On the Design of Double Serially Concatenated Codes with an Outer Repetition Code [1]

Francesca Vatta, DEEI, Università di Trieste, Trieste, Italy
Jörg Kliewer, Klipsch School of Electrical & Computer Engineering, New Mexico State University, Las Cruces, NM, U.S.A.
Christian Koller, Dept. of Electrical Engineering, University of Notre Dame, Notre Dame, IN, U.S.A.
Daniel J. Costello, Jr., Dept. of Electrical Engineering, University of Notre Dame, Notre Dame, IN, U.S.A.
Kamil Sh. Zigangirov, Dept. of Electrical Engineering, University of Notre Dame, Notre Dame, IN, U.S.A.

Abstract

Motivated by the recent discovery that repeat-accumulate-accumulate (RAA) codes with a repetition factor of three or larger are asymptotically good, we focus on the design of double-serially concatenated codes (DSCCs) with an outer repetition code. The main design goal is to obtain rate-1/2 codes with minimum distance approaching the Gilbert-Varshamov bound (GVB), while still maintaining a good convergence threshold with iterative decoding. To this end, the "accumulate" codes, used as middle and inner codes in RAA codes, are replaced with higher memory, rate-1 convolutional codes. We find that distance growth rates close to the GVB can be achieved in a DSCC with code memories as small as two or three. EXIT charts are used to design DSCCs with good convergence thresholds and distance growth properties. Finally, these codes are compared to the RAAA code and a punctured RAA code with a repetition factor of three, both of which exhibit linear distance growth with block length.

1 Introduction

Repeat-accumulate (RA) codes, consisting simply of a repetition code, an interleaver, and an accumulator are perhaps the simplest codes in the class of so-called "turbo-like" codes. Despite their simplicity, they were shown in [1] to meet the interleaver gain exponent conjecture, i.e., to achieve a vanishing word error probability as the interleaver length goes to infinity, for large enough signal-to-noise ratios (SNRs). The work reported in [1] led to the conjecture that powerful error-correcting codes can be obtained from simple component codes. Following this idea, Pfister et. al. [2] constructed "asymptotically good" binary linear block codes of any rate $r < 1$ by serially concatenating an arbitrary outer code of rate r with a large number of rate-1 accumulate codes using uniform random interleavers. Then they showed that the distance growth rate of these codes reaches the Gilbert-Varshamov bound (GVB) as the number of accumulators goes to infinity. [2] also contains numerical evaluations of minimum distance bounds for a variety of outer codes concatenated with one to four accumulators. Each additional accumulator increases the distance growth rate of the code, and for four accumulators the numerical bounds on minimum distance and the GVB are practically indistinguishable. Depending on the distance of the outer code, it can take more or less accumulators to obtain distance growth close to the GVB.

Recently in [3] it was shown that RAA codes with a repetition factor of 3 or larger exhibit linear distance growth as the block length tends to infinity, i.e., only two accumulators are needed to obtain "asymptotically good" codes. These results also hold if lower rate RAA codes are punctured to rate 1/2 using random puncturing, and the distance growth rate of these codes is very close to the GVB. Alternatively, in the case of a repeat-by-2 outer code, three accumulators are necessary to obtain linear distance growth. However, apart from the improved minimum distance, adding more concatenation stages causes degradation in the performance of iterative decoding. For example, the rate-1/2 RAAA code loses 1.5 dB in convergence threshold with respect to the RAA code [2].

The goal of this paper is to design double-serially-concatenated codes (DSCCs) with large minimum distance, while maintaining a reasonably good convergence threshold. As an alternative to adding more accumulator stages, we improve the distance properties of the code by replacing the "accumulate" codes, normally used as middle and inner codes, by higher memory rate-1 convolutional codes.

