MAT112 -Mr. José Pabón

Recitation will start soon.

We will pass this course with a great grade & meet our academic and professional goals

Finding Lengths of Curves

Find the lengths of the curves in Exercises 1–16. If you have graphing software, you may want to graph these curves to see what they look like.

10. $y = \frac{x^2}{2} - \frac{\ln x}{4}$ from $x = 1$ to $x = 3$

8. Bathroom scale A bathroom scale is compressed 1/16 in. when a 150-lb person stands on it. Assuming that the scale behaves like a spring that obeys Hooke's Law, how much does someone who compresses the scale 1/8 in. weigh? How much work is done compressing the scale 1/8 in.?

Find the areas of the surfaces generated by revolving the curves in Exercises 13–23 about the indicated axes. If you have a grapher, you may want to graph these curves to see what they look like.

13. $y = x^3/9, \quad 0 \le x \le 2; \quad x\text{-axis}$

14. $y = \sqrt{x}, \quad 3/4 \le x \le 15/4; \quad x\text{-axis}$

15. $y = \sqrt{2x - x^2}, \quad 0.5 \le x \le 1.5; \quad x\text{-axis}$

16. $y = \sqrt{x + 1}, \quad 1 \le x \le 5; \quad x\text{-axis}$

17. $x = y^3/3, \quad 0 \le y \le 1; \quad y\text{-axis}$

18. $x = (1/3)y^{3/2} - y^{1/2}, \quad 1 \le y \le 3; \quad y\text{-axis}$

19. $x = 2\sqrt{4 - y}, \quad 0 \le y \le 15/4; \quad y\text{-axis}$

- MAT112 T.A. José Pabón

We will be courteous, civil to each other.
NO SUCH THING AS AN OBVIOUS QUESTION
ask ask ask any doubt to clear up

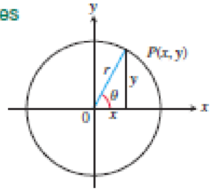
Trigonometry Formulas

Definitions and Fundamental Identities

Sine: $\sin\theta = \frac{y}{r} = \frac{1}{\csc\theta}$

Cosine: $\cos\theta = \frac{x}{r} = \frac{1}{\sec\theta}$

Tangent: $\tan\theta = \frac{y}{x} = \frac{1}{\cot\theta}$



Identities

$\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos\theta$

$\sin^2\theta + \cos^2\theta = 1, \quad \sec^2\theta = 1 + \tan^2\theta, \quad \csc^2\theta = 1 + \cot^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta, \quad \cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}, \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$

$\sin\left(A - \frac{\pi}{2}\right) = -\cos A, \quad \cos\left(A - \frac{\pi}{2}\right) = \sin A$

$\sin\left(A + \frac{\pi}{2}\right) = \cos A, \quad \cos\left(A + \frac{\pi}{2}\right) = -\sin A$

$\sin A\sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$

$\cos A\cos B = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)$

$\sin A\cos B = \frac{1}{2}\sin(A-B) + \frac{1}{2}\sin(A+B)$

$\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$

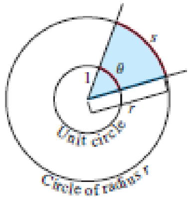
$\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$

$\cos A + \cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$

$\cos A - \cos B = -2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$

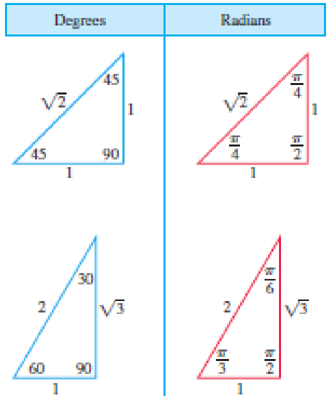
Trigonometric Functions

Radian Measure

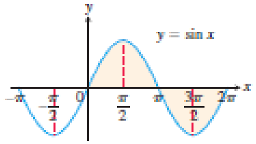


$\frac{s}{r} = \frac{\theta}{1} = \theta \quad \text{or} \quad \theta = \frac{s}{r},$

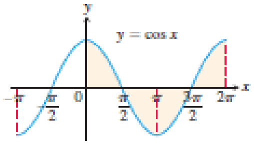
$180° = \pi$ radians.



The angles of two common triangles, in degrees and radians.



Domain: $(-\infty, \infty)$
Range: $[-1, 1]$



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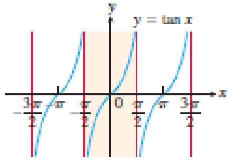
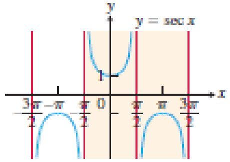


Domain: All real numbers except odd integer multiples of $\pi/2$
Range: $(-\infty, \infty)$



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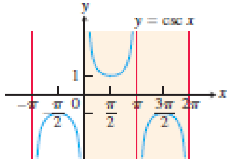
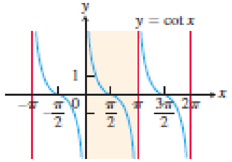


Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $(-\infty, -1] \cup [1, \infty)$



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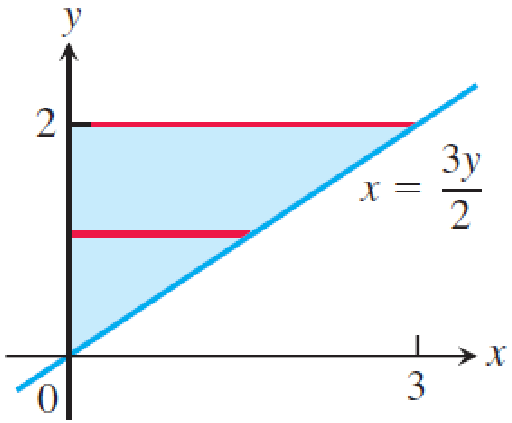
Volumes by the Disk Method

In Exercises 17–20, find the volume of the solid generated by revolving the shaded region about the given axis.

18. About the y-axis



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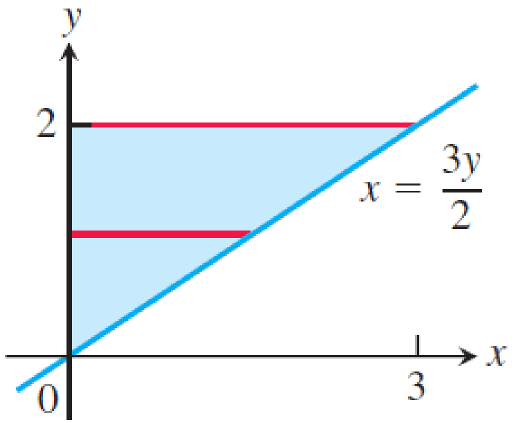

18. $R(y) = x = \frac{3y}{2} \Rightarrow$

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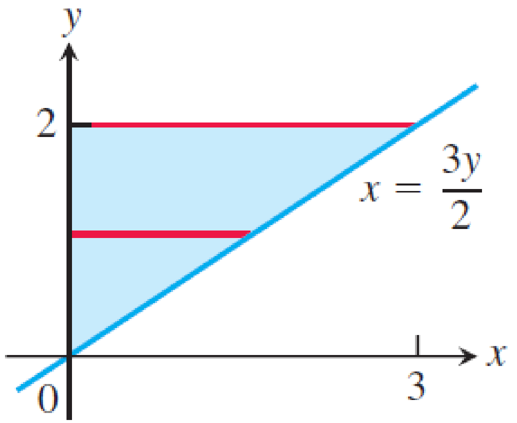

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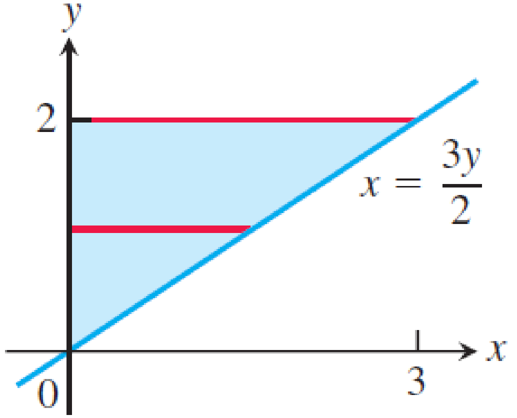
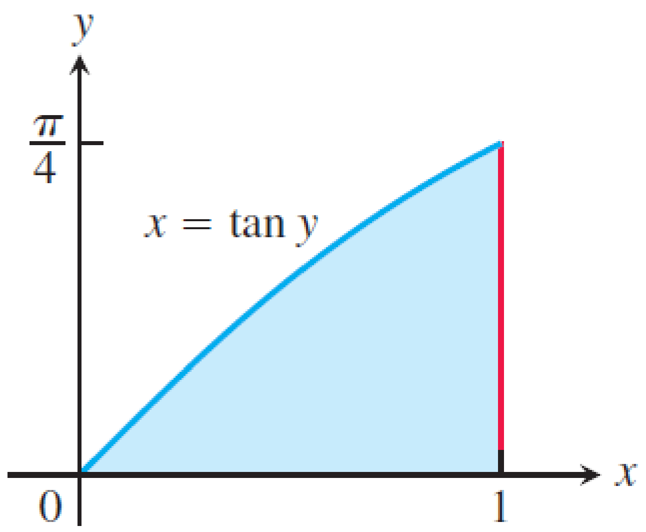


18. $R(y) = x = \frac{3y}{2} \Rightarrow V = \int_0^2 \pi [R(y)]^2 dy = \pi \int_0^2 \left(\frac{3y}{2}\right)^2 dy = \pi \int_0^2 \frac{9}{4} y^2 dy = \pi \left[\frac{3}{4} y^3\right]_0^2 = \pi \cdot \frac{3}{4} \cdot 8 = 6\pi$

Volumes by the Washer Method

Find the volumes of the solids generated by revolving the shaded regions in Exercises 39 and 40 about the indicated axes.

