# BME 301

5-Complex Circuits

# A Series RC circuit with a Constant Time Source (Battery) Charging a Capacitor

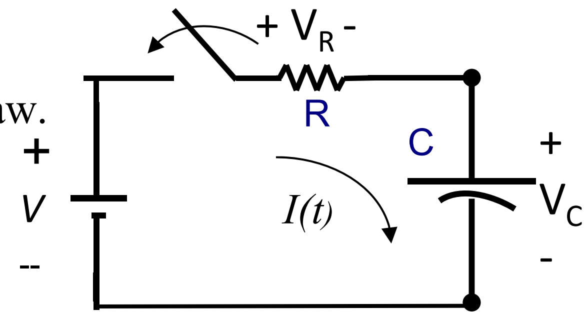
Since this is a Series Circuit, let's use KVL and we get

$V_{IN}(t) = V_R + V_C$ where $V_{IN}(t)$ is a battery or a constant voltage $= V$,

the voltage across the resistor is $V_R = IR$ from Ohm's Law and

the voltage across the capacitor is $V_C = \frac{1}{C}\int I\,dt$ from Gauss' Law.

Substituting we get: $V = IR + \frac{1}{C}\int I\,dt$.




At $t = 0$ when the switch is closed a current will flow to charge the capacitor until the voltage across it reaches $V$ volts.
At that point the current flow will end. So we say at t=0 the voltage across the capacitor is zero since there is no charge on its plates. Charge builds up across the capacitor its voltage equals the voltage of the source. This is called charging of a capacitor.

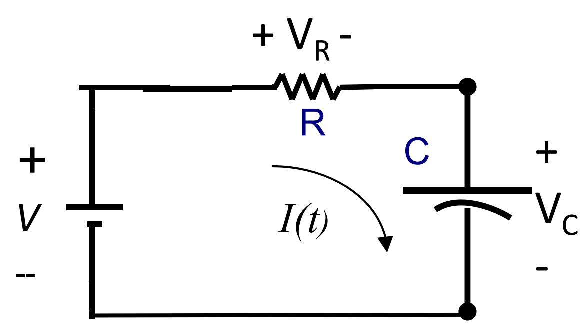
# A Series RC circuit with a Constant Time Source (Battery) Charging a Capacitor

To solve this 1st order differential equation; and since this is a linear equation then $I(t)$ must have the form $I(t) = K_1 + K_2 e^{-\frac{t}{\tau}}$;

where $K_1$ and $K_2$ are coefficents and constants and $\tau$ is called the time constant.

We can calculate can calculate the coefficents from the inital and final conditions.

$I(t) = K_1 + K_2 e^{-\frac{t}{\tau}}$;




The inital value of the current is $\frac{V}{R}$ and final value = 0.

$$I(0) = K_1 + K_2 = \frac{V}{R}$$

$$I(\infty) = K_1 = 0 \Rightarrow K_2 = \frac{V}{R}$$

It can also be shown that $\tau$, the time constant, equal to $RC$.

Therefore we get $I(t) = \frac{V}{R} e^{-\frac{t}{RC}}$.

# A Series RC circuit with a Constant Time Source (Battery) Charging a Capacitor

In a similar way we can calculate that the voltage across the capacitor is

$V_C(t) = K_1 + K_2 e^{-\frac{t}{RC}}$, where inital value of the voltage across the capacitor is zero and its final value is $V$.

where $K_1$ and $K_2$ are coefficents and constants and $\tau$ is called the time constant.

We can calculate can calculate the coefficents from the inital and final conditions.

$V_C(t) = K_1 + K_2 e^{-\frac{t}{\tau}};$

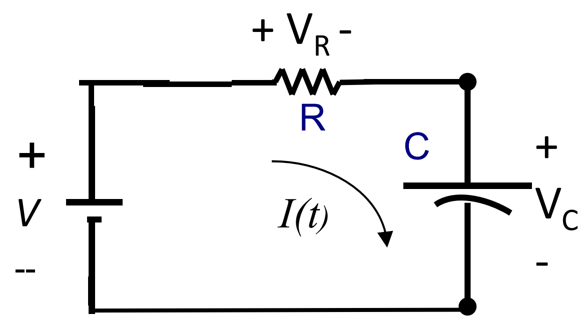
The inital value of the voltage is 0 and final value = $V$.

$V_C(0) = K_1 + K_2 = 0$

$V_C(\infty) = K_1 = V;$ therefore, $K_2 = -V$.

It can also be shown that $\tau$, the time constant, equal to $RC$.

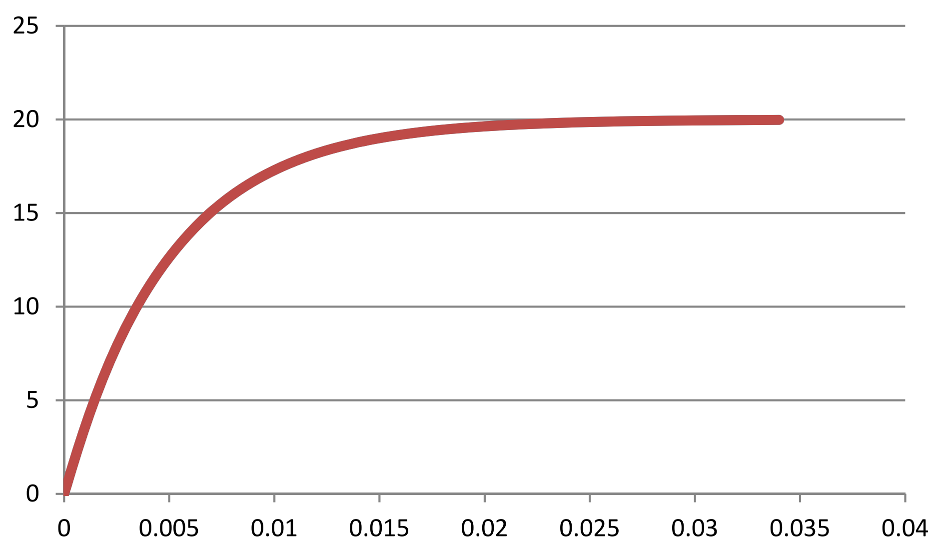
Therefore we get $V_C(t) = V(1 - e^{-\frac{t}{RC}})$.

