Numerical Analysis Previous Qual

José L. Pabón

Spring 2020

This work is based on the course textbook [1], the material discussed in lectures and office hours related to our course MAT614 and additional references. This is for the May 2018 qualifying exam.

1 Problem August 2015 Problem 6

We compute stuff and find that:

$$(1 - h\lambda\theta)r^{n+1} - (1 + ((h\lambda)(1-\theta))r^n = 0.$$

Which implies:

$$(r^n)[(1 - h\lambda\theta)r - (1 + ((h\lambda)(1-\theta))] = 0.$$

We discard trivial solutions and have that:

$$(1 - h\lambda\theta)r - (1 + ((h\lambda)(1-\theta)) = 0 \implies (1 - h\lambda\theta)r = (1 + ((h\lambda)(1-\theta)).$$

$$r = \frac{1}{(1 - h\lambda\theta)}(1 + ((h\lambda)(1-\theta))$$

2 Older stuff down here.

2.1 1a

Let A be a real symmetric 3×3 matrix with eigenvalues 0,3,5 and corresponding eigenvectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

2.1.1 Find a basis for the nullspace of A and a basis for the column space of A.

By definition, the nullspace of A is spanned by $c\mathbf{u}$, c any arbitrary scalar. Similarly, the column space is spanned by $d\mathbf{v}, e\mathbf{w}$ d, e any arbitrary scalars.

2.1.2 If possible find solution for $Ax = u$

Given that u spans the nullspace, there is no solution for $Ax = u$.

2.1.3 Is A Invertible?

Given that A is rank deficient, since it has one eigenvalue of 0, A is singular and not invertible, it is not full rank.

2.2 1b - A and B are similar matrices.

2.2.1 Show they have the same determinant

We have that:

$$A = SBS^{-1}.$$

$$AS = SB.$$

$$\det(AS) = \det(SB).$$

$$\det(A)\det(S) = \det(S)\det(B).$$

$$\det(A) = \det(B).$$

2.2.2 Show they have the same characteristic polynomial and eigenvalues.

We have that:

$$A = SBS^{-1}.$$

$$A - \lambda I = SBS^{-1} - \lambda I.$$

$$AS - \lambda IS = SB - \lambda IS.$$

$$\det\big((A-\lambda I)S\big) = \det\big(S(B-\lambda I)\big).$$

$$\det(A-\lambda I)\det(S) = \det(S)\det(B-\lambda I)).$$

$$\det(A-\lambda I) = \det(B-\lambda I)).$$

Thus, the characteristic polynomial of A, $\det(A-\lambda I)=0$ is the same as $\det(B-\lambda I) = 0$.

3 Problem May 2018 4:

3.1 REDACTED **function with $n+1$ continuous derivatives. Let $P_n(x)$ be the lagrange polynomial of degree n that interpolates as $P_n(x_i) = f(x_i)$.**

We are asked to show that:

$$\forall\ x \in \text{ domain of f }\ \exists\ \xi \in \mathbf{R}\ |\ f(x) - P_n(x) = \frac{1}{(n+1)!}(x-x_0)(x-x_1)...(x-x_n)f^{(n+1)}(\xi).$$

We notice the right hand side term looks suspiciously like a Taylor Remainder / end of a Taylor expansion.

We calculate the Taylor expansion around x_0, i.e., for some $\xi \in [x, x_0]$:

$$f(x) = f(x_0) + \frac{1}{1!}(x-x_0)f'(x_0) + \frac{1}{2!}(x-x_0)^2 f''(x_0) + ... + \frac{1}{(n+1)!}(x-x_0)^{(n+1)}f^{(n+1)}(\xi).$$

Similarly, we calculate the Taylor expansion around x_1, i.e., for some $\xi \in [x, x_1]$:

$$f(x) = f(x_1) + \frac{1}{1!}(x-x_1)f'(x_1) + \frac{1}{2!}(x-x_1)^2 f''(x_1) + ... + \frac{1}{(n+1)!}(x-x_1)^{(n+1)}f^{(n+1)}(\xi).$$

And so on; finally we calculate the Taylor expansion around x_n, i.e., for some $\xi \in [x, x_n]$:

$$f(x) = f(x_n) + \frac{1}{1!}(x-x_n)f'(x_n) + \frac{1}{2!}(x-x_n)^2 f''(x_n) + ... + \frac{1}{(n+1)!}(x-x_n)^{(n+1)}f^{(n+1)}(\xi).$$

It is at this point I realize that I cannot bootstrap or formulate a proof in this direction. Checking our lecture notes, I found a correct argument:

Sketch of proof:

- Construct a function $\phi(x)$ which is the collection of products of $(x-x_0)(x-x_1)...(x-x_n)$.
- Formulate the error function $E(x) = f(x) - P_n(x)$
- Construct another function G in the form of $G(x) = E(x) - \frac{1}{\phi(t)}\phi(x)E(t)$.
- Show that $G(x)$ has $n+2$ zeroes, and has $n+1$ derivatives.
- Invoke higher order mean value theorem to arrive at precise result requested.