40. The y-axis



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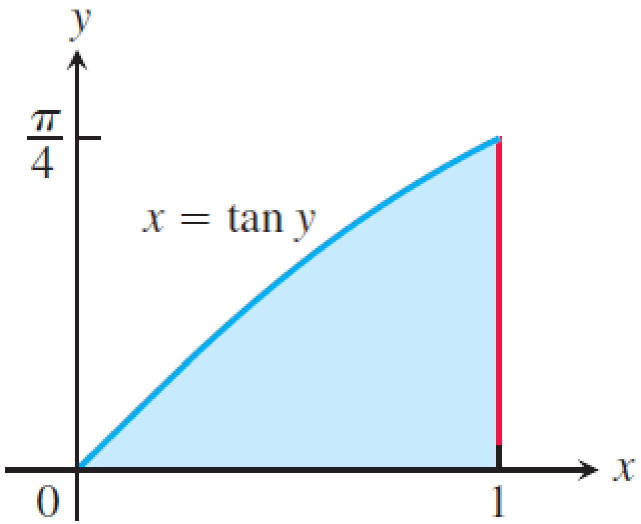

40. For the sketch given, $c = 0, d = \frac{\pi}{4}$; $R(y) = 1$, $r(y) = \tan y$; $V = \int_c^d \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

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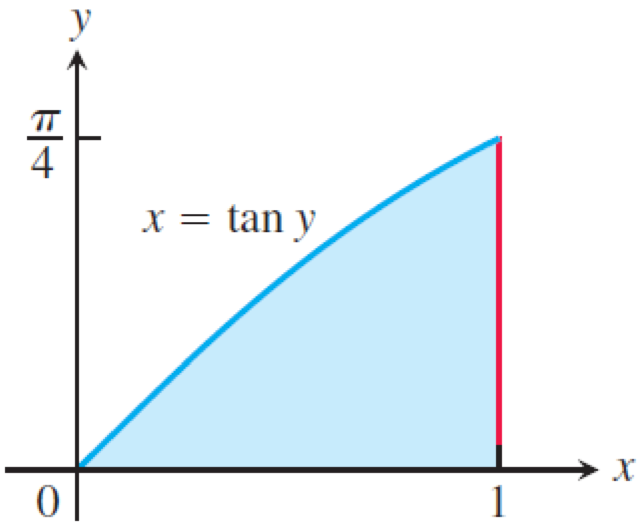

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$$= \pi\int_0^{\pi/4}\left(1-\tan^2 y\right) dy = \pi\int_0^{\pi/4}\left(2-\sec^2 y\right) dy = \pi\left[2y - \tan y\right]_0^{\pi/4} = \pi\left(\frac{\pi}{2}-1\right) = \frac{\pi^2}{2} - \pi$$

Summary of the Shell Method

Regardless of the position of the axis of revolution (horizontal or vertical), the steps for implementing the shell method are these.

1. *Draw the region and sketch a line segment* across it *parallel* to the axis of revolution. *Label* the segment's height or length (shell height) and distance from the axis of revolution (shell radius).
2. *Find the limits of integration* for the thickness variable.
3. *Integrate* the product 2π (shell radius) (shell height) with respect to the thickness variable (x or y) to find the volume.

EXAMPLE 2 The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution Sketch the region and draw a line segment across it *parallel* to the axis of revolution (Figure 6.20a). Label the segment's height (shell height) and distance from the axis of revolution (shell radius). (We drew the shell in Figure 6.20b, but you need not do that.)

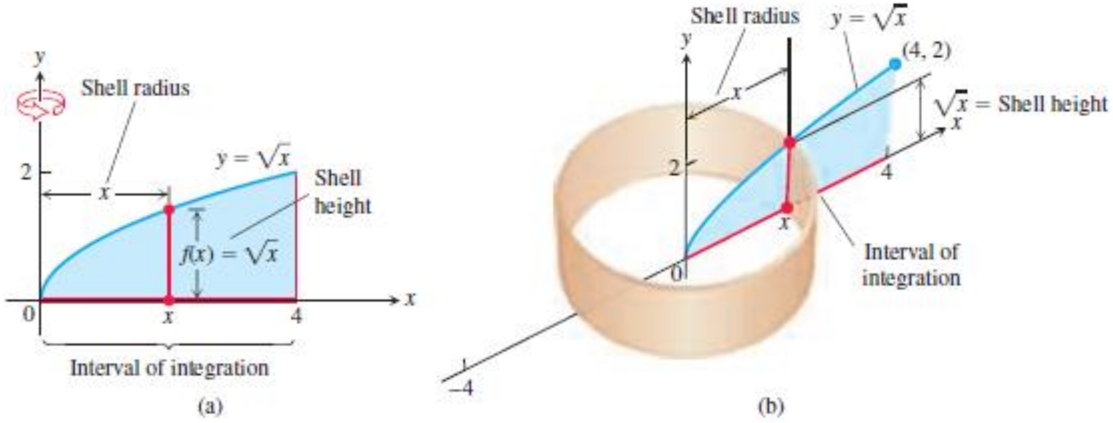


FIGURE 6.20 (a) The region, shell dimensions, and interval of integration in Example 2. (b) The shell swept out by the vertical segment in part (a) with a width Δx.

The shell thickness variable is x, so the limits of integration for the shell formula are $a = 0$ and $b = 4$ (Figure 6.20). The volume is

$$
\begin{aligned}
V &= \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx \\
&= \int_0^4 2\pi(x)\left(\sqrt{x}\right) dx \\
&= 2\pi \int_0^4 x^{3/2}\, dx = 2\pi\left[\frac{2}{5}x^{5/2}\right]_0^4 = \frac{128\pi}{5}.
\end{aligned}
$$

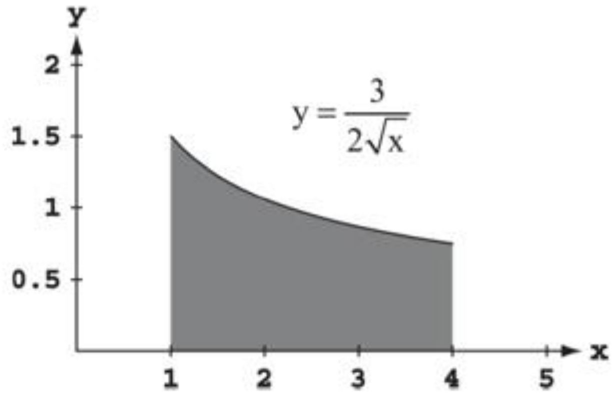
■

The shell method gives the same answer as the washer method when both are used to calculate the volume of a region. We do not prove that result here, but it is illustrated in Exercises 37 and 38. (Exercise 45 outlines a proof.) Both volume formulas are actually special cases of a general volume formula we will look at when studying double and triple integrals in Chapter 15. That general formula also allows for computing volumes of solids other than those swept out by regions of revolution.

Revolution About the y-Axis

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 7–12 about the y-axis.

12. $y = 3/(2\sqrt{x}),\quad y = 0,\quad x = 1,\quad x = 4$



12. $a = 1, b = 4;$

$$V = \int_a^b 2\pi\left(\begin{array}{c}\text{shell}\\ \text{radius}\end{array}\right)\left(\begin{array}{c}\text{shell}\\ \text{height}\end{array}\right)dx = \int_1^4 2\pi x\left(\tfrac{3}{2}x^{-1/2}\right)dx$$

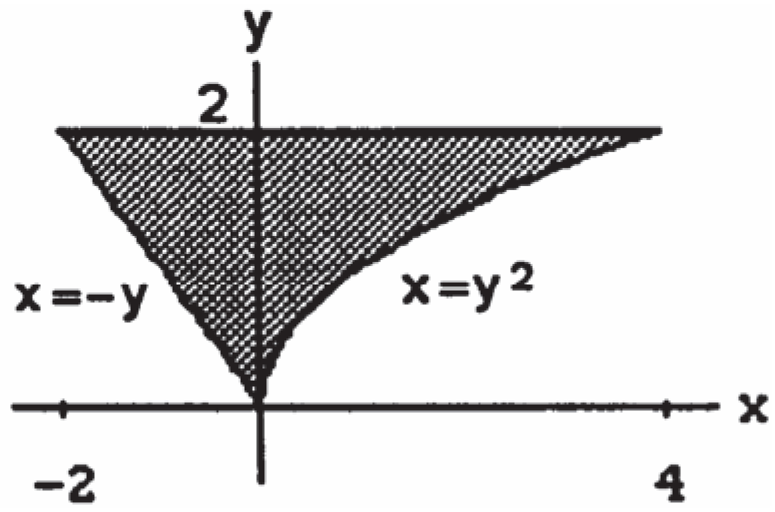
$$= 3\pi\int_1^4 x^{1/2}\,dx = 3\pi\left[\tfrac{2}{3}x^{3/2}\right]_1^4 = 2\pi\left(4^{3/2} - 1\right)$$

$$= 2\pi(8-1) = 14\pi$$

Revolution About the x-Axis

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 15–22 about the x-axis.