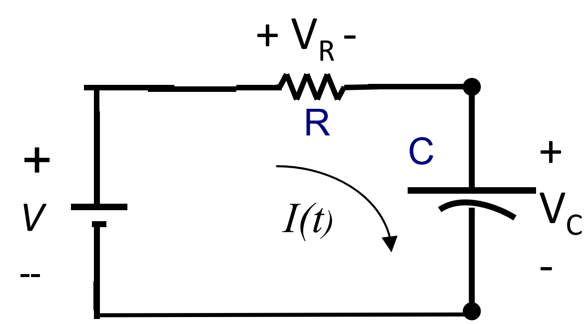


# A Series RC circuit with a Constant Time Source (Battery) Charging a Capacitor

*Note* : In general the time constant is equal the total resistance which is in series with the total capacitance.

Plotting $V_C(t) = V(1 - e^{-\frac{t}{RC}})$.



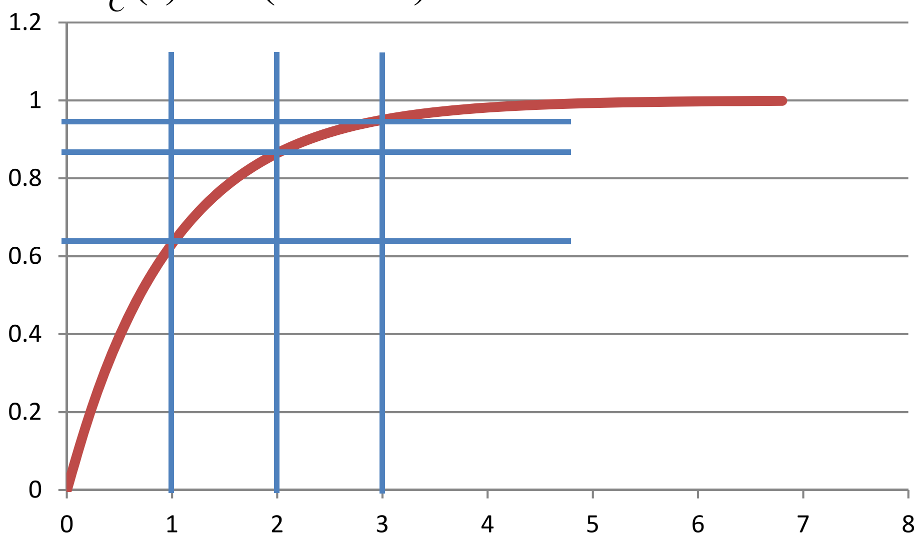



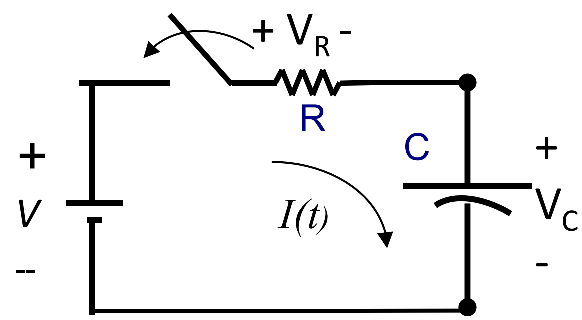

Voltage across the capacitor as a function of time for a circuit with R=1k ohm, C= 5 microfarads and V=20 volt.
Therefore, $\tau$, the time constant equals 0.005 second or 5 msec.

# A Series RC circuit with a Battery as the Source Charging a Capacitor

Voltage across the capacitor as a function of time
for a circuit with R=1 ohm, C= 1 farad and V=1 volt.
Therefore, $\tau$, the time constant equals 1 second.

$$V_C(t) = V(1 - e^{-\frac{t}{RC}}).$$




We say it takes 3 time constants to charge a capacitor.

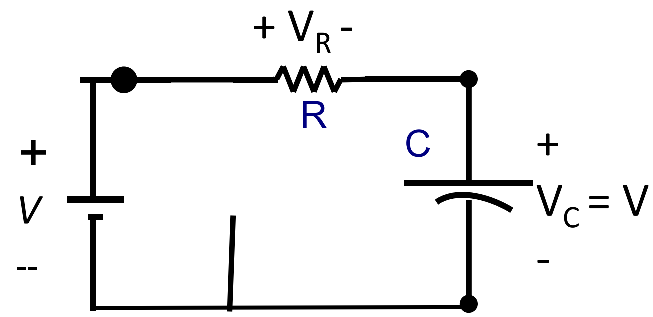
In 1 second, or 1 time constant, the voltage across the capacitor reaches 63% of it's final value.

In 2 seconds, or 2 time constants, the voltage across the capacitor reaches 83% of it's final value.