The paper is organized as follows. In the next section, we summarize the methods used for bounding the minimum distance of DSCCs and show how this distance can be im-

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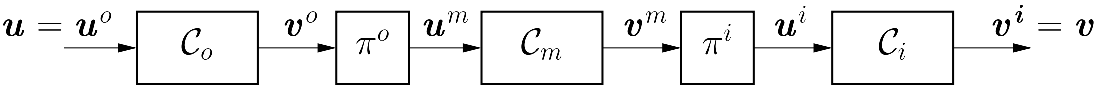

Figure 1: Encoder structure for a generic double serially concatenated code.

proved by replacing the middle and inner "accumulate" codes with higher memory convolutional codes. In Section 3, we present results on the convergence thresholds of several DCCCs using an EXIT chart analysis. In Section 4, we give simulation results for several of the proposed schemes. Finally, Section 5 summarizes the main results and anticipates some future work.

2 Minimum distance analysis

The encoder for a generic DSCC is shown in Fig. 1. The outer encoder $\mathcal{C}_o$ has a rate R^o and generates the code sequence v^o from the information sequence $u = u^o$. The outer permutation π^o then maps v^o to the sequence u^m, which is fed to the middle recursive convolutional code $\mathcal{C}_m$ of rate R^m. The resulting code sequence v^m is permuted by the inner permutation π^i, which provides the input sequence u^i for the inner rate R^i recursive convolutional code $\mathcal{C}_i$. The resulting code sequence is denoted by $v = v^i$.

2.1 Weight enumerators and the union bound

We follow the approach of Benedetto and Montorsi [4], [5] for serially concatenated and double serially concatenated codes. The component codes of the double-serially concatenated codes are connected through uniform random interleavers. The important property of the uniform interleaver is that its output only depends on the input weight w, not on the distribution of the weight within the input word. A uniform interleaver of length N' maps an input weight of w into all of its $\binom{N'}{w}$ possible permutations with equal probability. As a consequence, each codeword of the outer code $\mathcal{C}_o$ of weight l_1, through the action of the uniform interleaver, enters the middle encoder generating $\binom{N_1}{l_1}$ codewords of the inner code $\mathcal{C}_m$, and each codeword of the middle code $\mathcal{C}_m$ of weight l_2, through the action of the uniform interleaver, enters the inner encoder generating $\binom{N_2}{l_2}$ codewords of the inner code $\mathcal{C}_i$. The resulting Input-Output Weight Enumerating Function (IOWEF) $\bar{A}^{\mathcal{C}_s}_{w,h}$ is the average number of codewords in the overall DSCC $\mathcal{C}_s$of weight h associated with an input word of weight w over the ensemble of all interleavers and is given by

$$\bar{A}^{\mathcal{C}_s}_{w,h} = \sum_{l_1=0}^{N_1} \sum_{l_2=0}^{N_2} \frac{A^{\mathcal{C}_o}_{w,l_1}\, A^{\mathcal{C}_m}_{l_1,l_2}\, A^{\mathcal{C}_i}_{l_2,h}}{\binom{N_1}{l_1}\binom{N_2}{l_2}}, \qquad (1)$$

where $A^{\mathcal{C}_o}_{w,l_1}$, $A^{\mathcal{C}_m}_{l_1,l_2}$ and $A^{\mathcal{C}_i}_{l_2,h}$ are the IOWEFs of the outer, middle, and inner codes, respectively. From the average IOWEF we can derive the average Weight Enumerator Function (WEF) $\bar{A}^{\mathcal{C}_s}(h)$as

$$\bar{A}^{\mathcal{C}_s}_h = \sum_{w=1}^{K} \bar{A}^{\mathcal{C}_s}_{w,h} W^w \qquad (2)$$

where K is the input block length.

