

Numerical Analysis Previous Exam August 2018.

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This work is based on the course textbook [1], the material discussed in lectures and office hours related to our course MAT614 and additional references.

1 Problem August 2018 3 - REDACTED

IVP:

$$y' = 5y_1 - 6y_2.$$

$$y' = 3y_1 - 4y_2.$$

$$y_1(0) = 4, y_2(0) = 1$$

1.1 Study notes:

For IVP problems of this type, we have that our standard solutions relative to eigenvalues $\lambda = \mu + i\nu$ are:

$$y = e^{\mu t}(\alpha \sin(\nu t) + \beta \cos(\nu t)).$$

1.2 Solution proof

We have that our corresponding matrix representation for this system is:

$$A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

We calculate the eigenvalues of A by solving $\det(A - \lambda I) = 0 \implies (5 - \lambda)(-4 - \lambda) - (3(-6)) = 0 \implies \lambda^2 - \lambda - 2 = 0$. We use these two eigenvalues to solve $(A - \lambda I)v = 0$ to find the corresponding eigenvalues:

$$\text{For } \lambda_1 = 2, v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$$\text{For } \lambda_2 = -1, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We thus have our general solution:

$$y_{gen} = \alpha e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \beta e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We find our particular solution to this IVP by using the initial condition provided:

$$y_{part} = \alpha e^{20} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \beta e^{-0} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

We solve this system of equations to determine that $\alpha = 3, \beta = -2$.

Thus, our final particular solution to this particular initial value problem is:

$$y_{(part.final)} = 3e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Or, equivalently:

$$\therefore y_{(part.final)} = y_1 + y_2, y_1 = 6e^{2t} - 2e^{-t}, y_2 = 3e^{2t} - 2e^{-t}. \checkmark$$

2 Problem August 2018 4:

2.1 REDACTED

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2.2 Solution:

We have that Newton's method algorithm is of the form :

$$x_{n+1} = x_n - \frac{1}{f'(x_n)} f(x_n).$$

We insert our $g(x)$:

$$x_{n+1} = x_n - \frac{1}{2(x_n)}(x_n^2) = x_n - \frac{1}{2}(x_n) = \frac{1}{2}(x_n).$$

Thus, the iterations of Newton's method would yield for us that:

$$x_{n+1} = \left(\frac{1}{2}\right)^n x_0.$$

We note that we do have linear convergence:

$$\frac{1}{(x_n)} \left(\frac{1}{2}(x_n)\right) \leq 1 \implies \left(\frac{1}{2}\right) \leq 1.$$

Quadratic convergence would imply that:

$$\begin{aligned} \frac{1}{(x_n - \alpha)^2} (x_{n+1} - \alpha) &\leq k. \\ \implies \frac{1}{(x_n - \alpha)^2} \left(\frac{1}{2}(x_n) - \alpha\right) &\leq k. \end{aligned}$$

Given our root is $\alpha = 0$, we have that:

$$\frac{1}{(x_n)^2} \left(\frac{1}{2}(x_n)\right) \leq k \implies \frac{1}{(x_n)} \left(\frac{1}{2}\right) \leq k.$$

If Newton's method were to converge for this function, we'd have $\lim_{n \rightarrow \infty} x_n = 0$, thus:

$$\therefore \lim_{n \rightarrow \infty} \frac{1}{(x_n)} \left(\frac{1}{2}\right) = \infty \implies \text{no finite constant } k \text{ such that } : \frac{1}{(x_n)^2} \left(\frac{1}{2}(x_n)\right) \leq k. \checkmark$$

Thanks for the advice pointing me in the right direction!

2.3 4b - $f(x) \in \mathbf{C}^3$ q has degree less than or equal to two interpolating f at x_0, x_1, x_2 . Let $h = \max(x_1 - x_0, (x_2 - x_1))$, K be the max over the x 's in the interval of $|f'''(x)|$.