16. $x = y^2, \quad x = -y, \quad y = 2, \quad y \geq 0$



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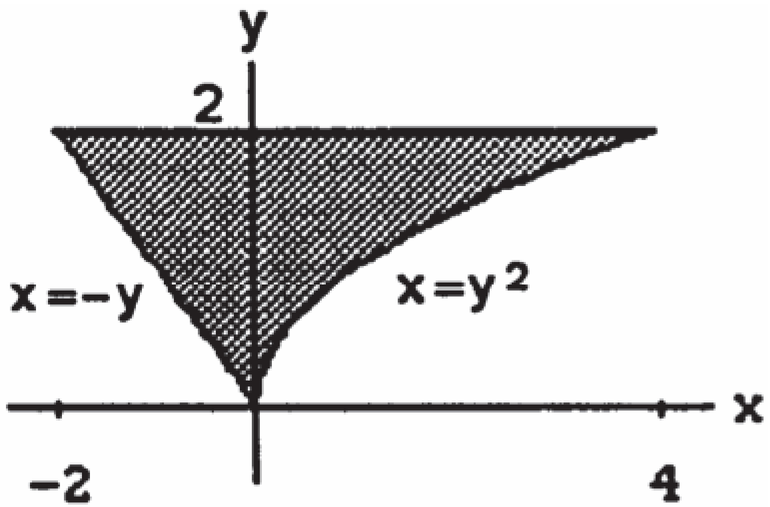

16. $c = 0, d = 2;$

$$V = \int_c^d 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dy = \int_0^2 2\pi y \left[y^2 - (-y) \right] dy$$

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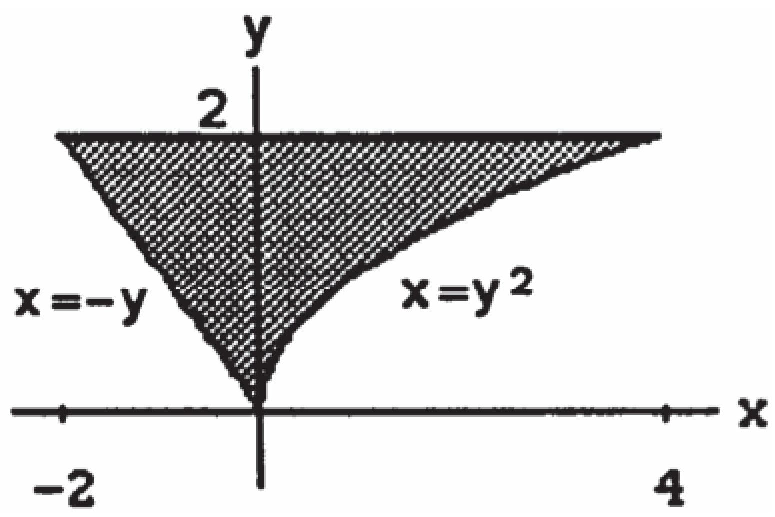

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$$= 2\pi \int_0^2 \left(y^3 + y^2\right) dy = 2\pi\left[\frac{y^4}{4} + \frac{y^3}{3}\right]_0^2 = 16\pi\left(\tfrac{2}{4} + \tfrac{1}{3}\right)$$

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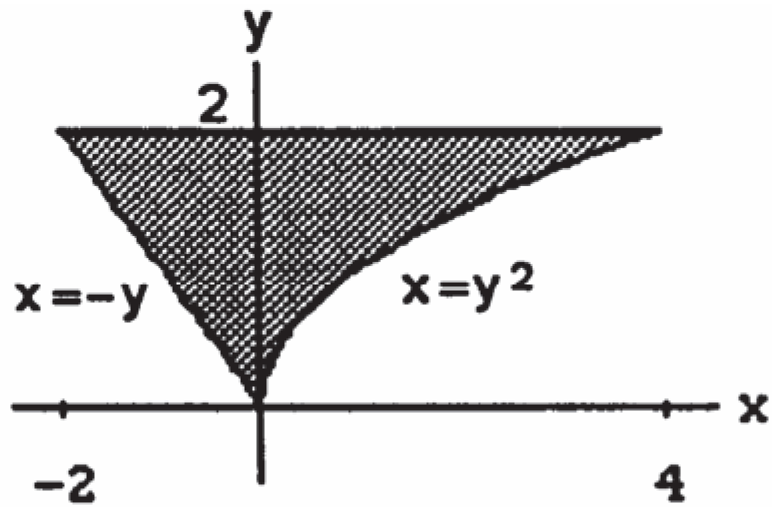

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$$= 16\pi\left(\tfrac{5}{6}\right) = \frac{40\pi}{3}$$

DEFINITION If f' is continuous on $[a, b]$, then the **length (arc length)** of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{3}$$

EXAMPLE 1 Find the length of the curve shown in Figure 6.24, which is the graph of the function

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \qquad 0 \le x \le 1.$$

Solution We use Equation (3) with $a = 0$, $b = 1$, and

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1$$

If $x = 1$, then $y \approx 0.89$

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}x^{1/2} = 2\sqrt{2}x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2}x^{1/2}\right)^2 = 8x.$$

The length of the curve over $x = 0$ to $x = 1$ is

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + 8x}\, dx$$

Eq. (3) with $a = 0, b = 1$.

$$= \frac{2}{3} \cdot \frac{1}{8}(1 + 8x)^{3/2}\Big]_0^1 = \frac{13}{6} \approx 2.17.$$

Let $u = 1 + 8x$, integrate, and replace u by $1 + 8x$.

Finding Lengths of Curves

Find the lengths of the curves in Exercises 1–16. If you have graphing software, you may want to graph these curves to see what they look like.

1. $y = (1/3)(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$

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3. $x = (y^3/3) + 1/(4y)$ from $y = 1$ to $y = 3$

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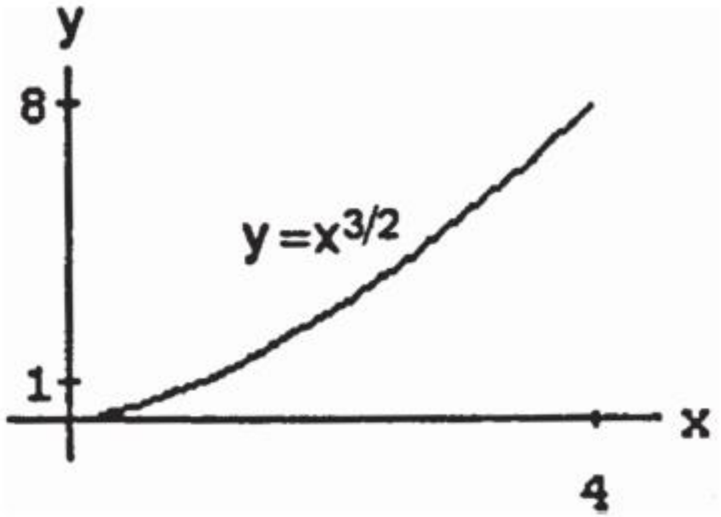

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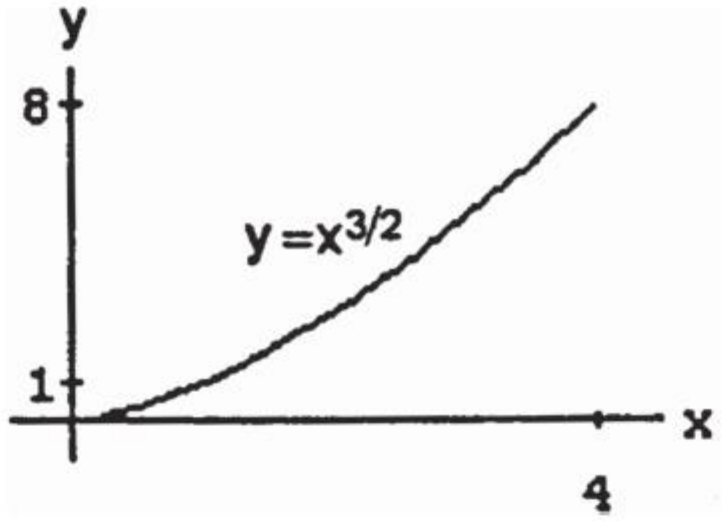
2. $\frac{dy}{dx} = \frac{3}{2}\sqrt{x} \Rightarrow L = \int_0^4 \sqrt{1+\frac{9}{4}x}\, dx;$