The average IOWEF can now be used to upper bound the error probability of the DSCC by employing the union bound. For an (N, K) DSCC the bit error probability on an additive white Gaussian noise (AWGN) channel with bit SNR equal to E_b/N_0 can, for maximum likelihood decoding, be upper bounded by

$$P_b(e) < \sum_{h=1}^{N} \sum_{w=1}^{K} \frac{w}{K} \bar{A}^{\mathcal{C}_s}_{w,h} \mathrm{e}^{-\frac{h\,K\,E_b}{N\,N_0}}. \qquad (3)$$

2.2 Asymptotic distance growth

Since the repetition code and the accumulator are very simple codes, their weight enumerators can be written as binomial coefficients. As given in [1], the IOWEF of an "accumulate" code for a input block length of N is

$$A^N_{w,h} = \binom{N-h}{\lfloor w/2 \rfloor}\binom{h-1}{\lceil w/2 \rceil - 1}, \qquad (4)$$

and the IOWEF of a repeat-by-q code with input block length K is

$$A^k_{w,h} = \binom{K}{w} \quad \text{if} \quad h = qw. \qquad (5)$$

Combining (4) and (5), and using the uniform interleaver analysis of [4] and [5], the average weight enumerator of serially concatenated codes consisting of repetition codes and accumulators can be written in terms of binomial coefficients.

Now following [6] and [7], we define the asymptotic spectral shape as

$$r(\rho) = \lim_{N\to\infty} \frac{\log \bar{A}^{\mathcal{C}_s}_{\rho N}}{N}, \qquad (6)$$

where $\rho = \frac{h}{N}$ is the normalized codeword weight. The important property of $r(\rho)$ in terms of minimum distance can be stated as follows: if the function becomes negative for some ρ, $\rho_0 > \rho > 0$, then crosses zero and is positive for $\rho > \rho_0$, it follows that almost all codes in the ensemble have at least a minimum distance of $\rho_0 N$ as the block length N tends to infinity.

Making use of the analysis presented recently in [3], Figure 2 shows the function $r(\rho)$ in the region of the zero crossing for a rate-1/2 RAAA code and an RAA code using a repeat-by-3 outer code, punctured to rate-1/2 using random puncturing. The zero crossings imply an asymptotic minimum distance growth rate of $\rho_0 = 0.1034$ for the RAAA code and $\rho_0 = 0.1036$ for the punctured RAA code. Both growth rates are very close to the GVB, which is $\rho_0 = 0.11$ for rate 1/2. Since this result applies only to asymptotically large block lengths, for finite block lengths we follow the method presented in [2] to bound the minimum distance of DSCCs having component encoders other than "accumulate" codes.

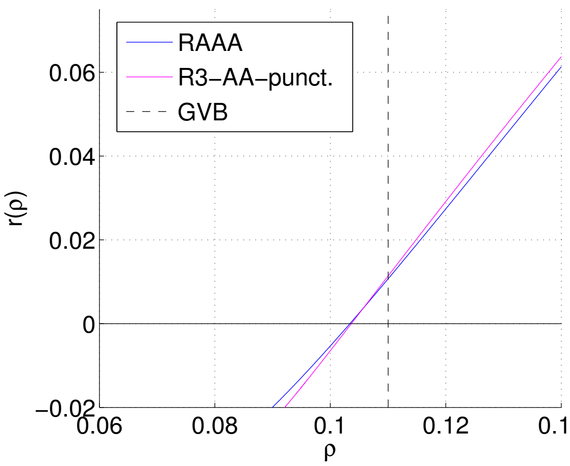

Figure 2: Asymptotic spectral shape function at the zero crossing for an RAAA code and a punctured repeat by 3-RAA code.

2.3 Minimum distance bounds

Since the uniform interleaver analysis yields the expected number of codewords of a certain weight, the probability that any code in the ensemble has a minimum distance d_{min} smaller than a value d can be upper bounded by [2]

$$\Pr(d_{\min} < d) \leq (\bar{A}_0^{\mathcal{C}_s} - 1) + \sum_{h=1}^{d-1} \bar{A}_h^{\mathcal{C}_s}. \quad (7)$$

To calculate the minimum distance bounds we set $\Pr(d_{\min} < d) = 1/2$. So we expect at least half of the codes in the ensemble to have a minimum distance of at least $d_{\min}$.