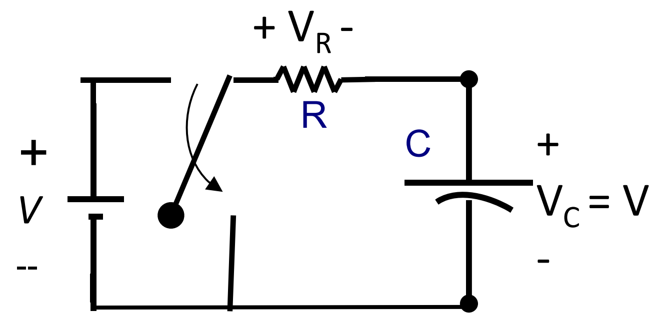
In 3 seconds, or 3 time constants, the voltage across the capacitor reaches 95% of it's final value.

# A Series RC circuit with a Battery as the Source Discharging a Capacitor

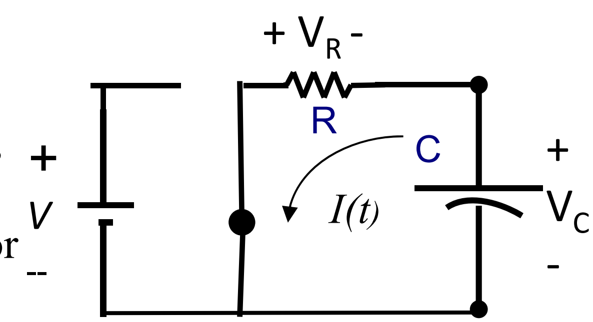
Let's assume that the capacitor has been charged up to $V$ volts and current is no longer flowing in the circuit. (Figure A)




At $t = 0$ the switch is thrown. (Figure B)




The capacitor which has charge across it (or a voltage across it) and with the switch thrown, there is a path for charge to flow out of the capacitor and through the resistor. Since the resistor does not store energy the current flow is disspated to heat until all of the charge is gone from the capacitor. (Figure C)




The equation of this circuit is $V_C + V_R = 0 \Rightarrow \frac{1}{C}\int I\,dt + I = 0$

Again $I(t) = K_1 + K_2 e^{-\frac{t}{\tau}}$; with the inital value of the current is

$\frac{V}{R}$ and final value $= 0$ and $I(t) = \frac{V}{R} e^{-\frac{t}{RC}}$.

In a similar way we can calculate that the voltage across the capacitor is

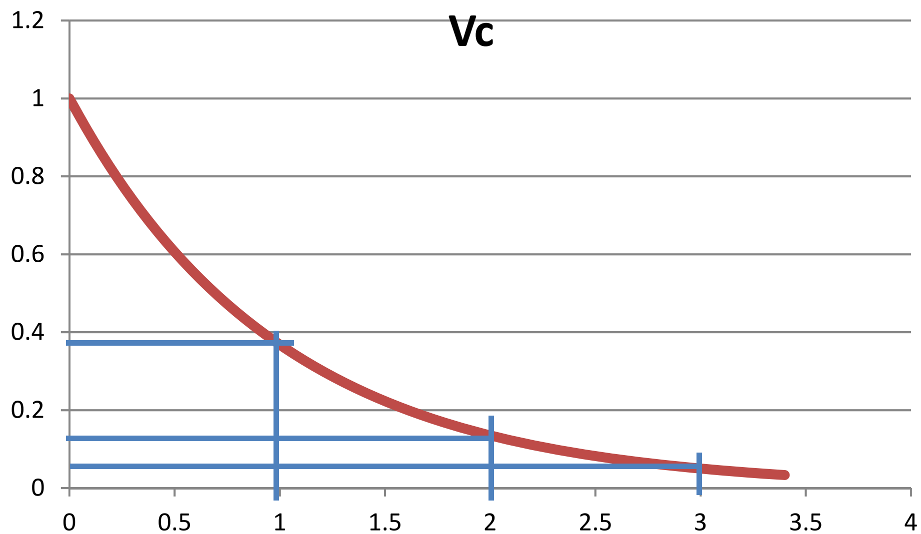
$V_C(t) = K_1 + K_2 e^{-\frac{t}{RC}}$, where inital value of the voltage across the capacitor

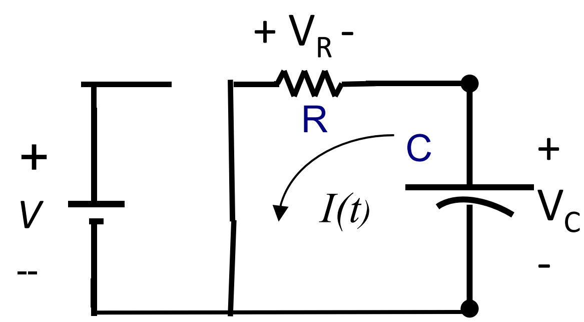
is $V$ and its final value is 0. And $V_C(t) = V e^{-\frac{t}{RC}}$.

# A Series RC circuit with a Battery as the Source Discharging a Capacitor

Voltage across the capacitor as a function of time for a circuit with R=1 ohm, C= 1 farad and V=1 volt. Therefore, $\tau$, the time constant equals 1 second.

$$V_C(t) = Ve^{-\frac{t}{RC}}.$$




We say it takes 3 time constants to discharge a capacitor.

In 1 second, or 1 time constant, the voltage across the capacitor reaches 37% of it's initial value.

In 2 seconds, or 2 time constants, the voltage across the capacitor reaches 13% of it's initial value.