3.2 REDACTED $\frac{1}{(2+x)}$ point in $x^* \in [0,1]$, describe the convergence of the iteration $x_{n+1} = g(x_n), x_0 = 1$.

REMEMBER HAVE TO HAVE $G(X)$ CONTINUOUS.

Via theorems from class, existence of fixed points occur when for any $x \in [0,1], g(x) \in [0,1]$. Furthermore, if $\left|g^{'}(x)\right| < 1$, then the fixed point in that interval is unique.

Also, due to a handy corollary to the contraction mapping theorem, we have that $\left|g^{'}(x)\right| < 1 \implies$ iterations of $x_{n+1} = g(x_n)$ for a starting $x_0 \in [0,1]$ will converge.

We have that for $\forall x \in [0,1], \frac{1}{3} < g(x) < \frac{1}{2} \implies g(x) \in [0,1]$. Then:

$$g^{'}(x) = -\frac{1}{(2+x)^2}.$$

In the interval $x \in [0,1]$, we have that $\frac{1}{9} < \left|g^{'}(x)\right| < \frac{1}{4} \implies \left|g^{'}(x)\right| < 1$, thus, via the theorems discussed in class the fixed point for this function exists, is unique and converges to some $x^* \;\; \forall \;\; x \in [0,1]$.

We will describe the convergence in more detail, we want to calculate how fast $|x_n - x^*|$ converges to zero. We have that:

$$x_{n+1} = g(x_n).$$

$$x^* = g(x^*).$$

We consider the absolute value of the difference of these two equations:

$$|x_n - x^*| = \left|g(x_{n-1}) - g(x^*)\right| \leq \frac{1}{4}|x_{n-1} - x^*| \implies |x_n - x^*| \leq (\frac{1}{4})^n|x_0 - x^*|.$$

Thus,

$$\implies \frac{1}{|x_0 - x^*|}|x_n - x^*| \leq (\frac{1}{4})^n.$$

Thus the convergence is linear and will converge as expected.

3.2.1 The shortest, most succinct proof:

In the interest of preparing for the limited time nature of our qualifying doctoral exams, we wonder if this proof could be as short as:

Via theorems from class, since $\forall x \in [0,1], \frac{1}{3} < g(x) < \frac{1}{2} \implies g(x) \in [0,1],$, and:

$$g^{'}(x) = -\frac{1}{(2+x)^2}.$$

such that $\frac{1}{9} < \left|g^{'}(x)\right| < \frac{1}{4} \implies \left|g^{'}(x)\right| < 1$, the fixed point is unique and will converge with:

$$|x_n - x^*| \approx (\frac{1}{4})^n.$$

New stuff above here.

Old stuff below here.

3.3 Solution:

We have that Newton's method algorithm is of the form :

$$x_{n+1} = x_n - \frac{1}{f'(x_n)} f(x_n).$$

We insert our $g(x)$:

$$x_{n+1} = x_n - \frac{1}{2(x_n)}(x_n^2) = x_n - \frac{1}{2}(x_n) = \frac{1}{2}(x_n).$$

Thus, the iterations of Newton's method would yield for us that:

$$x_{n+1} = (\frac{1}{2})^n x_0.$$

Quadratic convergence would imply that:

$$\frac{1}{(x_n - \alpha)^2}(x_{n+1} - \alpha) \leq k.$$

$$\implies \frac{1}{(x_n - \alpha)^2}(\frac{1}{2}(x_n) - \alpha) \leq k.$$

Given our root is $\alpha = 0$, we have that:

$$\frac{1}{(x_n)^2}(\frac{1}{2}(x_n)) \leq k \implies \frac{1}{(x_n)}(\frac{1}{2}) \leq k.$$