In Fig. 3, we plot the lower bound of (7) on d_{min} as a function of the block length N for several DSCCs together with the (finite block length) GVB for rate 1/2. As shown in the figure, using higher memory rate-1 codes instead of the accumulator is a good way to increase the distance growth rate of the code. We replaced the middle and/or inner "accumulate" codes (generator polynomial $(1,3)_8$) with higher memory convolutional codes, namely, we considered the rate-1 $(5,7)_8$, $(1,7)_8$, and $(1,13)_8$ recursive convolutional encoders (with generator polynomials in octal notation). Replacing only the middle accumulator proved more efficient in terms of distance than replacing only the inner accumulator. When both the middle and the inner "accumulate" codes are replaced, the resulting distance properties are very good. For the $(1,7)_8$ code, we almost reach the GVB, and when the $(5,7)_8$ code is used as the middle and inner encoder, d_{min} is practically indistinguishable from the GVB.

As anticipated from the asymptotic growth rates discussed in Section 2.2, the RAAA and the punctured repeat-by-3 RAA code also show very good distance growth behavior, and their distance growth rate bounds are almost identical[2]. (We note that the bounds in this section only apply to the best half of the ensemble, whereas the asymptotic growth rate ρ_0 from Section 2.2 states that the distance of almost all codes in the ensemble is at least ρ_0 times the block length as the block length tends to infinity.)

We expect the codes with large minimum distance growth rates to exhibit excellent error floor performance, but their convergence properties will not necessarily be good. This motivates the EXIT chart discussion in the next section.

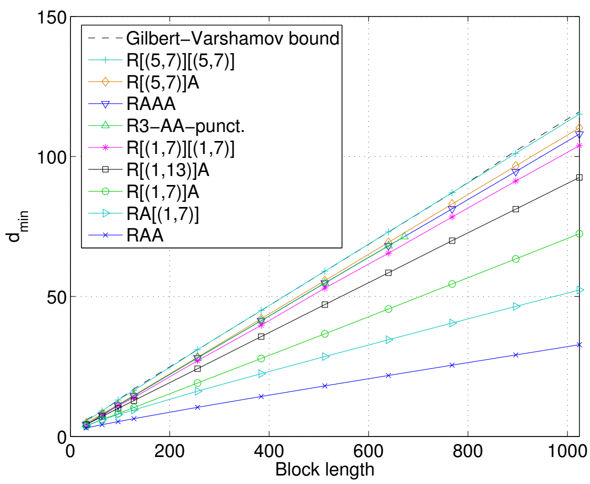

Figure 3: Lower bound on the minimum distance of several DSCCs versus the block length N.

3 Convergence analysis

Extrinsic information transfer (EXIT) charts have proved to be a useful tool for analyzing the convergence properties of iterative decoding for concatenated coding schemes

[2]The distance bound for the punctured repeat-by-3-RAA code is plotted only out to $N = 672$, which corresponds to $N = 1008$ before puncturing.

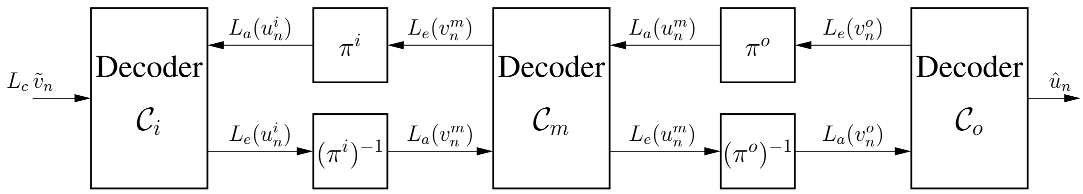

Figure 4: Iterative DSCC decoder structure.

with block lengths tending to infinity [8, 9] without requiring extensive decoder simulations. By considering the above minimum distance bounds alone, we might end up with code designs yielding a minimum distance close to the GVB that do not have good convergence properties; our goal is to find code designs with both good distance and convergence properties.