In 3 seconds, or 3 time constants, the voltage across the capacitor reaches 5% of it's initial value.

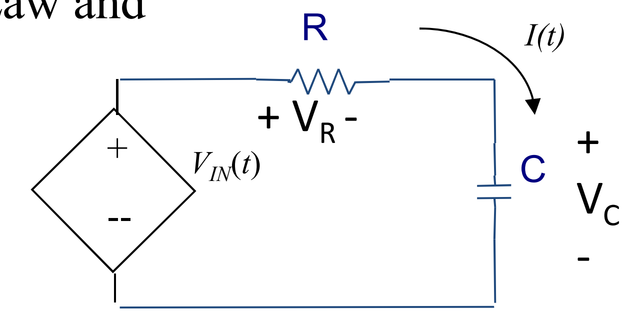
# A Series RC circuit with a Time Varying Source

Since this is a Series Circuit, let's use KVL and we get:

$V_{IN}(t) = V_R + V_C$ where $V_R = IR$ from Ohm's Law and

$V_C = \frac{1}{C}\int I dt$ from Gauss' Law

$V_{IN}(t) = IR + \frac{1}{C}\int I dt$




To solve this differential equation, let's assume that

$V_{IN}(t) = Ve^{5t}$; since this is a linear system then $I(t)$ must

also have the form $Ae^{5t}$ where A is unknown.

Substituting for $I$ and $V_{IN}(t)$ we get

$$Ve^{5t} = Ae^{5t}R + \frac{1}{5C}Ae^{5t}$$

$$V = A(R + \frac{1}{5C}) \Rightarrow A = \frac{V}{R + \frac{1}{5C}} \Rightarrow I(t) = \frac{V}{R + \frac{1}{5C}}e^{5t}$$

# Do we always have to solve Differential Equation?

There is a simpler way.

But first some assumptions.

Let's assume that most signals are of the form $V(t) = Ae^{st}$.

Then the current through a resistor is $I(t) = \dfrac{V(t)}{R} = \dfrac{Ae^{st}}{R}$ and $\dfrac{V(t)}{I(t)} = \dfrac{Ae^{st}}{\dfrac{Ae^{st}}{R}} = R$

Then the current through a capacitor is $I(t) = C\dfrac{dV(t)}{dt} = C\dfrac{d}{dt}Ae^{st} = sCAe^{st}$ and $\dfrac{V(t)}{I(t)} = \dfrac{Ae^{st}}{sCAe^{st}} = \dfrac{1}{sC}$

Then the current through an inductor is $I(t) = \dfrac{1}{L}\int V(t)dt = \dfrac{1}{L}\int Ae^{st}dt = \dfrac{1}{sL}Ae^{st}$ and $\dfrac{V(t)}{I(t)} = \dfrac{Ae^{st}}{\dfrac{1}{sL}Ae^{st}} = sL$

If we define $\dfrac{V(t)}{I(t)}$ as the impedance, then the impedance of

a resistor, $\dfrac{V(t)}{I(t)} = R$,

a capacitor $\dfrac{V(t)}{I(t)} = \dfrac{1}{sC}$, and

an inductor $\dfrac{V(t)}{I(t)} = sL$.

# Do we always have to solve Differential Equation? continued

There is a simpler way.

Now let's assume that most signals are either sinusoids or related to sinusoids.

So $V(t) = A\cos(\omega t)$; however, from Euler's formula we know that

$\cos(\theta) = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$ and that $\cos(\omega t) = \dfrac{e^{j\omega t} + e^{-j\omega t}}{2}$ where $j$ is the imaginary number $\sqrt{-1}$

So to make things simpler, let's concentrate on signals which have the form

$V(t) = Ve^{j\omega t}$

Then the impedance of

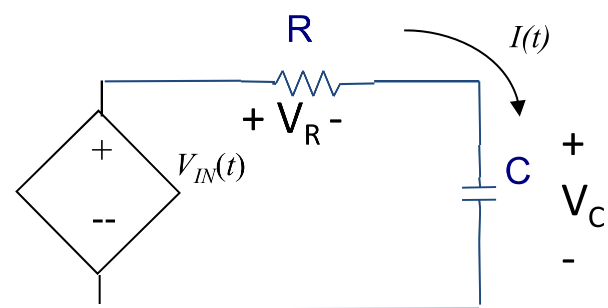
a resistor, $\dfrac{V(t)}{I(t)} = R$,

a capacitor $\dfrac{V(t)}{I(t)} = \dfrac{1}{j\omega C}$, and

an inductor $\dfrac{V(t)}{I(t)} = j\omega L$.

# Do we always have to solve Differential Equation? continued

Since this is a Series Circuit, let's use KVL and use impedances to solve for the current, we get:




$V_{IN}(t) = V_R(t) + V_C(\mathrm{t})$ where $V_R(t) = I(t)R$ and

$$V_C(t) = \frac{1}{C}\int I dt = \frac{1}{j\omega C} I(t)$$

$$V_{IN}(t) = I(t)R + \frac{1}{j\omega C} I(t) = (R + \frac{1}{j\omega C}) I(t)$$

This is just an arithmetic equation but with complex numbers, since

$$I(t) = \frac{V_{IN}(t)}{(R + \frac{1}{j\omega C})}$$

Before we go any further, let's review complex numbers.