Show that:

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2.3 REDACTED

polating f at x_0, x_1, x_2 . Let $h = \max(x_1 - x_0, (x_2 - x_1))$, K be the max over the x 's in the interval of $|f'''(x)|$. Show that:

$$\max_{x \in [x_0, x_2]} |f''(x) - q''(x)| = Ch^\alpha.$$

2.4 Solution proof:

Our general Newton's polynomial of second degree for the given points is:

$$N(x) = f(x_0) + [f(x_0), f(x_1)](x - x_0) + [f(x_0), f(x_1), f(x_2)](x - x_0)(x - x_1).$$

We have that:

$$[f(x_0), f(x_1)] = \frac{1}{(x_1 - x_0)}(f(x_1) - f(x_0)).$$

and:

$$[f(x_0), f(x_1), f(x_2)] = \frac{1}{(x_2 - x_1)(x_2 - x_0)}(f(x_2) - f(x_1)) - \frac{1}{(x_1 - x_0)(x_2 - x_0)}(f(x_1) - f(x_0)).$$

We put everything together to get our form for $q(x)$:

$$q(x) = f(x_0) + \frac{1}{(x_1 - x_0)}(f(x_1) - f(x_0))(x - x_0) + \left\{ \frac{1}{(x_2 - x_1)(x_2 - x_0)}(f(x_2) - f(x_1)) - \frac{1}{(x_1 - x_0)(x_2 - x_0)}(f(x_1) - f(x_0)) \right\} (x - x_0)(x - x_1).$$

We compute the second derivative of $q(x)$:

$$q''(x) = 2 \left(\frac{1}{(x_2 - x_1)(x_2 - x_0)}(f(x_2) - f(x_1)) - \frac{1}{(x_1 - x_0)(x_2 - x_0)}(f(x_1) - f(x_0)) \right).$$

Thus, we know that the second derivative of q is just a number. Perhaps computing all of the before is not a judicious use of time.

We will now show that the expression of $f''(x) - q''(x)$ has at least one zero in the domain for this problem. We use a clever new function suggested by Prof. Siegel, $m(x) = f(x) - q(x)$. By construction, $m(x) = 0$ at $x = x_0, x_1, x_2$. Via mean value theorem (for derivatives?), there exists some $\eta_1 \in [x_0, x_1]$ such that $m'(\eta_1) = 0$, $\eta_2 \in [x_1, x_2]$ such that $m'(\eta_2) = 0$. Thus, we have that $m'(x)$ has at least two zeroes in the interval $\in [x_0, x_2]$.

Using the same, or, you know, extremely similar mean value theorem argument, there exists some $\eta \in [x_0, x_2]$ such that $m''(\eta) = 0$. Thus, we know $f''(\eta) - q''(\eta) = m''(\eta) = 0$.

We follow the provided hint and consider the integral:

$$\int_{\eta}^x f'''(\xi) - q'''(\xi) d\xi = \int_{\eta}^x (f'''(\xi) - 0) d\xi = \left| f''(x) - q''(x) - (f''(\eta) - q''(\eta)) \right| = \left| f''(x) - q''(x) \right|.$$

Now we apply the constants provided in the problem K, h :

$$\int_{\eta}^x f'''(\xi) d\xi \leq (x - \eta) \max |f'''(x)|.$$

Given the x that maximizes this could be 'on the other side' of x_1 with respect to η , we put our inequalities together and have that $(x - \eta) \leq 2h$, $C = \frac{1}{2}K$, such that:

This equation:

$$\int_{\eta}^x f'''(\xi) - q'''(\xi) d\xi = \int_{\eta}^x (f'''(\xi) - 0) d\xi = \left| f''(x) - q''(x) \right|$$

Combined with this inequality:

$$\int_{\eta}^x f'''(\xi) d\xi \leq (x - \eta) \max |f'''(x)| \leq 2hK.$$

\implies for $\alpha = 1$, $(x - \eta) \leq 2h$, $C = \frac{1}{2}K$ provides the requested shown :

$$\therefore \max_{x \in [x_0, x_2]} \left| f''(x) - q''(x) \right| \leq Ch^\alpha. \checkmark$$

I'm unsure if the inequalities here can be considered equality for some x in the interval as requested by the original enunciation of the problem.

3 Problem 5 - Consider the scheme:

$$y_{n+1} = y_n + \frac{1}{2}h(y'_{n+1} + y'_n) + \frac{1}{12}h^2(y''_n - y''_{n+1})$$

3.1 REDACTED

We use Taylor expansions:

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''_n + \frac{1}{24}h^4y''''_n + O(h^5)$$

$$y'_{n+1} = y'_n + hy''_n + \frac{1}{2}h^2y'''_n + \frac{1}{6}h^3y''''_n + O(h^4)$$

$$y''_{n+1} = y''_n + hy'''_n + \frac{1}{2}h^2y''''_n + O(h^3)$$

We plug this into our scheme:

$$y_{n+1} = y_n + \frac{1}{2}h(y'_{n+1} + y'_n) + \frac{1}{12}h^2(y''_n - y''_{n+1}) \implies$$

$$\implies y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''_n + \frac{1}{24}h^4y''''_n + O(h^5) = y_n + \frac{1}{2}h(y'_{n+1} + y'_n) + \frac{1}{12}h^2(y''_n - y''_{n+1})$$