Let us first take a look at the iterative DSCC decoder in Fig. 4. The inner soft-input soft-output (SISO) decoder computes extrinsic log-likelihood ratios [10] (LLRs) $L_e(u_n^i)$ for the information bits u_n, where n is the bit-index, based on the LLRs $L_c\tilde{v}_n$ at the channel output, $L_c = 4\,r\,E_b/N_o$ and the *a priori* information $L_a(u_n^i)$. After deinterleaving with the permutation $(\pi^i)^{-1}$, the resulting LLRs are fed into the middle SISO decoder which, by additionally considering the *a priori* information $L_a(u_n^m)$ from the outer decoder, generates extrinsic LLRs $L_e(u_n^m)$ for its information bits. The outer SISO decoder computes extrinsic information $L_e(v_n^o)$ for its codebits by employing the *a priori* information $L_a(v_n^o)$ from the middle decoder. An iteration step is finished by again invoking the middle decoder, which calculates $L_e(v_n^m)$ based on $L_a(u_n^m)$, where the *a priori* information $L_a(v_n^m)$ is now provided by the inner decoder. After the iterations have converged, estimates $\hat{u}_n$ of the information bits u_n are generated at the output of the outer decoder.

The input-output behavior of a SISO decoder can be described by its EXIT characteristic. An EXIT characteristic specifies the relation between the symbol-wise mutual information between the transmitted bits u_n and the extrinsic values $L_e(u_n)$, $0 \le n \le N-1$, at the output of the decoder and the symbol-wise mutual information between the transmitted bits u_n and the *a priori* values $L_a(u_n)$ at the input to the decoder. By denoting the random variable for an information bit as U_n, the average mutual information or *a priori* information I_A at the *a priori* input and the average mutual information or extrinsic information I_E at the extrinsic output can be written as

$$I_A = \frac{1}{N}\sum_{n=0}^{N-1} I(U_n; L_a(U_n)), \tag{8}$$

$$I_E = \frac{1}{N}\sum_{n=0}^{N-1} I(U_n; L_e(U_n)), \tag{9}$$

respectively. For a double serially concatenated system we have four different mappings T from I_A to I_E, depending on the different decoding operations for the constituent decoders:

- For the inner decoder, the transfer function T_i from I_A^i to I_E^i depends on the communication channel capacity I_{ch}, which is a function of the channel SNR E_s/N_o. Thus, we can express the extrinsic information I_E^i as $I_E^i = T_i(I_A^i, I_{ch})$.
- In the first half iteration, where the middle decoder computes extrinsic LLRs $L_e(u_n^m)$ for the outer decoder, the transfer function depends on both the average mutual information $I_A^{o\to m}$ provided by the outer decoder and the average mutual information $I_A^{i\to m}$ provided by the inner decoder. The resulting extrinsic information is expressed as $I_E^{m\to o}$ and can be written as $I_E^{m\to o} = T_{m\to o}(I_A^{o\to m}, I_A^{i\to m})$.
- The transfer function of the outer decoder depends only on the *a priori* information I_A^o, and thus $I_E^o = T_o(I_A^o)$.
- In the second half iteration, the extrinsic information $I_E^{m\to i}$ provided by the middle decoder can be written as $I_E^{m\to i} = T_{m\to i}(I_A^{i\to m}, I_A^{o\to m})$.

In order to generate EXIT charts for the DSCC decoder which are easily interpretable, we follow the approach of [11]. In particular, we combine the transfer functions for the inner and middle decoders into a single transfer function $I_E^{m\to o} = T_{m,i}(I_A^{o\to m}, I_{ch})$ such that both decoders are considered as a single decoder for a single serially concatenated code. To this end, we express the transfer function for the inner decoder T_i solely as a function of $I_A^{o\to m}$, $I_E^{m\to o}$, and I_{ch} and denote the resulting transfer function as $\bar{T}_i$:

$$\begin{aligned} I_E^i &= T_i\left(T_{m\to i}\left[T_{m\to o}^{-1}(I_E^{m\to o}, I_A^{o\to m}), I_A^{o\to m}\right], I_{ch}\right) \\ &=: \bar{T}_i(I_E^{m\to o}, I_A^{o\to m}, I_{ch}). \end{aligned} \tag{10}$$

The transfer function $T_{m,i}(\cdot,\cdot)$ is now defined as the intersection of the two-dimensional functions $\bar{T}_i(I_E^{m\to o}, I_A^{o\to m}, I_{ch})$ and $\bar{T}_{m\to o}^{-1}(I_E^{m\to o}, I_A^{o\to m})$ projected onto the $I_E^{m\to o}, I_A^{o\to m}$ plane.