$\left[u = 1+\frac{9}{4}x \Rightarrow du = \frac{9}{4}dx \Rightarrow \frac{4}{9}du = dx;\right.$



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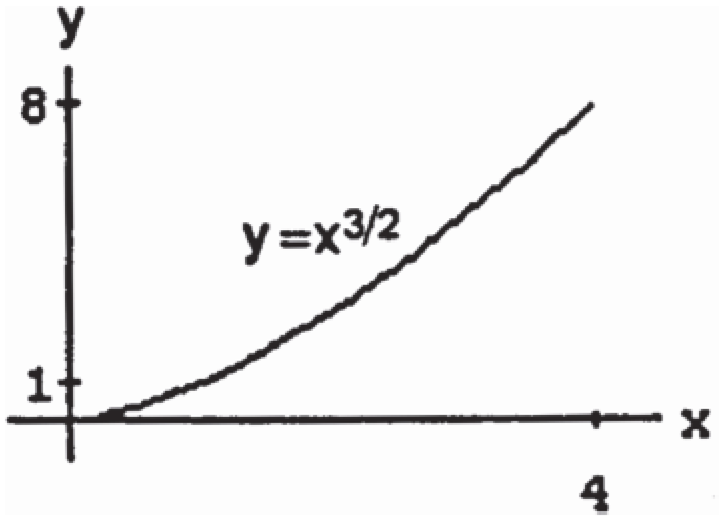
$\left[u = 1 + \frac{9}{4}x \Rightarrow du = \frac{9}{4}dx \Rightarrow \frac{4}{9}du = dx;\right.$

$x = 0 \Rightarrow u = 1;\ x = 4 \Rightarrow u = 10]$

$\rightarrow L = \int_1^{10} u^{1/2}\left(\frac{4}{9}du\right) = \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{10} = \frac{8}{27}\left(10\sqrt{10} - 1\right)$



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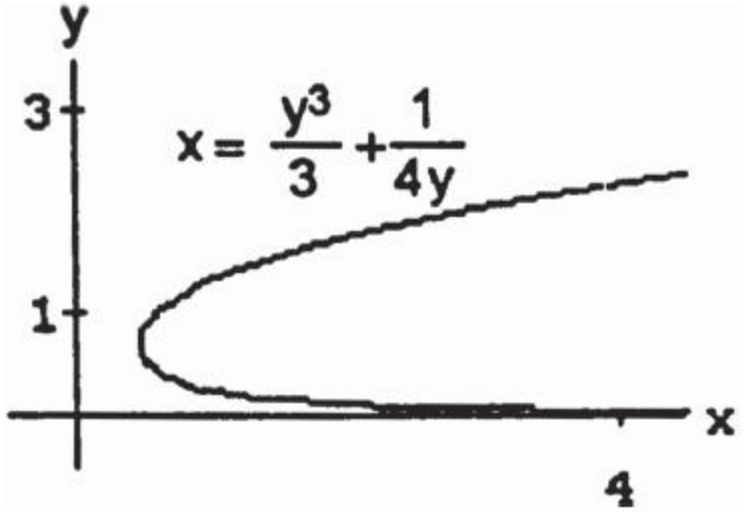


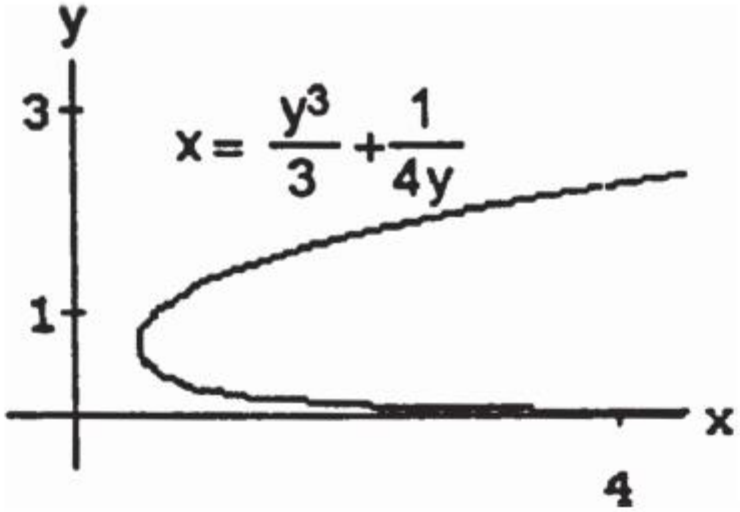
Finding Lengths of Curves

Find the lengths of the curves in Exercises 1–16. If you have graphing software, you may want to graph these curves to see what they look like.

1. $y = (1/3)(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$

2. $y = x^{3/2}$ from $x = 0$ to $x = 4$

3. $x = (y^3/3) + 1/(4y)$ from $y = 1$ to $y = 3$



3. $\frac{dx}{dy} = y^2 - \frac{1}{4y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = y^4 - \frac{1}{2} + \frac{1}{16y^4}$

$$\Rightarrow L = \int_1^3 \sqrt{1 + y^4 - \frac{1}{2} + \frac{1}{16y^4}}\, dy = \int_1^3 \sqrt{y^4 + \frac{1}{2} + \frac{1}{16y^4}}\, dy$$

$$= \int_1^3 \sqrt{\left(y^2 + \frac{1}{4y^2}\right)^2}\, dy = \int_1^3 \left(y^2 + \frac{1}{4y^2}\right) dy$$

$$= \left[\frac{y^3}{3} - \frac{y^{-1}}{4}\right]_1^3 = \left(\frac{27}{3} - \frac{1}{12}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) = 9 - \frac{1}{12} - \frac{1}{3} + \frac{1}{4}$$

$$= 9 + \frac{(-1-4+3)}{12} = 9 + \frac{(-2)}{12} = \frac{53}{6}$$

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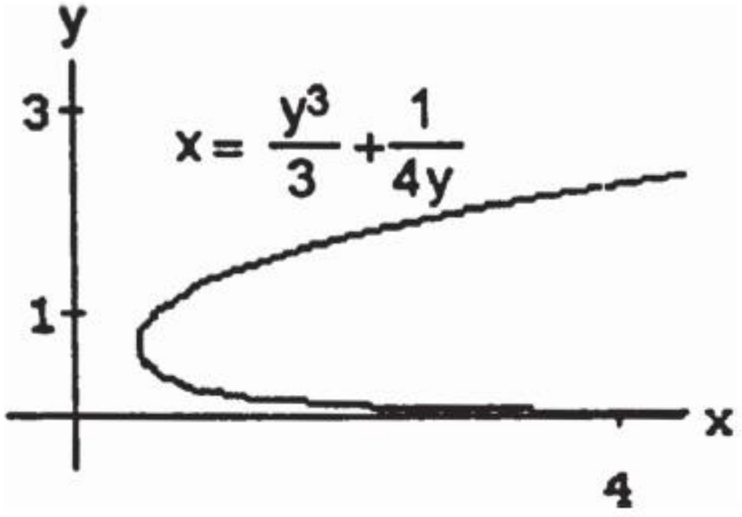

Finding Lengths of Curves

Find the lengths of the curves in Exercises 1–16. If you have graphing software, you may want to graph these curves to see what they look like.

10. $y = \frac{x^2}{2} - \frac{\ln x}{4}$ from $x = 1$ to $x = 3$

Finding Lengths of Curves

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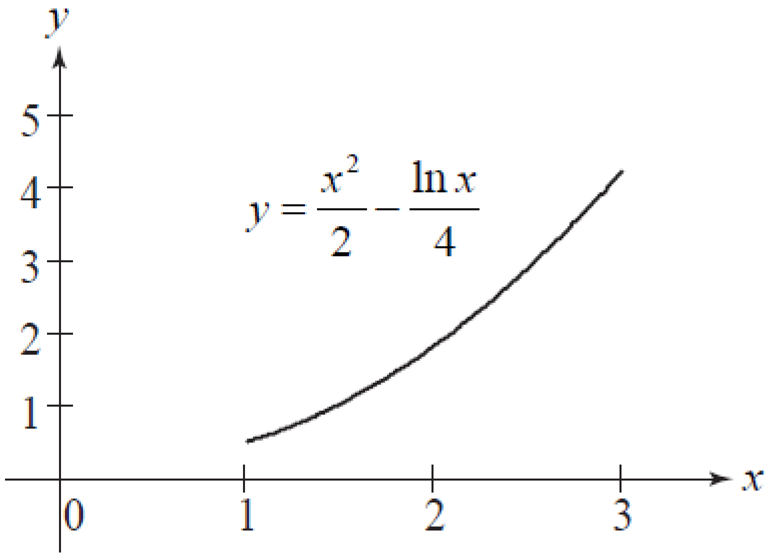
10. $y = \dfrac{x^2}{2} - \dfrac{\ln x}{4}$ from $x = 1$ to $x = 3$