We group and simplify:

$$\frac{1}{2}h(y'_{n+1} + y'_n) = \frac{1}{2}h(y'_n + y'_n + hy''_n + \frac{1}{2}h^2y'''_n + \frac{1}{6}h^3y''''_n + O(h^4))$$

We group and simplify:

$$\frac{1}{12}h^2(y''_n - y''_{n+1}) = \frac{1}{12}h^2(y''_n - y''_n - hy'''_n - \frac{1}{2}h^2y''''_n - O(h^3))$$

We group like terms:

$$y_n(1 - 1) = 0$$

$$y'_n(h - h) = 0$$

$$y''_nh^2(\frac{1}{2} - \frac{1}{2}) = 0$$

$$y'''_nh^3(\frac{1}{6} - \frac{1}{4} + \frac{1}{12}) = y'''_nh^3(0)$$

$$y_n'''' h^4 \left(\frac{1}{24} - \frac{1}{12} + \frac{1}{24} \right) = y_n'''' h^4(0)$$

At this point we realize that we haven't expanded far enough so we need to Taylor expand farther. Incoming:

We use Taylor expansions:

$$y_{n+1} = y_n + hy_n' + \frac{1}{2}h^2y_n'' + \frac{1}{6}h^3y_n''' + \frac{1}{24}h^4y_n'''' + \frac{1}{120}h^5y_n^V + O(h^6)$$

$$y_{n+1}' = y_n' + hy_n'' + \frac{1}{2}h^2y_n''' + \frac{1}{6}h^3y_n'''' + \frac{1}{24}h^4y_n^V + O(h^5)$$

$$y_{n+1}'' = y_n'' + hy_n''' + \frac{1}{2}h^2y_n'''' + \frac{1}{6}h^3y_n^V + O(h^4)$$

We plug this into our scheme:

$$y_{n+1} = y_n + \frac{1}{2}h(y_{n+1}' + y_n') + \frac{1}{12}h^2(y_n'' - y_{n+1}'') \implies$$

$$\implies y_n + hy_n' + \frac{1}{2}h^2y_n'' + \frac{1}{6}h^3y_n''' + \frac{1}{24}h^4y_n'''' + \frac{1}{120}h^5y_n^V + O(h^6) = y_n + \frac{1}{2}h(y_{n+1}' + y_n') + \frac{1}{12}h^2(y_n'' - y_{n+1}'')$$

We group and simplify:

$$\frac{1}{2}h(y_{n+1}' + y_n') = \frac{1}{2}h(y_n' + y_n' + hy_n'' + \frac{1}{2}h^2y_n''' + \frac{1}{6}h^3y_n'''' + \frac{1}{24}h^4y_n^V + O(h^5))$$

We group and simplify:

$$\frac{1}{12}h^2(y_n'' - y_{n+1}'') = \frac{1}{12}h^2(y_n'' - y_n'' - hy_n''' - \frac{1}{2}h^2y_n'''' - \frac{1}{6}h^3y_n^V + O(h^4))$$

We group like terms:

$$y_n(1 - 1) = 0$$

$$y_n'(h - h) = 0$$

$$y_n''h^2\left(\frac{1}{2} - \frac{1}{2}\right) = 0$$

$$y_n'''h^3\left(\frac{1}{6} - \frac{1}{4} + \frac{1}{12}\right) = (0)y_n'''h^3$$

3.2 5b - Show that the region of absolute stability contains the entire negative real axis of the $h\lambda$ plane.

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$$y_n'''' h^4 \left(\frac{1}{24} - \frac{1}{12} + \frac{1}{24} \right) = (0) y_n'''' h^4$$

$$y_n^V h^5 \left(\frac{1}{120} - \frac{1}{48} + \frac{1}{72} \right) = \left(\frac{1}{720} \right) y_n^V h^5.$$

We have that $48 = 2^4(3)$, $72 = 2^3(3^2)$, $120 = 2^3(3)(5)$, the least common multiple for the denominators in the fractions involved is $720 = 2^4(3^2)5$.