The resulting EXIT charts for several DSCCs are shown in Fig. 5 for a channel SNR of $E_b/N_0 = 1.3\,\text{dB}$ and Fig. 6 for a channel SNR of $E_b/N_0 = 2.7\,\text{dB}$, respectively. Fig. 5 depicts the results for DSCCs consisting of an inner “accumulate” code, an outer repetition code of rate $1/2$, and different middle codes. The transfer function for the repetition code is given by $I_E^o = T_o(I_A^o)$, whereas

$I_E^{m \to o} = T_{m,i}(I_A^{o \to m}, I_{ch})$ represents the transfer characteristic for the serial concatenation of the outer and middle codes. We see that the "accumulate" code achieves convergence at $E_b/N_0 = 1.3\,\text{dB}$, and the code with the $(1,7)_8$ encoders converges at $E_b/N_0 = 2.7\,\text{dB}$. The $(5,7)_8$ code is catastrophic and yields zero a posteriori estimates for zero a priori information [12]. So, to get the convergence started, we use systematic doping as in [13]. One out of every 16 code bits is replaced by its systematic counterpart. These bits are not included in the code constraints and thus help the convergence behavior of the code. However, if both (possibly doped) inner and middle codes with larger memory are used, convergence is much harder to achieve, as can be seen by comparing Figs. 5 and 6.

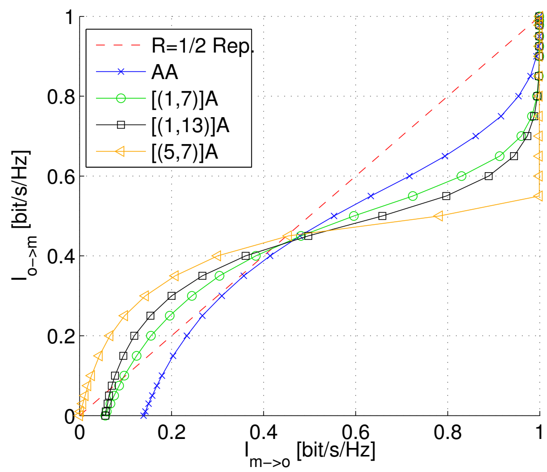

Figure 5: DSCC EXIT charts for $E_b/N_0 = 1.3\,\text{dB}$, an outer repetition code, different middle codes, and an inner accumulator.

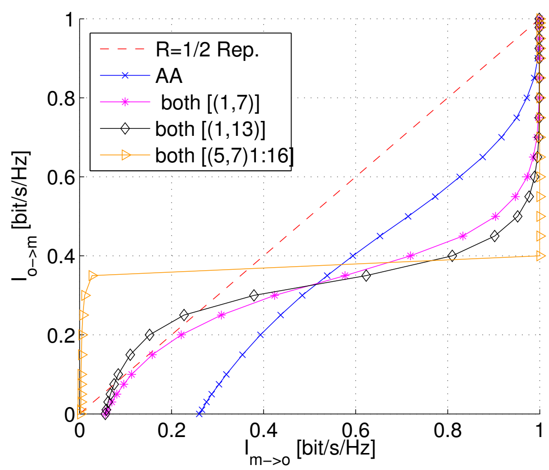

Figure 6: DSCC EXIT charts for $E_b/N_0 = 2.7\,\text{dB}$, an outer repetition code, and pairs of different middle/outer codes ("1:16" denotes systematic doping with a puncturing period of 16).