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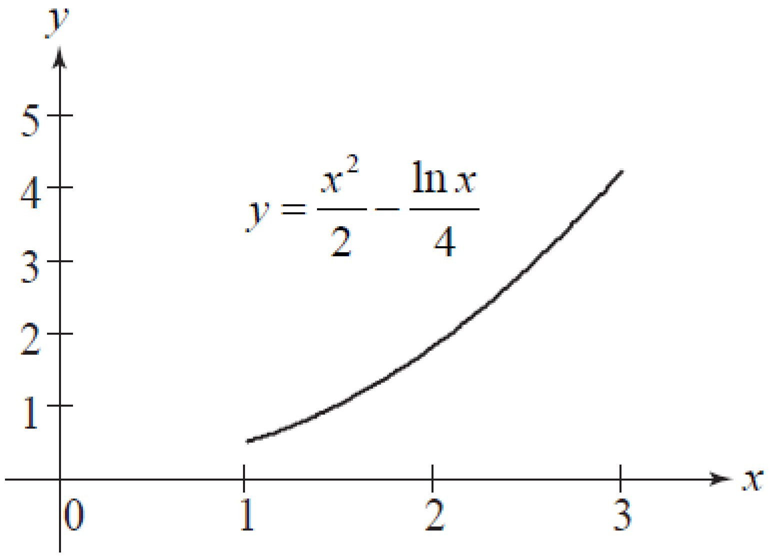

10. $\frac{dy}{dx} = x - \frac{1}{4x} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \left(x - \frac{1}{4x}\right)^2 = x^2 - \frac{1}{2} + \frac{1}{16x^2}$

$$\Rightarrow L = \int_1^3 \sqrt{1 + x^2 - \frac{1}{2} + \frac{1}{16x^2}}\, dx =$$

$$= \int_1^3 \sqrt{x^2 + \frac{1}{2} + \frac{1}{16x^2}}\, dx = \int_1^3 \sqrt{\left(x + \frac{1}{4x}\right)^2}\, dx =$$

Finding Lengths of Curves

Find the lengths of the curves in Exercises 1–16. If you have graphing software, you may want to graph these curves to see what they look like.

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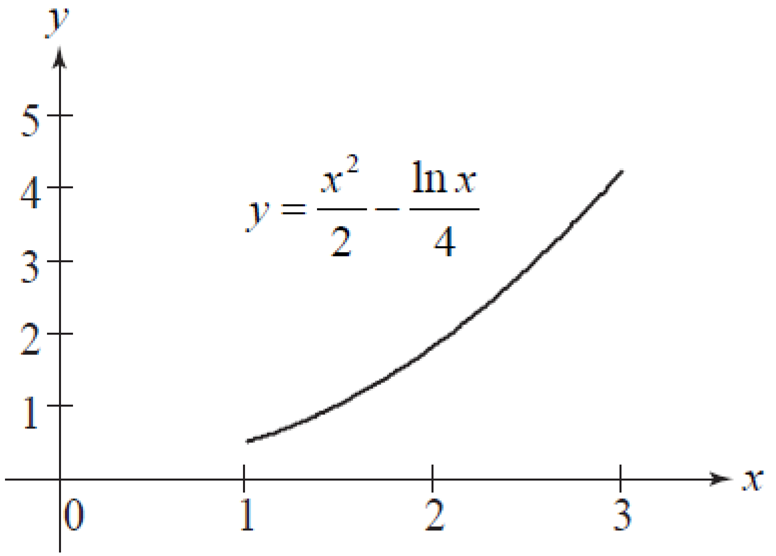

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$$= \int_1^3 \sqrt{x^2 + \frac{1}{2} + \frac{1}{16x^2}}\, dx = \int_1^3 \sqrt{\left(x + \frac{1}{4x}\right)^2}\, dx =$$

$$\int_1^3 \left(x + \frac{1}{4x}\right) dx = \left[\frac{x^2}{2} + \frac{1}{4}\ln|x|\right]_1^3 =$$

$$\left(\frac{9}{2} + \frac{1}{4}\ln 3\right) - \left(\frac{1}{2} + \frac{1}{4}\ln 1\right) = 4 + \frac{1}{4}\ln 3$$

6.4 Areas of Surfaces of Revolution

When you jump rope, the rope sweeps out a surface in the space around you similar to what is called a *surface of revolution*. The surface surrounds a volume of revolution, and many applications require that we know the area of the surface rather than the volume it encloses. In this section we define areas of surfaces of revolution. More general surfaces are treated in Chapter 16.

Defining Surface Area

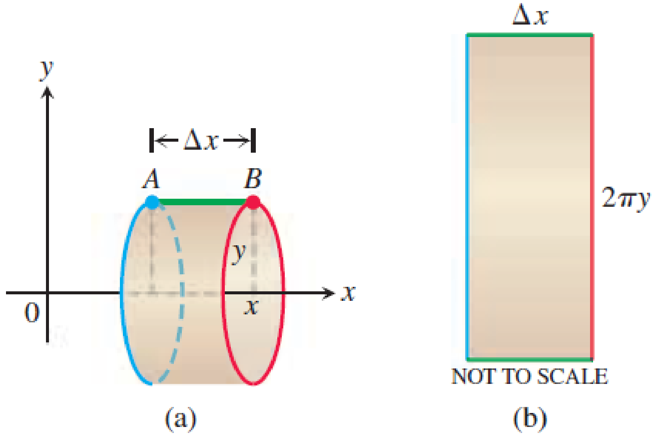


FIGURE 6.28 (a) A cylindrical surface generated by rotating the horizontal line segment AB of length Δx about the x-axis has area $2\pi y \Delta x$. (b) The cut and rolled-out cylindrical surface as a rectangle.

If you revolve a region in the plane that is bounded by the graph of a function over an interval, it sweeps out a solid of revolution, as we saw earlier in the chapter. However, if you revolve only the bounding curve itself, it does not sweep out any interior volume but rather a surface that surrounds the solid and forms part of its boundary. Just as we were interested in defining and finding the length of a curve in the last section, we are now interested in defining and finding the area of a surface generated by revolving a curve about an axis.

Before considering general curves, we begin by rotating horizontal and slanted line segments about the x-axis. If we rotate the horizontal line segment AB having length Δx about the x-axis (Figure 6.28a), we generate a cylinder with surface area $2\pi y \Delta x$. This area is the same as that of a rectangle with side lengths Δx and $2\pi y$ (Figure 6.28b). The length $2\pi y$ is the circumference of the circle of radius y generated by rotating the point (x, y) on the line AB about the x-axis.

Suppose the line segment AB has length L and is slanted rather than horizontal. Now when AB is rotated about the x-axis, it generates a frustum of a cone (Figure 6.29a). From

DEFINITION If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area of the surface** generated by revolving the graph of $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx. \qquad (3)$$

Note that the square root in Equation (3) is similar to the one that appears in the formula for the arc length differential of the generating curve in Equation (6) of Section 6.3.

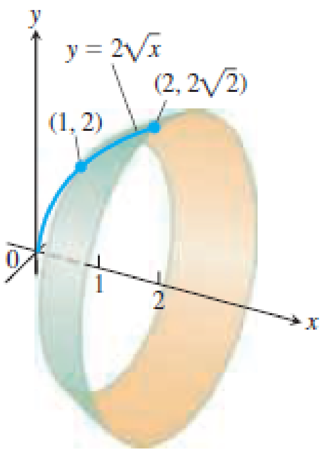


FIGURE 6.34 In Example 1 we calculate the area of this surface.

EXAMPLE 1 Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \le x \le 2$, about the x-axis (Figure 6.34).

Solution We evaluate the formula

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{Eq. (3)}$$

with

$$a = 1, \qquad b = 2, \qquad y = 2\sqrt{x}, \qquad \frac{dy}{dx} = \frac{1}{\sqrt{x}}.$$

First, we perform some algebraic manipulation on the radical in the integrand to transform it into an expression that is easier to integrate.

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}$$

$$= \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x + 1}{x}} = \frac{\sqrt{x + 1}}{\sqrt{x}}$$

With these substitutions, we have

$$S = \int_1^2 2\pi \cdot 2\sqrt{x}\frac{\sqrt{x + 1}}{\sqrt{x}}\, dx = 4\pi\int_1^2 \sqrt{x + 1}\, dx$$

$$= 4\pi \cdot \frac{2}{3}(x + 1)^{3/2}\Big]_1^2 = \frac{8\pi}{3}\left(3\sqrt{3} - 2\sqrt{2}\right).$$ ■

Revolution About the y-Axis

For revolution about the y-axis, we interchange x and y in Equation (3).

