Complex Analysis

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These problems are usually from the course textbook [1], but sometimes directly from Prof. Blackmore. This exercise is from prelim exam May 2017, problem 6:

(a) Use Rouché's theorem to prove that Φ has a unique fixed point as long as $|z| \leq 1$ and $|\mu| < \sqrt{2}$ and $z(\mu)$ is an analytic function of the parameter.

(b) Use a variant of the zero-pole theorem for meromorphic functions ($\int_C (f'/f)dz = 2\pi i(\#\text{zeros of } f - \#\text{poles of } f)$) to find the power series representation of the solution $z = z(\mu)$ in (a).

1 Proposed partial solution:

Note to future readers, including my esteemed friend Austin - I communicated with NJIT faculty about this problem and was told this problem has a typo that should read - $|\mu| \leq \frac{1}{\sqrt{2}}$. The problem is much more solvable after this correction, in fact it is downright simple. The rest of this exposition has not been reconfigured in light of this information.

For part (a), if we define dominating function $D(z) = 5z - \mu^2$ and $h(z) = -\mu^2 4z^4$, then $\Phi(z, \mu) = D(z) + h(z)$ and, if the dominating function $D(z)$ dominates (is greater than in modulus) the function $h(z)$ over the prescribed region of $|z| \leq 1$, then $\Phi(z, \mu)$ has the same amount of zeroes as does the function $D(z)$ in the region $|z| \leq 1$, which is exactly one.

We take absolute values $|D(z)| = |5z| + |-\mu^2|$ which for $|\mu| \leq \sqrt{2}$ is greater than $|-\mu^2 4z^4| = |h(z)|$ over the domain of $|z| \leq 1$.

Thus, this one zero is exactly equal to $h(z, \mu) = -\mu^2 4z^4$ and thus we have that $h(z, \mu)$ has a unique fixed point in the region $|z| \leq 1$.

As I understand currently, an analytic function has a power series representation on its domain of analyticity, but I am unsure how to prove that $z(\mu)$ is analytic nor how to use the zero pole theorem to find the power series representation.

I appreciate any advice, direction of clarification you may be able to provide.
Best wishes, -José

2 Problems 1 and 2 - evaluate integrals

These are from MAT656 homework from Ablowitz and Fokas, section 2.6.

See original text for full conditions on the problem, including the Cauchy Integral Formula, theorem 2.6.2 from the textbook (equation 2.6.5).

Theorem 1. *Let f be analytic interior to and on a closed contour C and let $C \subset \mathbf{C}$. Then*

$$\int_C \frac{f(s)ds}{(s-z)^{n+1}} = \frac{2\pi i}{n!} f^n(z).$$

2.1 1c

Solution:
$1c \rightarrow f(z) = \frac{1}{(2z-1)^3} = \frac{1}{8}\frac{1}{(z-\frac{1}{2})^3}$. We have $f(s) = 1$ analytic in and on the unit circle, thus

$$\int_C \frac{f(s)ds}{(s-z)^3} = k\pi i f^{''}(s = (0.5)) = k\pi i(0) = 0$$

given $f^{''}(s = (0.5) = 0$, k a constant ($k = \frac{16}{2!} = 8$).
□

2.2 1d

Solution:
$1d \rightarrow f(z) = \frac{e^z}{(z)}$. We have $f(s) = e^s$ analytic in and on the unit circle, thus

$$\int_C \frac{f(s)ds}{(s-z)} = 2\pi i f(s = 0) = 2\pi i e^0 = 2\pi i.$$

□

2.3 2a

Solution:
$2a \rightarrow f(z) = \frac{e^z}{(z-\frac{ai\pi}{4})}$. We have $f(s) = e^s$ analytic in and on square contour, thus

$$\int_C \frac{f(s)ds}{(s-z)} = 2\pi i f(s = \frac{ai\pi}{4})) = 2\pi i e^{\frac{ai\pi}{4}}.$$

□

2.4 2c

Solution:
$2c \rightarrow f(z) = \frac{z^2}{(2z+a)} = \frac{1}{2}\frac{z^2}{(z+\frac{a}{2})}$. We have $f(s) = s^2$ analytic in and on the square contour, thus

$$\int_C \frac{1}{2}\frac{f(s)ds}{(s-z)} = 2\pi i f(s = \frac{-a}{2})) \rightarrow 2 * 2\pi i(\frac{-a}{2})^2 = \pi i a^2.$$

□

2.5 2d

Solution:
$2d \rightarrow f(z) = \frac{\sin z}{(z^2} =$. We have $f(s) = \sin s$ analytic in and on the square contour, thus

$$\int_C \frac{f(s)ds}{(s-z)^2 =} = 2\pi i f^{'}(z = (0)) = 2\pi i \cos 0 = 2\pi i.$$

□

3 Problem 7 - REDACTED

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We know that:

$$\int_C \frac{f(s)ds}{(s-z)^3 =} = \pi i f^{"'}(z)$$

$$\left| \int_C \frac{f(s)ds}{(s-z)^3 =} \right| = \left| \pi i f^{"'}(z) \right|$$

The scenario given is that f(z) is entire and $\left|f(z)\right| = C|z|\ \forall z$ with C constant.
We consider the contour C defined by $C = Re^{i\theta} = |s-z|$, which implies $\left|Re^{i\theta} + z\right| = |s|$, and $ds = iRd\theta$. Then

$$\left| \frac{1}{\pi i} \int_C \frac{C|s|ds}{(s-z)^3 =} \right| = \left| f^{"'}(z) \right|$$

$$\frac{C}{\pi} \left| \int_C \frac{\left|R(Re^{i\theta} + z)d\theta\right|}{((Re^{i\theta})^3} \right| = \left| f^{"'}(z) \right|$$

Via triangle inequality we have that:

$$\frac{C}{\pi} \left| \int_C \frac{\left|R(Re^{i\theta} + z)\right|}{((Re^{i\theta})^3} \right| \leq \frac{C}{\pi} \int_C \frac{R(R+|z|)d\theta}{((R)^3} = 2C\frac{R=|z|}{R^2}.$$

(1)

Thus, $\left| f^{''}(z) \right| \leq 2C\frac{R=|z|}{R^2}$.
$R \to \infty \longrightarrow \left| f(z)^{''} \right| = 0$.
This implies $f(z) = Az + B\ \forall z$, with A,B constants. Then $\left|f(z)\right| = |Az+B|$ and we are given that $\left|f(z)\right| = C|z| = C|Az+B|$.
We evaluate this expression with $z = 0$, and have that $B = 0$, thus,
$f(z) = Az$ with A constant.

□

4 Conclusion

Thank you to Prof. Blackmore for his instruction, lectures and office hours effort. I look forward to any feedback and learning more of the material in this course.

This humble student thanks anyone reading this work and welcomes any feedback. Also, this template format had a figure and conclusion field... I decided they add a nice ending to the document. I look forward to your feedback and learning more from you. Best wishes, - José

References

[1] Mark J Ablowitz, Athanassios S Fokas, and AS Fokas. *Complex variables: introduction and applications*. Cambridge University Press, 2003.