# Complex Numbers

- Complex numbers: What are they?
- What is the solution to this equation?

$$ax^2+bx+c=0$$

- This is a second order equation whose solution is:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

# What is the solution to?

1. $x^2+4x+3=0$

$$x_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4\times 3}}{2} = \frac{-4 \pm \sqrt{16-12}}{2}$$

$$= \frac{-4 \pm \sqrt{4}}{2} = \frac{-4 \pm 2}{2} = -1, -3$$

# What is the solution to?

*2.* $x^2+4x+5=0$

$$x_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4 \times 5}}{2} = \frac{-4 \pm \sqrt{16 - 20}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2} \quad ?????$$

# What is the Square Root of a Negative Number?

- We define the square root of a negative number as an imaginary number
- We define

$$\sqrt{-1} \Rightarrow j \text{ for engineers } (i \text{ for mathematicans})$$

- Then our solution becomes:

$$x_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4 \times 5}}{2} = \frac{-4 \pm \sqrt{16 - 20}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm j\sqrt{4}}{2} = \frac{-4 \pm j2}{2} = -2 + j1, -2 - j1$$

# The Complex Plane

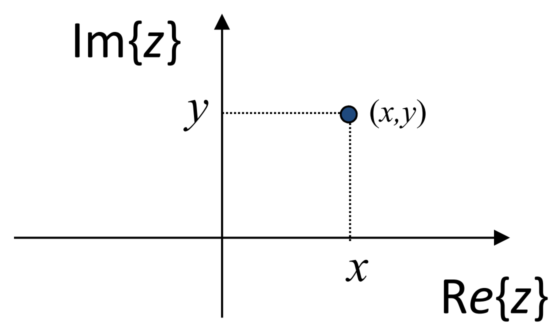
- $z = x+jy$ is a complex number where:

  $x = \text{Re}\{z\}$ is the real part of $z$

  $y = \text{Im}\{z\}$ is the imaginary part of $z$

- We can define the complex plane and we can define 2 representations for a complex number:

$$z = x+jy$$



# Rectangular Form

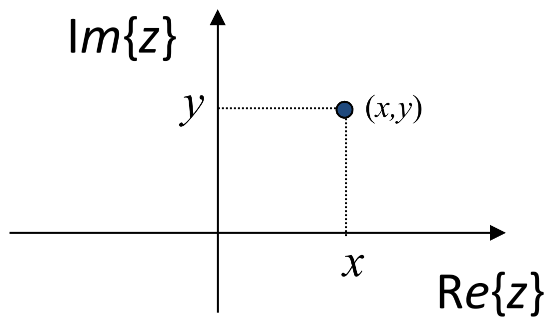
- Rectangular (or cartesian) form of a complex number is given as

$$z = x + jy$$

$x = \mathrm{Re}\{z\}$ is the real part of $z$

$y = \mathrm{Im}\{z\}$ is the imaginary part of $z$

$z = x + jy$

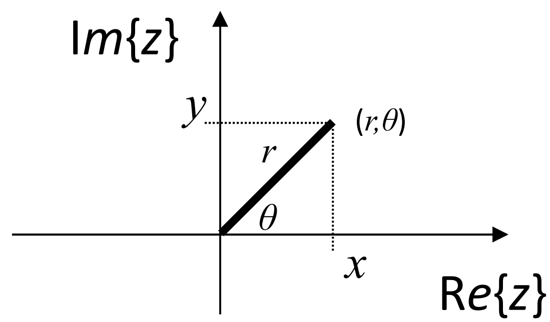



Rectangular or Cartesian

# Polar Form

- $z = re^{j\theta} = r\angle\theta$ is a complex number where:
- $r$ is the magnitude of $z$
- $\theta$ is the angle or argument of $z$ (arg $z$)

$$z = r\,e^{j\theta}$$




Polar

# Relationships between the Polar and Rectangular Forms

$$z = x + jy = r\,e^{j\theta}$$

- Relationship of Polar to the Rectangular Form:

$$x = \mathrm{R}e\{z\} = r\cos\theta$$

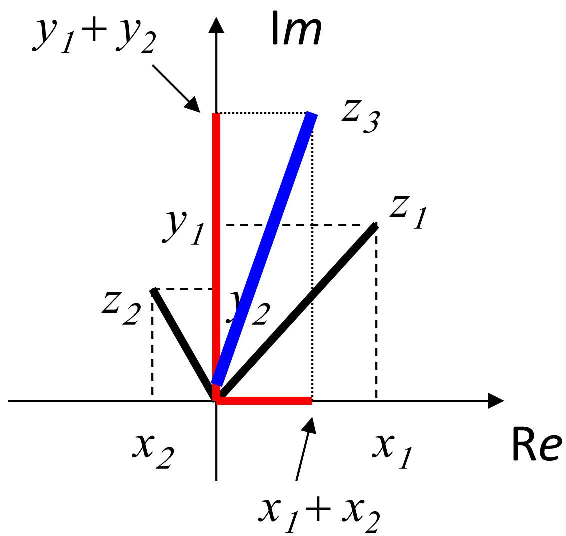
$$y = \mathrm{I}m\{z\} = r\sin\theta$$

- Relationship of Rectangular to Polar Form:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arctan(\frac{y}{x})$$