Surface Area for Revolution About the y-Axis

If $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the graph of $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\, dy. \qquad (4)$$

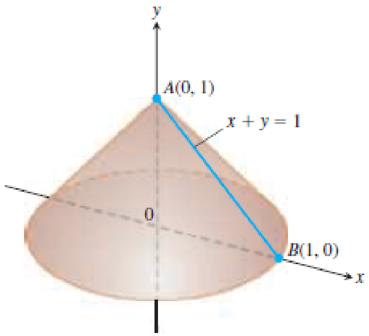


FIGURE 6.35 Revolving line segment AB about the y-axis generates a cone whose lateral surface area we can now calculate in two different ways (Example 2).

EXAMPLE 2 The line segment $x = 1 - y$, $0 \leq y \leq 1$, is revolved about the y-axis to generate the cone in Figure 6.35. Find its lateral surface area (which excludes the base area).

Solution Here we have a calculation we can check with a formula from geometry:

$$\text{Lateral surface area} = \frac{\text{base circumference}}{2} \times \text{slant height} = \pi\sqrt{2}.$$

To see how Equation (4) gives the same result, we take

$$c = 0, \quad d = 1, \quad x = 1 - y, \quad \frac{dx}{dy} = -1,$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + (-1)^2} = \sqrt{2}$$

and calculate

$$\begin{aligned} S &= \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 2\pi(1 - y)\sqrt{2}\, dy \\ &= 2\pi\sqrt{2}\left[y - \frac{y^2}{2}\right]_0^1 = 2\pi\sqrt{2}\left(1 - \frac{1}{2}\right) \\ &= \pi\sqrt{2}. \end{aligned}$$

The results agree, as they should. ■

Find the areas of the surfaces generated by revolving the curves in Exercises 13–23 about the indicated axes. If you have a grapher, you may want to graph these curves to see what they look like.

13. $y = x^3/9, \quad 0 \le x \le 2; \quad x\text{-axis}$

14. $y = \sqrt{x}, \quad 3/4 \le x \le 15/4; \quad x\text{-axis}$

15. $y = \sqrt{2x - x^2}, \quad 0.5 \le x \le 1.5; \quad x\text{-axis}$

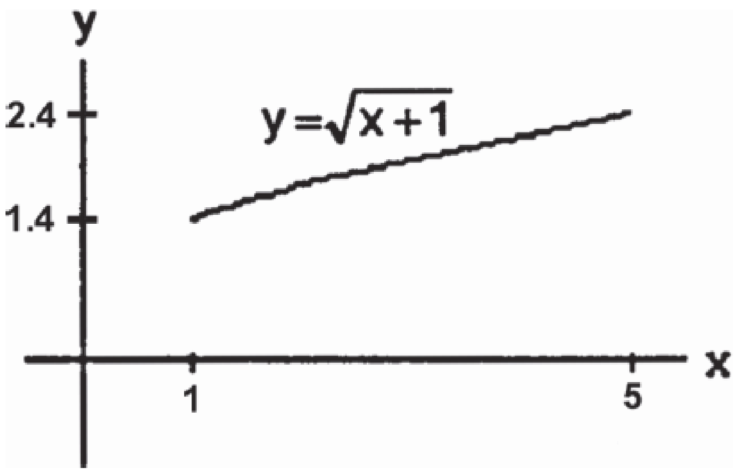
16. $y = \sqrt{x + 1}, \quad 1 \le x \le 5; \quad x\text{-axis}$

17. $x = y^3/3, \quad 0 \le y \le 1; \quad y\text{-axis}$

18. $x = (1/3)y^{3/2} - y^{1/2}, \quad 1 \le y \le 3; \quad y\text{-axis}$

19. $x = 2\sqrt{4 - y}, \quad 0 \le y \le 15/4; \quad y\text{-axis}$

16:



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17. $x = y^3/3, \quad 0 \le y \le 1; \quad y$-axis

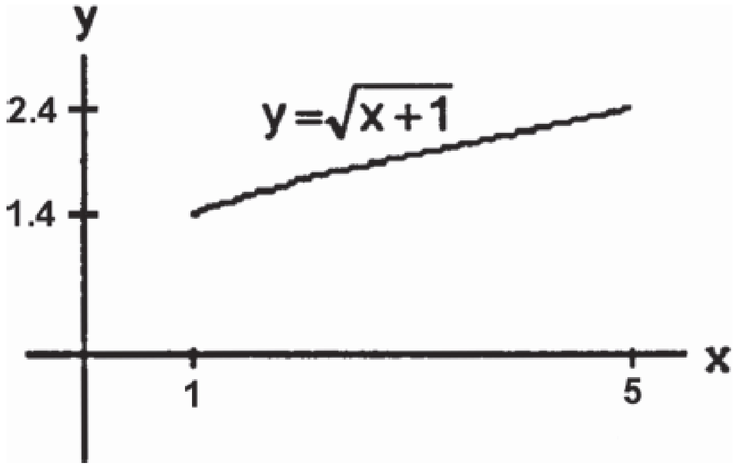
18. $x = (1/3)y^{3/2} - y^{1/2}, \quad 1 \le y \le 3; \quad y$-axis

19. $x = 2\sqrt{4 - y}, \quad 0 \le y \le 15/4; \quad y$-axis

16. $$\frac{dy}{dx} = \frac{1}{2\sqrt{x+1}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4(x+1)}$$

$$\Rightarrow S = \int_1^5 2\pi\sqrt{x+1}\sqrt{1+\frac{1}{4(x+1)}}\, dx = 2\pi\int_1^5 \sqrt{(x+1)+\frac{1}{4}}\, dx$$

$$= 2\pi\int_1^5 \sqrt{x+\frac{5}{4}}\, \mathrm{dx} = 2\pi\left[\frac{2}{3}\left(x+\frac{5}{4}\right)^{3/2}\right]_1^5$$



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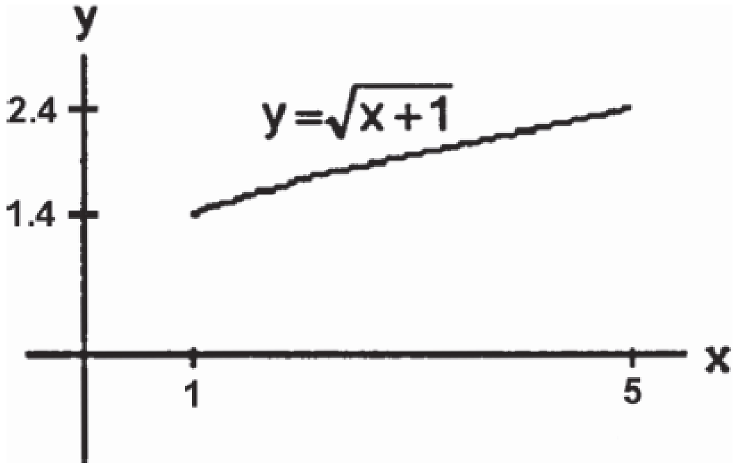
$$\Rightarrow S = \int_1^5 2\pi\sqrt{x+1}\sqrt{1+\frac{1}{4(x+1)}}\,dx = 2\pi\int_1^5 \sqrt{(x+1)+\frac{1}{4}}\,dx$$

$$= 2\pi\int_1^5 \sqrt{x+\frac{5}{4}}\,\text{dx} = 2\pi\left[\frac{2}{3}\left(x+\frac{5}{4}\right)^{3/2}\right]_1^5$$

$$= \frac{4\pi}{3}\left[\left(5+\frac{5}{4}\right)^{3/2} - \left(1+\frac{5}{4}\right)^{3/2}\right] = \frac{4\pi}{3}\left[\left(\frac{25}{4}\right)^{3/2} - \left(\frac{9}{4}\right)^{3/2}\right]$$

$$= \frac{4\pi}{3}\left(\frac{5^3}{2^3} - \frac{3^3}{2^3}\right) = \frac{\pi}{6}(125-27) = \frac{98\pi}{6} = \frac{49\pi}{3}$$



Find the areas of the surfaces generated by revolving the curves in Exercises 13–23 about the indicated axes. If you have a grapher, you may want to graph these curves to see what they look like.