We note that we do have linear convergence:

$$\frac{1}{(x_n)}(\frac{1}{2}(x_n)) \leq 1 \implies (\frac{1}{2}) \leq 1.$$

3.4 REDACTED-polating f at x_0, x_1, x_2,. Let $h = \max{(x_1 - x_0), (x_2 - x_1)}$, K be the max over the x's in the interval of $\left|f'''(x)\right|$. Show that:

$$\max_{x \in [x_0, x_2]} \left|f''(x) - q''(x)\right| = Ch^{\alpha}.$$

3.5 Solution proof:

Our general Newton's polynomial of second degree for the given points is:

$$N(x) = f(x_0) + [f(x_0), f(x_1)](x - x_0) + [f(x_0), f(x_1), f(x_2)](x - x_0)(x - x_1).$$

We have that:

$$[f(x_0), f(x_1)] = \frac{1}{(x_1 - x_0)}(f(x_1 - f(x_0)).$$

and:

$$[f(x_0), f(x_1), f(x_2)] = \frac{1}{(x_2 - x_1)(x_2 - x_0)}(f(x_2) - f(x_1)) - \frac{1}{(x_1 - x_0)(x_2 - x_0)}(f(x_1) - f(x_0)).$$

We put everything together to get our form for $q(x)$:

$$q(x) = f(x_0) + \frac{1}{(x_1 - x_0)}(f(x_1) - f(x_0))(x - x_0)$$

$$+\{\frac{1}{(x_2 - x_1)(x_2 - x_0)}(f(x_2) - f(x_1)) - \frac{1}{(x_1 - x_0)(x_2 - x_0)}(f(x_1) - f(x_0))\}(x - x_0)(x - x_1).$$

We compute the second derivative of $q(x)$:

$$q^{''}(x) = 2(\frac{1}{(x_2 - x_1)(x_2 - x_0)}(f(x_2) - f(x_1)) - \frac{1}{(x_1 - x_0)(x_2 - x_0)}(f(x_1) - f(x_0))).$$

We follow the provided hint and consider the integral:

$$\int_{\eta}^{x} f^{'''}(\xi)d\xi = f^{''}(x) - f^{''}(\eta).$$

4 REDACTED:

$$y_{n+1} = y_n + \frac{1}{2}h(y^{'}_{n+1} + y^{'}_n) + \frac{1}{12}h^2(y^{''}_n - y^{''}_{n+1})$$

.

4.1 REDACTED.

We use Taylor expansions:

$$y_{n+1} = y_n + hy^{'}_n + \frac{1}{2}h^2y^{''}_n + \frac{1}{6}h^3y^{'''}_n + \frac{1}{24}h^4y^{''''}_n + O(h^5)$$

$$y'_{n+1} = y'_n + hy''_n + \frac{1}{2}h^2y'''_n + \frac{1}{6}h^3y''''_n + O(h^4)$$

$$y''_{n+1} = y''_n + hy'''_n + \frac{1}{2}h^2y''''_n + O(h^3)$$

We plug this into our scheme:

$$y_{n+1} = y_n + \frac{1}{2}h(y'_{n+1} + y'_n) + \frac{1}{12}h^2(y''_n - y''_{n+1}) \implies$$

$$\implies y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''_n + \frac{1}{24}h^4y''''_n + O(h^5) = y_n + \frac{1}{2}h(y'_{n+1} + y'_n) + \frac{1}{12}h^2(y''_n - y''_{n+1})$$

We group and simplify:

$$\frac{1}{2}h(y'_{n+1} + y'_n) = \frac{1}{2}h(y'_n + y'_n + hy''_n + \frac{1}{2}h^2y'''_n + \frac{1}{6}h^3y''''_n + O(h^4))$$

We group and simplify:

$$\frac{1}{12}h^2(y''_n - y''_{n+1}) = \frac{1}{12}h^2(y''_n - y''_n - hy'''_n - \frac{1}{2}h^2y''''_n - O(h^3))$$

We group like terms:

$$y_n(1-1) = 0$$

$$y'_n(h-h) = 0$$

$$y''_nh^2(\frac{1}{2} - \frac{1}{2}) = 0$$

$$y'''_nh^3(\frac{1}{6} - \frac{1}{4} + \frac{1}{12}) = y'''_nh^3(0)$$

$$y''''_nh^4(\frac{1}{24} - \frac{1}{12} + \frac{1}{24}) = y''''_nh^4(0)$$

At this point we realize that we haven't expanded far enough so we need to Taylor expand farther. Incoming:

We use Taylor expansions:

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''_n + \frac{1}{24}h^4y''''_n + \frac{1}{120}h^5y^V_n + O(h^6)$$

$$y'_{n+1} = y'_n + hy''_n + \frac{1}{2}h^2y'''_n + \frac{1}{6}h^3y''''_n + \frac{1}{24}h^4y^V_n + O(h^5)$$

$$y''_{n+1} = y''_n + hy'''_n + \frac{1}{2}h^2y''''_n + \frac{1}{6}h^3y^V_n + O(h^4)$$

We plug this into our scheme:

$$y_{n+1} = y_n + \frac{1}{2}h(y'_{n+1} + y'_n) + \frac{1}{12}h^2(y''_n - y''_{n+1}) \implies$$

$$\implies y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''_n + \frac{1}{24}h^4y''''_n + \frac{1}{120}h^5y^V_n + O(h^6) = y_n + \frac{1}{2}h(y'_{n+1} + y'_n) + \frac{1}{12}h^2(y''_n - y''_{n+1})$$

We group and simplify:

$$\frac{1}{2}h(y'_{n+1} + y'_n) = \frac{1}{2}h(y'_n + y'_n + hy''_n + \frac{1}{2}h^2y'''_n + \frac{1}{6}h^3y''''_n + \frac{1}{24}h^4y^V_n + O(h^5))$$

We group and simplify:

$$\frac{1}{12}h^2(y''_n - y''_{n+1}) = \frac{1}{12}h^2(y''_n - y''_n - hy'''_n - \frac{1}{2}h^2y''''_n - \frac{1}{6}h^3y^V_n + O(h^4)))$$

We group like terms:

$$y_n(1-1) = 0$$

$$y'_n(h-h) = 0$$

$$y''_nh^2(\frac{1}{2} - \frac{1}{2}) = 0$$

$$y'''_nh^3(\frac{1}{6} - \frac{1}{4} + \frac{1}{12}) = (0)y'''_nh^3$$

$$y''''_nh^4(\frac{1}{24} - \frac{1}{12} + \frac{1}{24}) = (0)y''''_nh^4$$

$$y^V_nh^5(\frac{1}{120} - \frac{1}{48} + \frac{1}{72}) = (\frac{1}{720})y^V_nh^5.$$

We have that $48 = 2^4(3), 72 = 2^3(3^2), 120 = 2^3(3)(5)$, the least common multiple for the denominators in the fractions involved is $720 = 2^4(3^2)5$.

Thus the computed coefficient is:

$$\frac{6}{720} - \frac{15}{720} + \frac{10}{720} = \frac{1}{720}.$$

Thus, the lowest term of nonzero coefficients are $(\frac{1}{720})y^V_nh^5$, i.e. of order $O(h^5)$ in our computation of $y_{n+1} - y_n$. Now our local truncation error is:

$$\tau_{n+1} = \frac{1}{h}(y_{n+1} - y_n) \implies \tau_{n+1} = \frac{1}{h}(\frac{1}{720})y_n^V h^5$$

Thus, the method is of fourth order $O(h^4)$.

4.2 5b - Show that the region of absolute stability contains the entire negative real axis of the $h\lambda$ plane.

We define $y^{'} = \lambda y, y(0) = y_0$, and we consider the method applied to this function. We will use notation of $f_n \equiv f^n$, which will not denote the nth derivative. We have that:

$$y_n^{''} = \frac{df(x_n, y_n)}{dx} + f(x_n, y_n)\frac{df(x_n, y_n)}{dx}.$$

$$f^n = \lambda y_n.$$

$$f_x^n = \lambda y_{'}(x_n) = \lambda f^n = \lambda^2 y_n.$$

$$f_y^n = \lambda.$$

We examine the method within this framework:

$$y_{n+1} = y_n + \frac{h}{2}[y_n^{'} + y_{n+1}^{'}] + \frac{h^2}{12}[y_n^{''} - y_{n+1}^{''}].$$

$$y_{n+1} = y_n + \frac{h}{2}[f^n + f^{n+1}] + \frac{h^2}{12}[f_x^n + f^n f_y^n - f_x^{n+1} + f^{n+1} f_y^{n+1}].$$

$$y_{n+1} = y_n + \frac{h}{2}[\lambda y_n + \lambda y_{n+1}] + \frac{h^2}{12}[\lambda^2 y_n + \lambda^2 y_n - \lambda^2 y_{n+1} - \lambda^2 y_{n+1}].$$