# Addition of 2 complex numbers

- When two complex numbers are added, it is best to use the rectangular form.
- The real part of the sum is the sum of the real parts and imaginary part of the sum is the sum of the imaginary parts.
- Example: $z_3 = z_1 + z_2$




$$z_1 = x_1 + jy_1; z_2 = x_2 + jy_2$$

$$z_3 = z_1 + z_2 = x_1 + jy_1 + x_2 + jy_2$$

$$= x_1 + x_2 + jy_1 + jy_2$$

$$= (x_1 + x_2) + j(y_1 + y_2)$$

# Multiplication of 2 complex numbers

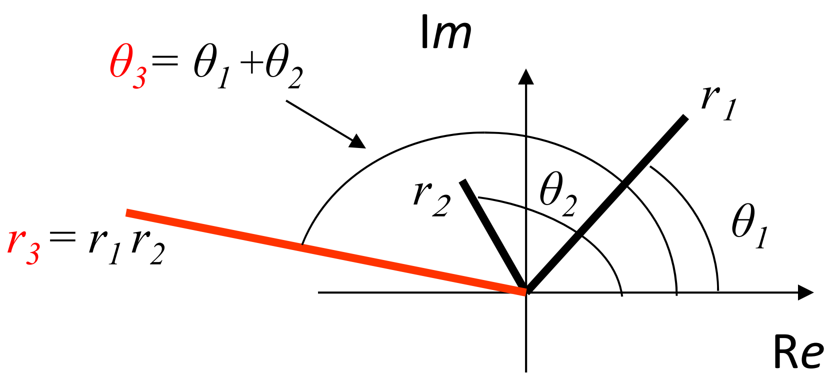
- When two complex numbers are multiplied, it is best to use the polar form:
- Example: $z_3 = z_1 \times z_2$

$$z_1 = r_1 e^{j(\theta_1)}; z_2 = r_2 e^{j(\theta_2)}$$

$$z_3 = z_1 \times z_2 = r_1 e^{j(\theta_1)} \times r_2 e^{j(\theta_2)}$$

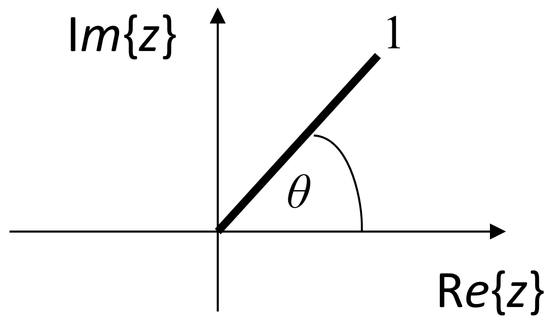
$$= r_1 r_2 e^{j(\theta_1)} e^{j(\theta_2)} = r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

- We multiply the magnitudes and add the phase angles



# Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$




- We can use Euler's Formula to define complex numbers

$$z = r\,e^{j\theta} = r\cos\theta + j\,r\sin\theta$$
$$= x + j\,y$$

# Some shorthand

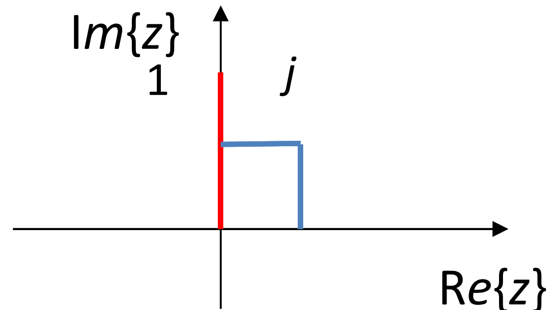
Instead of this form $Ae^{j\theta}$

we can write phasors using this shorthand

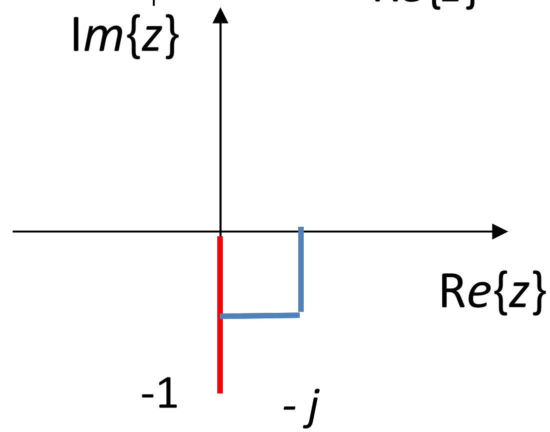
$$Ae^{j\theta} \Rightarrow A\angle\theta$$

# Some Examples

$$j = e^{j\frac{\pi}{2}} = 1\angle\frac{\pi}{2}$$

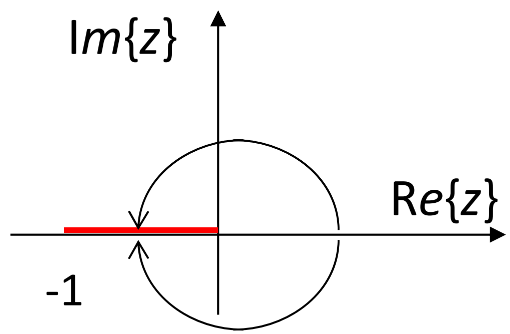



$$\frac{1}{j} = \frac{j}{j}\frac{1}{j} = \frac{j}{-1} = -j = e^{-j\frac{\pi}{2}} = 1\angle-\frac{\pi}{2}$$