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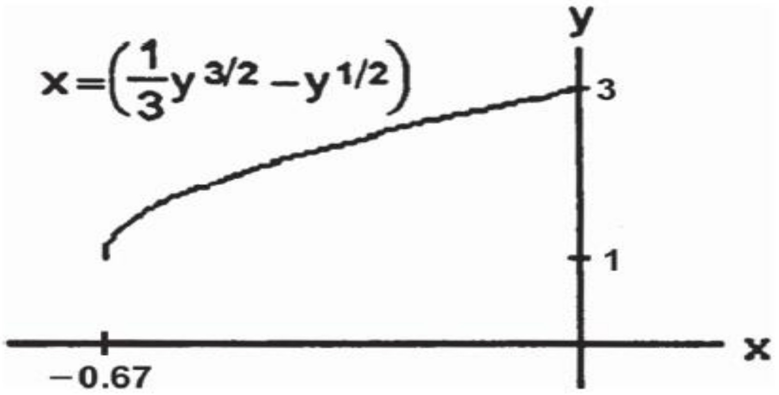

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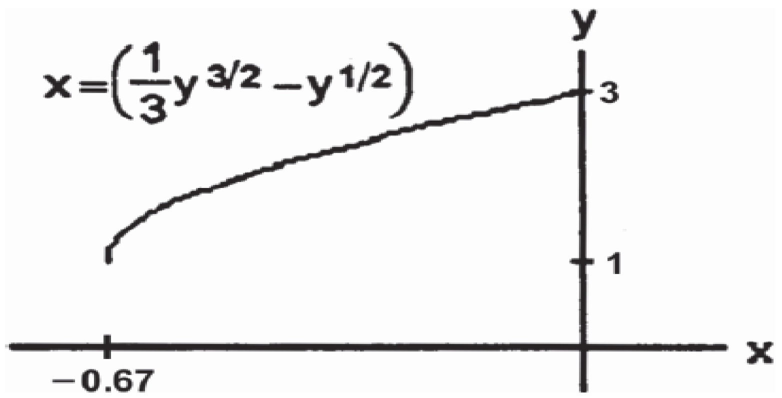
18. $x = \left(\frac{1}{3}y^{3/2} - y^{1/2}\right) \le 0$, when $1 \le y \le 3$. To get positive area, we take $x = -\left(\frac{1}{3}y^{3/2} - y^{1/2}\right)$

$$\Rightarrow \frac{dx}{dy} = -\frac{1}{2}\left(y^{1/2} - y^{-1/2}\right) \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{4}\left(y - 2 + y^{-1}\right)$$

$$\Rightarrow S = -\int_1^3 2\pi\left(\frac{1}{3}y^{3/2} - y^{1/2}\right)\sqrt{1 + \frac{1}{4}\left(y - 2 + y^{-1}\right)}\, dy$$

$$= -2\pi\int_1^3 \left(\frac{1}{3}y^{3/2} - y^{1/2}\right)\sqrt{\frac{1}{4}\left(y + 2 + y^{-1}\right)}\, dy$$



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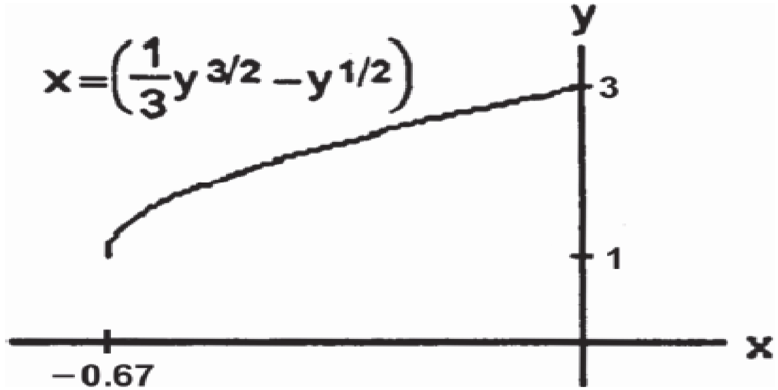
$$\Rightarrow S = -\int_1^3 2\pi\left(\frac{1}{3}y^{3/2} - y^{1/2}\right)\sqrt{1 + \frac{1}{4}\left(y - 2 + y^{-1}\right)}\, dy$$

$$= -2\pi\int_1^3\left(\frac{1}{3}y^{3/2} - y^{1/2}\right)\sqrt{\frac{1}{4}\left(y + 2 + y^{-1}\right)}\, dy$$

$$= -2\pi\int_1^3\left(\frac{1}{3}y^{3/2} - y^{1/2}\right)\frac{\sqrt{\left(y^{1/2} + y^{-1/2}\right)^2}}{2}\, dy = -\pi\int_1^3 y^{1/2}\left(\frac{1}{3}y - 1\right)\left(y^{1/2} + \frac{1}{y^{1/2}}\right) dy = -\pi\int_1^3\left(\frac{1}{3}y - 1\right)(y+1)\, dy$$

$$= -\pi\int_1^3\left(\frac{1}{3}y^2 - \frac{2}{3}y - 1\right) dy = -\pi\left[\frac{y^3}{9} - \frac{y^3}{3} - y\right]_1^3 = -\pi\left[\left(\frac{27}{9} - \frac{9}{3} - 3\right) - \left(\frac{1}{9} - \frac{1}{3} - 1\right)\right] = -\pi\left(-3 - \frac{1}{9} + \frac{1}{3} + 1\right)$$

$$= -\frac{\pi}{9}(-18 - 1 + 3) = \frac{16\pi}{9}$$



```
In[13]:= m
```

Out[13]= $$\frac{1}{\sqrt{y}} + \sqrt{y}$$

```
In[14]:= m^2
```

Out[14]= $$\left(\frac{1}{\sqrt{y}} + \sqrt{y}\right)^2$$

```
In[15]:= Simplify[m^2]
```

Out[15]= $$2 + \frac{1}{y} + y$$

6.5 Work and Fluid Forces

In everyday life, *work* means an activity that requires muscular or mental effort. In science, the term refers specifically to a force acting on an object and the object's subsequent displacement. This section shows how to calculate work. The applications run from compressing railroad car springs and emptying subterranean tanks to forcing subatomic particles to collide and lifting satellites into orbit.

Work Done by a Constant Force

When an object moves a distance d along a straight line as a result of being acted on by a force of constant magnitude F in the direction of motion, we define the **work** W done by the force on the object with the formula

$$W = Fd \qquad \text{(Constant-force formula for work).} \qquad (1)$$

From Equation (1) we see that the unit of work in any system is the unit of force multiplied by the unit of distance. In SI units (SI stands for *Système International*, or International System), the unit of force is a newton, the unit of distance is a meter, and the unit of work is a newton-meter ($N \cdot m$). This combination appears so often it has a special name, the **joule**. Taking gravitational acceleration at sea level to be 9.8 m/sec^2, to lift one kilogram one meter requires work of 9.8 joules. This is seen by multiplying the force of 9.8 newtons exerted on one kilogram by the one-meter distance moved. In the British system, the unit of work is the foot-pound, a unit sometimes used in applications. It requires one foot-pound of work to lift a one pound weight a distance of one foot.

Joules
The joule, abbreviated J, is named after the English physicist James Prescott Joule (1818–1889). The defining equation is

$$1 \text{ joule} = (1 \text{ newton})(1 \text{ meter}).$$

In svmbols. $1\ J = 1\ N \cdot m$.

EXAMPLE 1 Suppose you jack up the side of a 2000-lb car 1.25 ft to change a tire. The jack applies a constant vertical force of about 1000 lb in lifting the side of the car (but because of the mechanical advantage of the jack, the force you apply to the jack itself is only about 30 lb). The total work performed by the jack on the car is $1000 \times 1.25 = 1250$ ft-lb. In SI units, the jack has applied a force of 4448 N through a distance of 0.381 m to do $4448 \times 0.381 \approx 1695$ J of work. ■

> **DEFINITION** The **work** done by a variable force $F(x)$ in moving an object along the x-axis from $x = a$ to $x = b$ is
>
> $$W = \int_a^b F(x)\,dx. \tag{2}$$

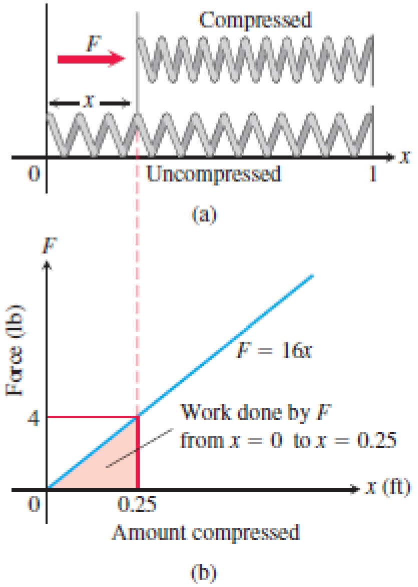


FIGURE 6.36 The force F needed to hold a spring under compression increases linearly as the spring is compressed (Example 2).

Hooke's Law for Springs: $F = kx$

One calculation for work arises in finding the work required to stretch or compress a spring. **Hooke's Law** says that the force required to hold a stretched or compressed spring x units from its natural (unstressed) length is proportional to x. In symbols,

$$F = kx. \tag{3}$$

The constant k, measured in force units per unit length, is a characteristic of the spring, called the **force constant** (or **spring constant**) of the spring. Hooke's Law, Equation (3), gives good results as long as the force doesn't distort the metal in the spring. We assume that the forces in this section are too small to do that.

EXAMPLE 2 Find the work required to compress a spring from its natural length of 1 ft to a length of 0.75 ft if the force constant is $k = 16$ lb/ft.