4 Simulation results

In Section 2, the bit error probability $P_b(E)$ of a DSCC was upper bounded using (3). $P_b(E)$ results for several rate-$1/2$ DSCCs are given in Fig. 7 together with iterative decoding simulations using 20 iterations, an input block length of $K = 1024$, and random interleavers on the AWGN channel. The simulation results are plotted as solid lines and the corresponding union bounds are plotted as dotted lines. Attention was focused on the $(1,7)_8$ and $(1,13)_8$ codes, since Figs. 5 and 6 indicate poor convergence behavior for the $(5,7)_8$ code.

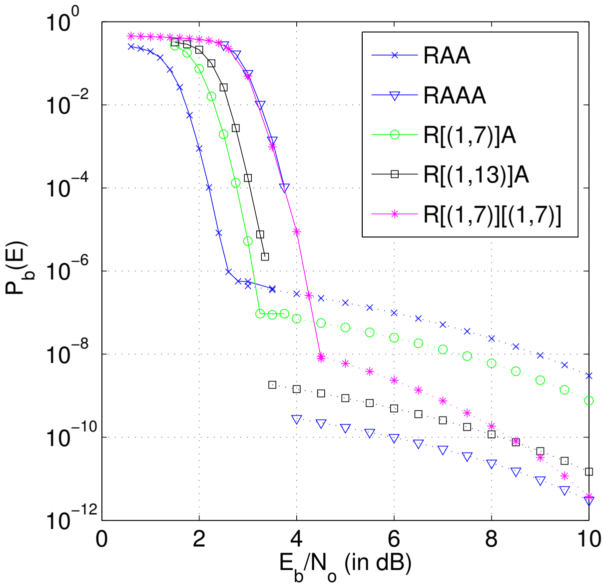

Figure 7: Bit error probability performance for rate-$r = 1/2$ DSCCs: union bounds and simulations, AWGN channel.

As anticipated from the results of Sections 2 and 3, replacing both "accumulate" codes with the $(1,7)_8$ code (R[(1,7)][(1,7)]) yields a very good error-floor, but a convergence threshold that is 1.5 dB worse than the RAA code. So both in its distance properties as well as in its convergence, it is comparable to the RAAA code. On the other hand, when replacing only the middle "accumulate" code with the $(1,7)_8$ code, the (R[(1,7)]A) code only loses about 0.5 dB w.r.t. the RAA code in convergence threshold, but achieves a lower error floor. The code with the middle $(1,13)_8$ code (R[(1,13)]A) has a slightly worse convergence threshold than the R[(1,7)]A code, as predicted by the EXIT chart in Fig. 5, but at 3.5 dB we were not able to observe any bit errors in 1,000,000 transmitted frames, which is an indication of excellent error floor performance. Overall, this code seems to achieve a good compromise between error floor performance and convergence threshold.

In general, the results exhibit a trade-off between minimum distance and convergence threshold. Considering the rather large gap between the thresholds of the RAA and RAAA codes, we have designed DSCCs with very good error floor performance and thresholds between those of the RAA and RAAA codes, as seen in Fig. 7.

5 Conclusions and future work

In this paper, we have demonstrated that DSCCs with minimum distance approaching the GVB can be designed by replacing the middle and/or inner "accumulator" in an RAA code with higher memory rate-1 convolutional encoders. In particular, distance growth close to the GVB can be achieved in a DSCC with code memories as small as 2 or 3. Based on the minimum distance results, EXIT charts were then used to design DSCCs with low error floors and convergence thresholds that lie between those of RAA and RAAA codes. In general, we observed that it is not possible to optimize both design criteria simultaneously, so that in practice it is necessary to seek a compromise between error floor performance and convergence threshold. Finally, simulation results were included to support the conclusions drawn from the minimum distance and EXIT chart analysis.

An important conclusion to be drawn from the paper is that DSCCs can achieve good convergence thresholds and distance growth approaching the GVB with very modest code memories, thus making it unnecessary to consider codes with multiple (more than two) levels of serial or parallel concatenation. Future work includes extensions of these code designs to higher rates and comparisons to other high rate code designs, such as those presented by Chugg et. al. [14], as well as an analytical treatment of the apparent trade-off between achievable minimum distance and convergence threshold.

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