2M{-}1$. Subsampling of $\hat{g}(n)$ by a factor $2M$ without phase offset now extracts the relevant samples $g(n+\lambda D)$, $\lambda \in \mathbb{Z}$, for satisfying the Nyquist property in (6):

$$\mathcal{Z}\{\hat{g}(2Mn)\} =$$
$$z^{-1} \sum_{\ell=0}^{2M-1} P_{2M-1-\ell}(z)\, P_\ell(z) \stackrel{!}{=} z^{-\frac{D+1}{2M}} = z^{-d_0-1}. \quad (13)$$

We can see that the right hand side of (13) can be written as the PR condition for $N = 1$ in (8), which confirms the identity between the pseudo-QMF and PR approaches in the non-subsampled case. Clearly, for $E^{(N)}(z) \neq 0$ this error term also affects the right hand side of (13) and thus also denotes the approximation error in the Nyquist condition (6).

4.2. PR Error Bound for Given Linear Distortions

As an important consequence we can design pseudo-QMF prototypes via an evaluation of the PR condition (8) for $L = M$. The simple relation

$$|\varepsilon_r| \leq \frac{|E_{T_{\max}}|}{2M\,Q}, \quad \text{where} \quad Q = \left\lfloor \frac{2L_p - 1}{2M} \right\rfloor, \quad (14)$$

can be used to roughly estimate the required PR approximation error $\pm\varepsilon_r$, when $|E_{T_{\max}}|$ denotes the maximum allowed deviation from the ideal flat overall frequency response of the filter bank.

Proof of eq. (14): In the non-subsampled case only the error term

$$E^{(N)}(z^{2M}) = E_g(z) = \sum_{n=0}^{Q-1} \varepsilon(n)\, z^{-2Mn}$$

is present in (12), where Q, given in (14), denotes the number of zeros plus one in the Nyquist impulse response $g(n)$, when (6) is ideally satisfied. Under the simplifying assumption that all coefficients $\varepsilon(n)$ are of the same size, i.e. $\varepsilon(n) = \varepsilon$ for all $n = 0, \ldots, Q-1$, we can calculate the frequency response of the error sequence $\epsilon(n)$ according to

$$E_g(e^{j\Omega}) = \varepsilon\, e^{-jM\Omega(Q-1)}\, \frac{\sin(M\Omega Q)}{\sin(M\Omega)}. \quad (15)$$

An evaluation of the maximum value of $|E_g(e^{j\Omega})|$ leads to

$$\max_{\Omega\in[0,2\pi]} |E_g(e^{j\Omega})| = |\varepsilon|\, Q \;\; \text{for} \;\; \Omega = 0, \frac{\pi}{M}, \frac{2\pi}{M}, \ldots \quad (16)$$

The error contribution to the (ideal) overall frequency response is given with (5), $D = n_a + n_s$, $k_0 = 0$ and $z = e^{j\Omega}$ as

$$E_T(e^{j\Omega}) = \frac{1}{N} \sum_{k=0}^{2M-1} W_{2M}^{Dk}\, E_g(e^{j(\Omega - k\frac{\pi}{M})}).$$

A coarse upper bound of the error magnitude frequency response $|E_T(e^{j\Omega})|$ can now be estimated as

$$|E_{T_{\max}}| = \max_{\Omega\in[0,2\pi]} |E_T(e^{j\Omega})| \leq \frac{1}{N} \sum_{k=0}^{2M-1} |E_g(e^{j(\Omega - k\frac{\pi}{M})})|,$$

which leads with (16) to the relation

$$|E_{T_{\max}}| \leq 2\,L\,Q\,|\varepsilon|. \quad (17)$$

We finally obtain (14) with $L{=}M$, when $|E_{T_{\max}}|$ is regarded as the independent variable and the equality in (17) indicates the upper bound for the searched PR error $|\varepsilon_r| = |\varepsilon|$.

4.3. Results

As an example a prototype for $L_p = 512$, $M = 32$, $D = 447$, $\Omega_s = 0.029\pi$ is designed, where the PR condition (8) for $N = 1$ is solved via a numerical optimization approach. As a design goal the maximum error in the overall magnitude frequency response $|T(e^{j\Omega})|$ should be kept below $20\log_{10}(1 + |E_{T_{\max}}|) = 10^{-3}$ dB. From (14) we then obtain the maximal allowable PR approximation error as $|\varepsilon_r| = 1.2 \cdot 10^{-7}$. The design result is depicted in Figure 3, where (a) shows the magnitude frequency response $|P(e^{j\Omega})|$ and (b) one period of $|T(e^{j\Omega})|$. We can see that $|T(e^{j\Omega})|$ matches the given design goal.