$$y_{n+1} = y_n + \frac{h\lambda}{2}[y_n + y_{n+1}] + \frac{h^2\lambda^2}{6}[y_n - y_{n+1}].$$

$$\implies (1 - \frac{1}{2}h\lambda + \frac{1}{6}h^2\lambda^2)y_{n+1} = (1 + \frac{1}{2}h\lambda + \frac{1}{6}h^2\lambda^2)y_n \implies (6 - 3h\lambda + h^2\lambda^2)y_{n+1} = (6 + 3h\lambda + h^2\lambda^2)y_n.$$

Thus we have our relation:

$$y_{n+1} = \frac{(6 + 3h\lambda + h^2\lambda^2)}{(6 - 3h\lambda + h^2\lambda^2)}y_n.$$

We verify the stability of the method for this function by studying:

$$y_n = (\frac{(6 + 3h\lambda + h^2\lambda^2)}{(6 - 3h\lambda + h^2\lambda^2)})^n y_0.$$

By definition and convention, we have step size of $h > 0$ and we define $\lambda < 0 \implies h\lambda < 0$. For the method to be stable, we find we need the condition:

$$\left|\frac{(6 + 3h\lambda + h^2\lambda^2)}{(6 - 3h\lambda + h^2\lambda^2)}\right| < 1.$$

$h\lambda < 0 \implies (6 - 3h\lambda + h^2\lambda^2) > 0$

So we have that

$$-1 < \frac{(6 + 3h\lambda + h^2\lambda^2)}{(6 - 3h\lambda + h^2\lambda^2)} < 1.$$

$$-1(6 - 3h\lambda + h^2\lambda^2) < (6 + 3h\lambda + h^2\lambda^2) < (6 - 3h\lambda + h^2\lambda^2).$$

We examine these inequalities and have that:

$$-1(6 - 3h\lambda + h^2\lambda^2) < (6 + 3h\lambda + h^2\lambda^2) \implies -6 < (6 + 2h^2\lambda^2).\checkmark$$

This inequality holds for all values of h, λ. For the second set of inequalities we find that:

$$(6 + 3h\lambda + h^2\lambda^2) < (6 - 3h\lambda + h^2\lambda^2) \implies 3h\lambda < -3h\lambda.\checkmark$$

This inequality holds for $h\lambda < 0$.

Thus

$$\left|\frac{(6 + 3h\lambda + h^2\lambda^2)}{(6 - 3h\lambda + h^2\lambda^2)}\right| < 1.$$

When $h > 0$, $\lambda < 0 \implies h\lambda < 0$ giving us our region of stability.

$\therefore$ The region of absolute stability contains the entire negative real axis of the $h\lambda$ plane.

5 Problem 6

5.1 a - REDACTED $T_{\frac{1}{2}h}(f) = \frac{1}{2}(T_h(f) + M_h(f))$.

5.2 b - Given $T_h(f) = I(f) + k_2h^2 + k_4h^4 + O(h^6)$, find similar rule for $M_h(f)$.

From part a we know that $2T_{\frac{1}{2}h}(f) = (T_h(f) + M_h(f)) \implies M_h(f) = 2T_{\frac{1}{2}h}(f) - T_h(f)$.

We are given:

$$T_h(f) = I(f) + k_2h^2 + k_4h^4 + O(h^6).$$

We compute:

$$T_{\frac{1}{2}h}(f) = I(f) + k_2\frac{1}{4}h^2 + k_4\frac{1}{16}h^4 + O(h^6).$$

$$2T_{\frac{1}{2}h}(f) - T_h(f) = 2I(f) + 2k_2\frac{1}{4}h^2 + 2k_4\frac{1}{16}h^4 + O(h^6) - I(f) - k_2h^2 - k_4h^4 - O(h^6)$$

.

$$\therefore 2T_{\frac{1}{2}h}(f) - T_h(f) = I(f) - k_2\frac{1}{2}h^2 - k_4\frac{7}{8}h^4 + O(h^6) = M_h.\checkmark$$

5.3 c -

6 Conclusion

Thank you to Prof. Hamfeldt, neé Froese, for reading this work, for her instruction, lectures and future office hours efforts. I look forward to any feedback and learning more of the material in this course.

References

[1] Kendall E Atkinson. *An introduction to numerical analysis.* John wiley & sons, 2008.