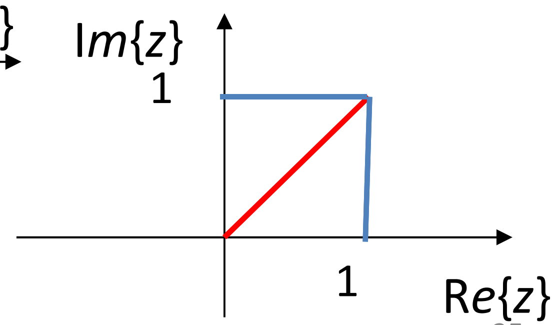



$$-1 = e^{j\pi} = 1\angle\pi$$

$$-1 = e^{-j\pi} = 1\angle-\pi$$




$$1 + j = \sqrt{1^2 + 1^2}\angle\tan^{-1}(1) = \sqrt{2}\angle\frac{\pi}{4}$$



# Some Examples

$$2e^{-j\pi/4} + 2e^{j\pi/2}$$

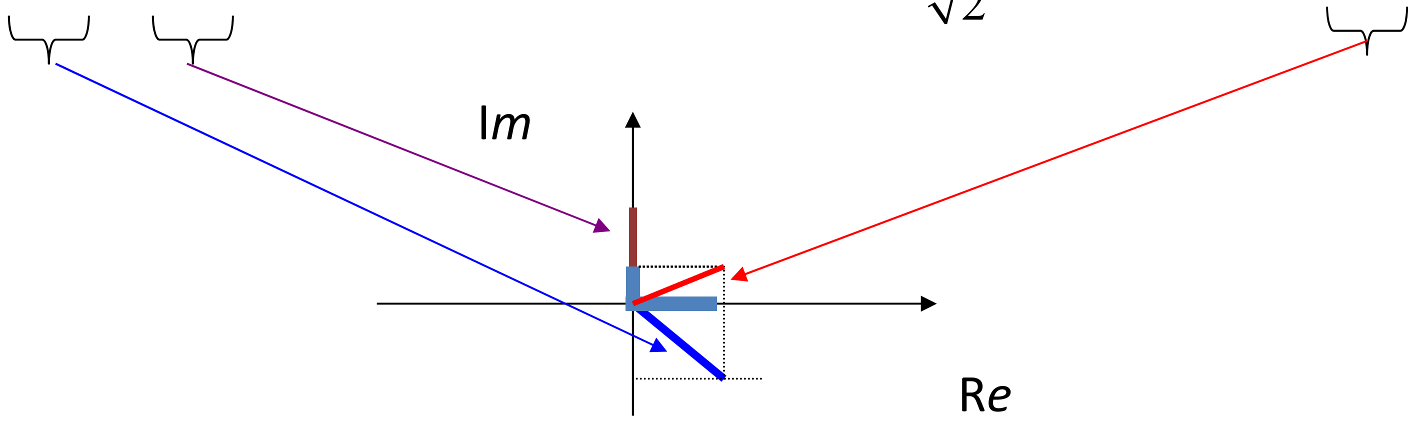
$$1)\ 2e^{-j\pi/4} = 2[\cos(-\pi/4) + j\sin(-\pi/4)]$$

$$= 2[\cos(\pi/4) - j\sin(\pi/4)] = 2[\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}] = \sqrt{2} - j\sqrt{2}$$

$$2)\ 2e^{j\pi/2} = 0 + 2j$$

$$3)\ \sqrt{2} - j\sqrt{2} + 0 + 2j = \sqrt{2} - j(1.41-2) = \sqrt{2} + j(1.41-2) = \sqrt{2} + j0.586$$

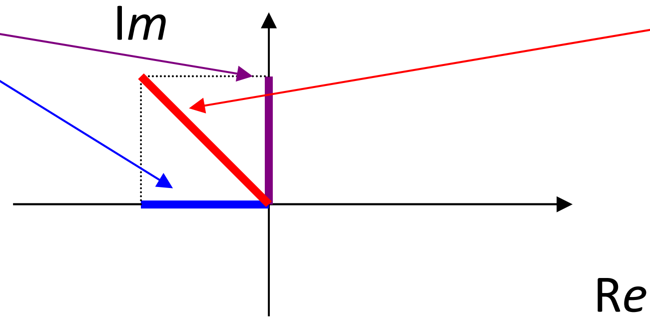
$$\therefore \underbrace{2e^{-j\pi/4}} + \underbrace{2e^{j\pi/2}} = \sqrt{(\sqrt{2})^2 + (0.586)^2}\angle\tan^{-1}(\frac{0.586}{\sqrt{2}}) = 1.53\angle\underbrace{\tan^{-1}(0.41)}$$




$$\text{Note } \frac{\pi}{4} = 0.785; \frac{\pi}{8} = 0.392; 0.692 \Rightarrow 22.5^\circ$$

# Some Examples

$$5e^{j\frac{\pi}{2}} + 5e^{j\pi} = j5 - 5 = 5(-1 + j) = 5 \times \sqrt{(-1)^2 + 1^2}\, e^{j\tan^{-1}(-1)} = 5\sqrt{2}e^{j\frac{3\pi}{4}}$$



# Complex Exponential Signals

- Since we are dealing with sinusoid signal let's look at the complex exponential ***signal*** which is defined as:

$$z(t) = Ae^{j(\omega t+\theta)}$$

- Note that it is defined in polar form where
  - the magnitude of $z(t)$ is $|z(t)| = A$
  - the angle (or argument, arg $z(t)$ ) of $z(t) = (\omega_o t + \theta)$
    - Where $\omega$ is called the radian frequency and $\theta$ is the phase angle (phase shift)

# Complex Exponential Signals

- Note that by using Euler's formula, we can rewrite the complex exponential signal in rectangular form as:

$$z(t) = Ae^{j(\omega t+\theta)}$$
$$= A\cos(\omega t + \theta) + jA\sin(\omega t + \theta)$$