Thus the computed coefficient is:

$$\frac{6}{720} - \frac{15}{720} + \frac{10}{720} = \frac{1}{720}.$$

Thus, the lowest term of nonzero coefficients are $\left(\frac{1}{720}\right)y_n^V h^5$, i.e. of order $O(h^5)$ in our computation of $y_{n+1} - y_n$. Now our local truncation error is:

$$\tau_{n+1} = \frac{1}{h}(y_{n+1} - y_n) \implies \tau_{n+1} = \frac{1}{h} \left(\frac{1}{720} \right) y_n^V h^5$$

Thus, the method is of fourth order $O(h^4)$ ✓.

3.2 5b - Show that the region of absolute stability contains the entire negative real axis of the $h\lambda$ plane.

We define $y' = \lambda y$, $y(0) = y_0$, and we consider the method applied to this function. We will use notation of $f_n \equiv f^n$, which will not denote the n th derivative. We have that:

$$y_n'' = \frac{df(x_n, y_n)}{dx} + f(x_n, y_n) \frac{df(x_n, y_n)}{dx}.$$

$$f^n = \lambda y_n.$$

$$f_x^n = \lambda y'(x_n) = \lambda f^n = \lambda^2 y_n.$$

$$f_y^n = \lambda.$$

We examine the method within this framework:

$$y_{n+1} = y_n + \frac{h}{2}[y_n' + y_{n+1}'] + \frac{h^2}{12}[y_n'' - y_{n+1}''].$$

$$y_{n+1} = y_n + \frac{h}{2}[f^n + f^{n+1}] + \frac{h^2}{12}[f_x^n + f_y^n f_y^n - f_x^{n+1} + f^{n+1} f_y^{n+1}].$$

$$y_{n+1} = y_n + \frac{h}{2}[\lambda y_n + \lambda y_{n+1}] + \frac{h^2}{12}[\lambda^2 y_n + \lambda^2 y_n - \lambda^2 y_{n+1} - \lambda^2 y_{n+1}].$$

$$y_{n+1} = y_n + \frac{h\lambda}{2}[y_n + y_{n+1}] + \frac{h^2\lambda^2}{6}[y_n - y_{n+1}].$$

3.2 5b - Show that the region of absolute stability contains the entire negative real axis of the $h\lambda$ plane.

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$$\implies \left(1 - \frac{1}{2}h\lambda + \frac{1}{6}h^2\lambda^2\right)y_{n+1} = \left(1 + \frac{1}{2}h\lambda + \frac{1}{6}h^2\lambda^2\right)y_n \implies (6 - 3h\lambda + h^2\lambda^2)y_{n+1} = (6 + 3h\lambda + h^2\lambda^2)y_n.$$

Thus we have our relation:

$$y_{n+1} = \frac{(6 + 3h\lambda + h^2\lambda^2)}{(6 - 3h\lambda + h^2\lambda^2)}y_n.$$

We verify the stability of the method for this function by studying:

$$y_n = \left(\frac{(6 + 3h\lambda + h^2\lambda^2)}{(6 - 3h\lambda + h^2\lambda^2)}\right)^n y_0.$$

By definition and convention, we have step size of $h > 0$ and we define $\lambda < 0 \implies h\lambda < 0$. For the method to be stable, we find we need the condition:

$$\left| \frac{(6 + 3h\lambda + h^2\lambda^2)}{(6 - 3h\lambda + h^2\lambda^2)} \right| < 1.$$

$$h\lambda < 0 \implies (6 - 3h\lambda + h^2\lambda^2) > 0$$

So we have that

$$-1 < \frac{(6 + 3h\lambda + h^2\lambda^2)}{(6 - 3h\lambda + h^2\lambda^2)} < 1.$$

$$-1(6 - 3h\lambda + h^2\lambda^2) < (6 + 3h\lambda + h^2\lambda^2) < (6 - 3h\lambda + h^2\lambda^2).$$

We examine these inequalities and have that:

$$-1(6 - 3h\lambda + h^2\lambda^2) < (6 + 3h\lambda + h^2\lambda^2) \implies -6 < (6 + 2h^2\lambda^2). \checkmark$$

This inequality holds for all values of h, λ . For the second set of inequalities we find that:

$$(6 + 3h\lambda + h^2\lambda^2) < (6 - 3h\lambda + h^2\lambda^2) \implies 3h\lambda < -3h\lambda. \checkmark$$

This inequality holds for $h\lambda < 0$.

Thus

$$\left| \frac{(6 + 3h\lambda + h^2\lambda^2)}{(6 - 3h\lambda + h^2\lambda^2)} \right| < 1.$$

When $h > 0, \lambda < 0 \implies h\lambda < 0$ giving us our region of stability.