Solution We picture the uncompressed spring laid out along the x-axis with its movable end at the origin and its fixed end at $x = 1$ ft (Figure 6.36). This enables us to describe the force required to compress the spring from 0 to x with the formula $F = 16x$. To compress the spring from 0 to 0.25 ft, the force must increase from

$$F(0) = 16 \cdot 0 = 0 \text{ lb} \quad \text{to} \quad F(0.25) = 16 \cdot 0.25 = 4 \text{ lb}.$$

The work done by F over this interval is

$$W = \int_0^{0.25} 16x\,dx = 8x^2\Big]_0^{0.25} = 0.5 \text{ ft-lb}.$$

Eq. (2) with $a = 0, b = 0.25,$ $F(x) = 16x$ ■

6. **Stretching a spring** If a force of 90 N stretches a spring 1 m beyond its natural length, how much work does it take to stretch the spring 5 m beyond its natural length?

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6. We find the force constant from Hooke's law: $F = kx \Rightarrow k = \frac{F}{x} \Rightarrow k = \frac{90}{1} \Rightarrow k = 90\,\frac{\text{N}}{\text{m}}$. The work done to stretch the spring 5 m beyond its natural length is $W = \int_0^5 kx\,dx = 90\int_0^5 x\,dx = 90\left[\frac{x^2}{2}\right]_0^5 = (90)\left(\frac{25}{2}\right) = 1125\text{ J}$

8. Bathroom scale A bathroom scale is compressed $1/16$ in. when a 150-lb person stands on it. Assuming that the scale behaves like a spring that obeys Hooke's Law, how much does someone who compresses the scale $1/8$ in. weigh? How much work is done compressing the scale $1/8$ in.?

8. Bathroom scale A bathroom scale is compressed 1/16 in. when a 150-lb person stands on it. Assuming that the scale behaves like a spring that obeys Hooke's Law, how much does someone who compresses the scale 1/8 in. weigh? How much work is done compressing the scale 1/8 in.?

First, we find the force constant from Hooke's law: $F = kx \Rightarrow k = \frac{F}{x} = \frac{150}{\left(\frac{1}{16}\right)} = 16 \cdot 150 = 2{,}400\ \frac{\text{lb}}{\text{in}}$. If someone

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7.3 Hyperbolic Functions

The hyperbolic functions are formed by taking combinations of the two exponential functions e^x and e^{-x}. The hyperbolic functions simplify many mathematical expressions and occur frequently in mathematical and engineering applications.

Definitions and Identities

The hyperbolic sine and hyperbolic cosine functions are defined by the equations

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

We pronounce $\sinh x$ as "cinch x," rhyming with "pinch x," and $\cosh x$ as "kosh x," rhyming with "gosh x." From this basic pair, we define the hyperbolic tangent, cotangent, secant, and cosecant functions. The defining equations and graphs of these functions are shown in Table 7.4. We will see that the hyperbolic functions bear many similarities to the trigonometric functions after which they are named.

Hyperbolic functions satisfy the identities in Table 7.5. Except for differences in sign, these resemble identities we know for the trigonometric functions. The identities are proved directly from the definitions, as we show here for the second one:

$$\begin{aligned} 2\sinh x \cosh x &= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) \\ &= \frac{e^{2x} - e^{-2x}}{2} && \text{Simplify.} \\ &= \sinh 2x. && \text{Definition of sinh} \end{aligned}$$

The other identities are obtained similarly, by substituting in the definitions of the hyperbolic functions and using algebra.

For any real number u, we know the point with coordinates $(\cos u, \sin u)$ lies on the unit circle $x^2 + y^2 = 1$. So the trigonometric functions are sometimes called the *circular* functions. Because of the first identity

$$\cosh^2 u - \sinh^2 u = 1,$$

with u substituted for x in Table 7.5, the point having coordinates $(\cosh u, \sinh u)$ lies on the right-hand branch of the hyperbola $x^2 - y^2 = 1$. This is where the *hyperbolic* functions get their names (see Exercise 86).

TABLE 7.5 Identities for hyperbolic functions

$\cosh^2 x - \sinh^2 x = 1$
$\sinh 2x = 2 \sinh x \cosh x$
$\cosh 2x = \cosh^2 x + \sinh^2 x$
$\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$
$\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$
$\tanh^2 x = 1 - \operatorname{sech}^2 x$
$\coth^2 x = 1 + \operatorname{csch}^2 x$

TABLE 7.6 Derivatives of hyperbolic functions

$\dfrac{d}{dx}(\sinh u) = \cosh u \dfrac{du}{dx}$
$\dfrac{d}{dx}(\cosh u) = \sinh u \dfrac{du}{dx}$
$\dfrac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \dfrac{du}{dx}$
$\dfrac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \dfrac{du}{dx}$
$\dfrac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \dfrac{du}{dx}$
$\dfrac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \dfrac{du}{dx}$

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Hyperbolic functions are useful in finding integrals, which we will see in Chapter 8. They play an important role in science and engineering as well. The hyperbolic cosine describes the shape of a hanging cable or wire that is strung between two points at the same height and hanging freely (see Exercise 83). The shape of the St. Louis Arch is an inverted hyperbolic cosine. The hyperbolic tangent occurs in the formula for the velocity of an ocean wave moving over water having a constant depth, and the inverse hyperbolic tangent describes how relative velocities sum according to Einstein's Law in the Special Theory of Relativity.

Derivatives and Integrals of Hyperbolic Functions

The six hyperbolic functions, being rational combinations of the differentiable functions e^x and e^{-x}, have derivatives at every point at which they are defined (Table 7.6). Again, there are similarities with trigonometric functions.

The derivative formulas are derived from the derivative of e^u:

$$\frac{d}{dx}(\sinh u) = \frac{d}{dx}\left(\frac{e^u - e^{-u}}{2}\right) \qquad \text{Definition of } \sinh u$$

$$= \frac{e^u\, du/dx + e^{-u}\, du/dx}{2} \qquad \text{Derivative of } e^u$$

$$= \cosh u \frac{du}{dx}. \qquad \text{Definition of } \cosh u$$

TABLE 7.7 Integral formulas for hyperbolic functions

$$\int \sinh u\, du = \cosh u + C$$

$$\int \cosh u\, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u\, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u\, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

This gives the first derivative formula. From the definition, we can calculate the derivative of the hyperbolic cosecant function, as follows:

$$\begin{aligned}
\frac{d}{dx}(\operatorname{csch} u) &= \frac{d}{dx}\left(\frac{1}{\sinh u}\right) && \text{Definition of csch } u \\
&= -\frac{\cosh u}{\sinh^2 u}\frac{du}{dx} && \text{Quotient Rule for derivatives} \\
&= -\frac{1}{\sinh u}\frac{\cosh u}{\sinh u}\frac{du}{dx} && \text{Rearrange terms.} \\
&= -\operatorname{csch} u \coth u \frac{du}{dx} && \text{Definitions of csch } u \text{ and coth } u
\end{aligned}$$

The other formulas in Table 7.6 are obtained similarly.

The derivative formulas lead to the integral formulas in Table 7.7.

EXAMPLE 1 We illustrate the derivative and integral formulas.

(a) $$\frac{d}{dt}\left(\tanh\sqrt{1+t^2}\right) = \operatorname{sech}^2\sqrt{1+t^2}\cdot\frac{d}{dt}\left(\sqrt{1+t^2}\right)$$

$$= \frac{t}{\sqrt{1+t^2}}\operatorname{sech}^2\sqrt{1+t^2}$$

TABLE 7.10 Integrals leading to inverse hyperbolic functions

1. $$\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C, \qquad a > 0$$

2. $$\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C, \qquad u > a > 0$$

3. $$\int \frac{du}{a^2 - u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C, & u^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C, & u^2 > a^2 \end{cases}$$

4. $$\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C, \qquad 0 < u < a$$

5. $$\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C, \qquad u \neq 0 \text{ and } a > 0$$

EXAMPLE 3 Evaluate

$$\int_0^1 \frac{2\,dx}{\sqrt{3 + 4x^2}}.$$

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$$\int_0^1 \frac{2\,dx}{\sqrt{3 + 4x^2}}.$$

Solution The indefinite integral is

$$\int \frac{2\,dx}{\sqrt{3 + 4x^2}} = \int \frac{du}{\sqrt{a^2 + u^2}} \qquad u = 2x, \quad du = 2\,dx, \quad a = \sqrt{3}$$

$$= \sinh^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Formula from Table 7.10}$$

$$= \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C.$$

EXAMPLE 3 Evaluate

$$\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}}.$$

Solution The indefinite integral is

$$\begin{aligned}\int \frac{2\,dx}{\sqrt{3+4x^2}} &= \int \frac{du}{\sqrt{a^2+u^2}} && u = 2x,\quad du = 2\,dx,\quad a = \sqrt{3}\\ &= \sinh^{-1}\left(\frac{u}{a}\right) + C && \text{Formula from Table 7.10}\\ &= \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}} = \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\Bigg]_0^1 &= \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - \sinh^{-1}(0)\\ &= \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - 0 \approx 0.98665.\end{aligned}$$

■

- Questions?

We’re here to help.
Remember the tutoring center is open!
Study hard, best of luck!