In another example we test the validity of the estimation (17) for $L = M$[1]. Figure 4 shows the obtained upper bounds for different PR approximation errors ε and prototype designs with different system delays, when we again design the pseudo-QMF prototype via the condition in (12). In Figure 4(a) the remaining design parameters are chosen as $L_p = 160$, $M = 16$ and $\Omega_s = 0.046\pi$, in Figure 4(b) they are $L_p = 512$, $M = 32$ and $\Omega_s = 0.023\pi$. One can see that for nearly all chosen ε eq. (17) represents an upper bound for the linear distortions. In Figure 4(b) for small ε, however, one encounters the limits of the utilized optimization algorithm, since here the required precision for satisfying the PR condition is not reached anymore.

5. Conclusion

As a novelty we have shown that the classical pseudo-QMF approach based on approximate square-root Nyquist

[1] It is only possible to verify (14) indirectly via (17), since $|E_{T_{\max}}|$ cannot be chosen as an independent parameter in the design process.

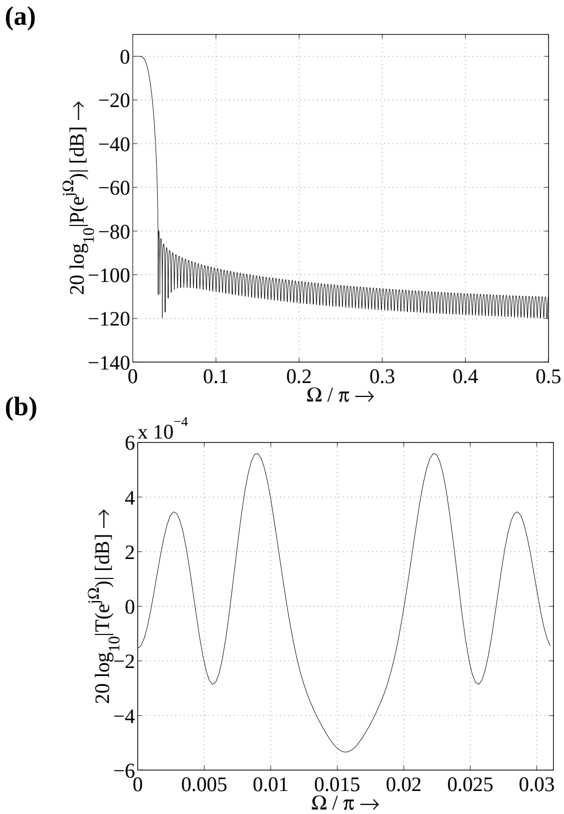

Figure 3. Pseudo-QMF prototype design based on the PR conditions for $L = M$ ($L_p = 512$, $M = 32$, $D = 447$, $\Omega_s = 0.029\pi$).

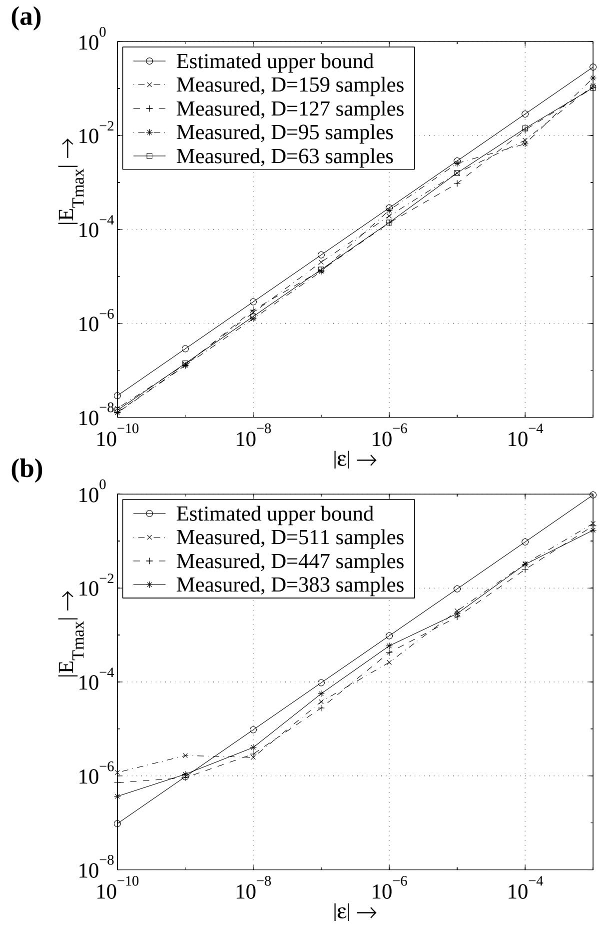

Figure 4. Verification of PR approximation bound, design parameter (a) $L_p = 160$, $M = 16$, $\Omega_s = 0.046\pi$, (b) $L_p = 512$, $M = 32$, $\Omega_s = 0.023\pi$.

prototypes can be regarded as a special case of the generalzed PR description, which is additionally extended by some error terms. This allows us to use the complete framework for PR modulated filter banks also in the pseudo-QMF case. For example, in the case of a system being free of linear distortions, lattice structures [8] or lifting schemes [5] could be used for an NPR prototype design with possibly better frequency selectivity compared to pure PR solutions.

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