\therefore The region of absolute stability contains the entire negative real axis of the $h\lambda$ plane.

4 Problem August 2018 6 - Show the degree of precision is less than or equal to $2n - 1$, and then that it is no more than the same.

Suppose inner product with weight is defined by:

$$\langle f, g \rangle = \int_a^b f(x)g(x)w(x)dx.$$

Consider quadrature formula:

$$I(f) = \int_a^b f(x)w(x)dx \approx Q(f) = \sum_{j=1}^n w_j f(x_j).$$

We have that:

$$w_j = \int_a^b l_j(x)w(x)dx.$$

And

$$l_j(x) = \prod_{i=1,2,3,\dots,n-i \neq j} \frac{(x - x_i)}{(x_j - x_i)}.$$

4.1 Solution, proof:

Degree of precision is defined as the highest order polynomial that the quadrature will return an exact answer for. Study note - for Simpsons rule this is three, for trapezoid rule this is one.

The key idea of this proof is division of the polynomial f . We define polynomials such that:

$$\frac{f(x)}{p_n(x)} - \frac{r(x)}{p_n(x)} = q(x) \implies f(x) = p_n(x)q(x) + r(x).$$

Where rational expression is shorthand for polynomial division. We construct this with the degrees of f, p, q, r being $\leq 2n - 1, n, n - 1, n - 1$ and by construction and the interpolation of $I(f)$, *we have that $r(x)$ is exact and there is no further remainder.*

Note to reader - I went to Prof. Hamfeldt's office hour to talk about this problem, she mentioned this claim in italics is not clear or obvious. I thus proffer the follow explanation:

We have that $r(x)$ is exact without a further remainder because it is of degree less than or equal to $n - 1$, thus there via fundamental properties of polynomials, there is an expression for $r(x)$ in terms of basis function monomials of up to n degree, thus for some constants a_i , $r(x) = a_0 + a_1x + a_2x^2 + \dots$ such that $r(x)$ is an exact $n - 1$ degree polynomial

in x .

We have that $p_n x$ is orthogonal to all polynomials of degree $\leq n - 1$, thus we have that:

$$I(f) = I(qp_n + r) = \int_a^b q(x)p_n(x) + r(x) dx + \int_a^b r(x)w(x) dx = 0 + \int_a^b r(x)w(x) dx = Q(r).$$

Thus we find that:

$$I(f) = Q(r) = \sum w_j r(x_j) = \sum w_j (p_n(x_j)r(x_j) + r(x_j)) = \sum 0 + w_j r(x_j) = Q(f).$$

Need to show the remainder polynomials are exact.

Thus, we have that the precision of this method (Gaussian quadrature) is $\leq 2n - 1$.

4.2 Show that the precision is no greater than $2n - 1$.

We assume the same construction as in the previous argument, except for the degrees of the polynomials:

$$\frac{f(x)}{p_n(x)} - \frac{r(x)}{p_n(x)} = q(x) \implies f(x) = p_n(x)q(x) + r(x).$$

Where rational expression is shorthand for polynomial division. We construct this with the degrees of f, p, q, r being $\leq 2n, n, n, n - 1$ and by construction and the interpolation of $I(f)$, we have that $r(x)$ is still exact and there is no further remainder.

We have that $p_n x$ is orthogonal to all polynomials of degree $\leq n - 1$ but not orthogonal to all polynomials of degree n , thus we have that:

$$I(f) = I(qp_n + r) = \int_a^b q(x)p_n(x) + r(x) dx + \int_a^b r(x)w(x) dx = 0 + \int_a^b r(x)w(x) dx = Q(r) + \langle q, p_n \rangle.$$

Thus, $\langle q, p_n \rangle \neq 0$.

We still have the same zeros of our p_n , so thus we find that:

$$Q(f) = \sum w_j r(x_j) = Q(qp_n + r) = \sum w_j (p_n(x_j)r(x_j) + r(x_j)) = \sum 0 + r(x_j) = Q(r).$$

Thus,

$$Q(f) \neq Q(r)$$

Thus, for all polynomials f of degree $2n$ or higher, the quadrature formula is not precise.

5 Conclusion

Thank you to Prof. Hamfeldt, neé Froese, as well as anyone else for reading this work, as well as any instruction, lectures and future office hours efforts. I look forward to any feedback and learning more of the material in this course and qualifying exams.

References

- [1] Kendall E Atkinson. *An introduction to numerical analysis*. John wiley & sons, 2008.

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