Methods of Applied Math Exam Prep 2018.

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Spring 2020

This work is based on the course textbook [1], the material discussed in lectures and office hours related to our course MAT614 and additional references.

1 Problem August 2018 4:

$$x^2y^{''} - (2x + 2x^2)y^{'} + (x^2 + 2x + 2)y = 0.$$

We substitute:

$$y = vx, y^{'} = v^{'}x + v, y^{''} = v^{''}x + 2v^{'}.$$

$$x^2(v^{''}x + 2v^{'}) - (2x + 2x^2)(v^{'}x + v) + (x^2 + 2x + 2)(vx) = 0.$$

$$(v^{''}x^3 + 2v^{'}x^2) - (2v^{'}x^2 + 2x^2v + 2x^3v^{'} + 2xv) + v(x^3 + 2x^2 + 2x) = 0.$$

Quite a few terms add to zero, leaving us with:

$$v^{''}x^3 - 2x^3v^{'} + vx^3 = 0$$

$\implies$ Non trivial solution: $v^{''} - 2v^{'} + v = 0 = \implies (r-1)^2 = 0 \implies r = 1$ with double multiplicity .

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1.1 Frobenius practice

Frobenius form requires:

$$x^2y^{''} + (-2 - 2x)xy^{'} + (x^2 + 2x + 2)1y = 0.$$

$$x^2(R(x))y^{''} + x(P(x))y^{'} + ((Q(x)))y = 0.$$

$R_0 = 1, R_1, R_2, ..., = 0.$
$P_0 = -2, P_1 = -2, P_2 = ... = 0,$
$Q_0 = 2, Q_1 = 2, Q_2 = 1, Q_3 = ... = 0.$ Indicial equation is:

$$R_0 s(s-1) + P_0 s + Q_0 = 0 \implies s^2 - 3s + 2 = 0 \implies (s-1)(s-2) = 0 \to s_+ = 2, s_- = 1.$$

The difference of $s_+ - s_- = 1 = N$, thus we need to further test to see which case of Frobenius we have:

$$A_n^-[(n+s)(n+s-1)+P_0(n+s)+Q_0] = -\sum_{m=1}^{n} A_{n-m}^-[R_m(n-m+s)(n-m+s-1)+P_m(n-m+s)+Q_m].$$

We take our sum up to the N we found, with $s_- = 1$:
Our left hand side is:

$$A_n^-[(n+1)(n+1-1)+P_0(n+1)+Q_0] = \text{ Substitute 1 for n } = A_1^-[2(1)+(-2)(2)+2] = A_1^-(0) = 0.$$

Our right hand side is:

$$-\sum_{m=1}^{N} A_{n-m}^-[R_m(n-m+s)(n-m+s-1) + P_m(n-m+s) + Q_m] = \star$$

$$\star = A_0^-[0((n-m+s)(n-m+s-1)) - 2(1) + 2(1)] = A_0^-(0). \implies A_1^- \text{ is arbitrary.}$$

For our larger root $s_+ = 2$ we have that:
Left hand side:

$$A_n^+[R_0(n+2)(n+2-1)+P_0(n+2)+Q_0] = A_n^+[(n+2)(n+2)-2(n+2)+2] = A_n^+[n(n+1)]$$

Right hand side:

$$-\sum_{m=1}^{N} A_{n-m}^+[R_m(n-m+2)(n-m+2-1) + P_m(n-m+2) + Q_m] = \star$$

For $m = 1$:

$$-A_{n-1}^+[R_1(n-1+2)(n-1+2-1) + P_1(n-1+2) + Q_1] = -A_{n-1}^+[(0) + -2(n+1) + 2].$$

For $m = 2$:

$-A^+_{n-2}[R_2(n-2+2)(n-2+2-1) + P_2(n-2+2) + Q_2] = -A^+_{n-2}[(0) + 0 + 1].$

Thus, we put it all together:

$$A^+_n[n(n+1)] = +2nA^+_{n-1} - A^+_{n-2} \implies A^+_n = \frac{1}{n+1}2A^+_{n-1} - \frac{1}{n(n+1)}A^+_{n-2}.$$

For$n = 1$:

$$\implies A^+_1 = \frac{2}{2}A^+_0$$

.

For$n = 2$:

$$\implies A^+_2 = \frac{2}{3}A^+_1 - \frac{1}{(2)(3)}A^+_0\frac{1}{2}A^+_0$$

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For$n = 3$:

$$\implies A^+_3 = \frac{2}{4}A^+_2 - \frac{1}{(4)(3)}A^+_1 \implies \frac{1}{4}A^+_0 - \frac{1}{3(4)}A^+_0 = \frac{1}{6}A^+_0$$

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Thus, we have that:

$$A^+_n = \frac{1}{n!}A^+_0 \implies y = A^+_0 x^2 \sum_{n=0}^{\infty} \frac{1}{n!}x^n \implies y = A^+_0 x^2 e^2.$$

2 Conclusion

Thank you to Prof. Hamfeldt, neé Froese, as well as anyone else for reading this work, as well as any instruction, lectures and future office hours efforts. I look forward to any feedback and learning more of the material in this course and qualifying exams.

References

[1] Kendall E Atkinson. *An introduction to numerical analysis.* John wiley & sons, 2008.