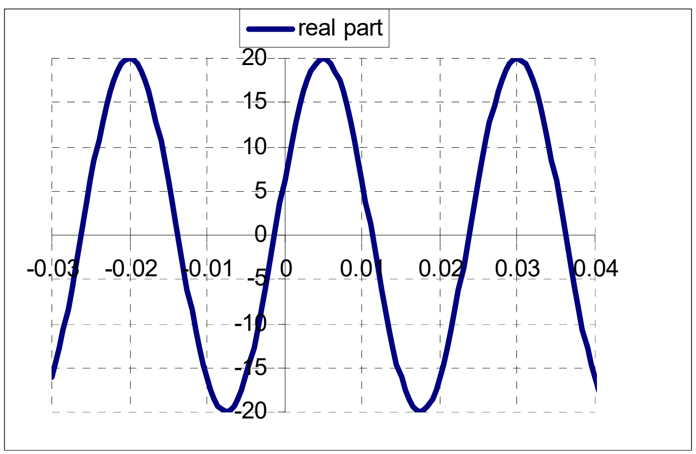
- Therefore real part is the cosine signal and imaginary part is a sine signal both of radial frequency $\omega$ and phase angle of $\theta$.

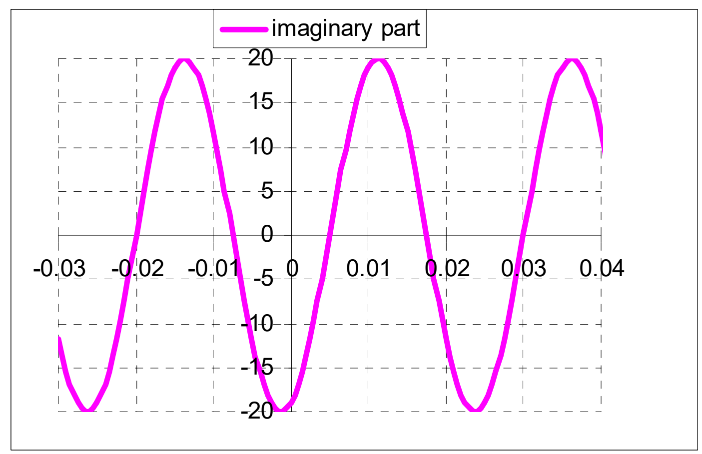
## Plotting the waveform of a complex exponential signal

- For an complex signal, we plot the real part and the imaginary part separately.
- Example:

$$z(t) = 20e^{j(2\pi(40)t-0.4\pi)} = 20e^{j(80\pi t-0.4\pi)}$$

$$= 20\cos(80\pi t-0.4\pi) + j20\sin(80\pi t-0.4\pi)$$



# NOTE!!!!

- The reason why we prefer the complex exponential representation of the real cosine signal:

$$x(t) = \Re e\{z(t)\} = \Re e\{Ae^{j(\omega t+\theta)}\}$$
$$= A\cos(\omega t + \theta)$$

- In solving equations and making other calculations, it easier to use the complex exponential form and then take the Real Part.

# Phasor Representation of a Complex Exponential Signal

- Using the multiplication rule, we can rewrite the complex exponential signal as

$$z(t) = Ae^{j(\omega t+\theta)} = Ae^{j\omega t}e^{j\theta} = Ae^{j\theta}e^{j\omega t} = \mathbf{X}e^{j\omega t}$$

where **X** is a complex number equal to

$$\mathbf{X} = Ae^{j\theta}$$

- **X** is complex amplitude of the complex exponential signal and is also called a phasor

# Rotating Phasor

- Let's look at this $z(t) = Ae^{j(2\pi t+\frac{\pi}{4})} = Ae^{j\frac{\pi}{4}}e^{j2\pi t} = \mathbf{X}e^{j2\pi t}$

$$\mathbf{X} = Ae^{j\frac{\pi}{4}}$$

*t=0 seconds*

*arg(z(t))=2π x0+π/4 = π/4*

*t=1/8 seconds*

*arg(z(t))=2π x1/8+π/4 = π/2*

*t=1/4 seconds*

*arg(z(t))=2π x1/4+π/4 = 3π/4*

*t=3/8 seconds*

*arg(z(t))=2π x3/8+π/4 = 4π/4 = π*

*t=1/2 seconds*

*arg(z(t))=2π x1/2+π/4 = 5π/4*

*t=5/8 seconds*

*arg(z(t))=2π x5/8+π/4 = 6π/4 = 3π/2*

*t=3/4 seconds*

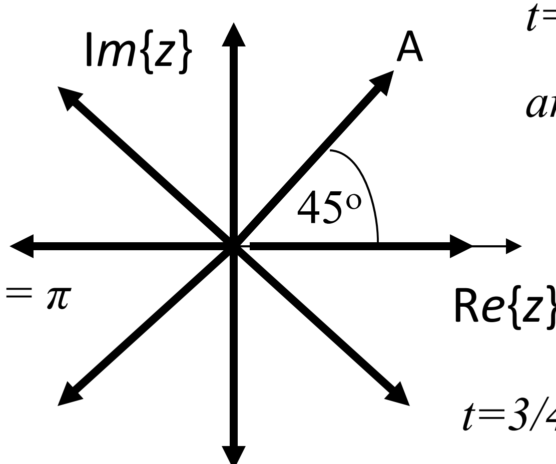
*arg(z(t))=2π x3/4+π/4 = 7π/4*

*t=7/8 seconds*

*arg(z(t))=2π x7/8+π/4 = 8π/4 = 2π*

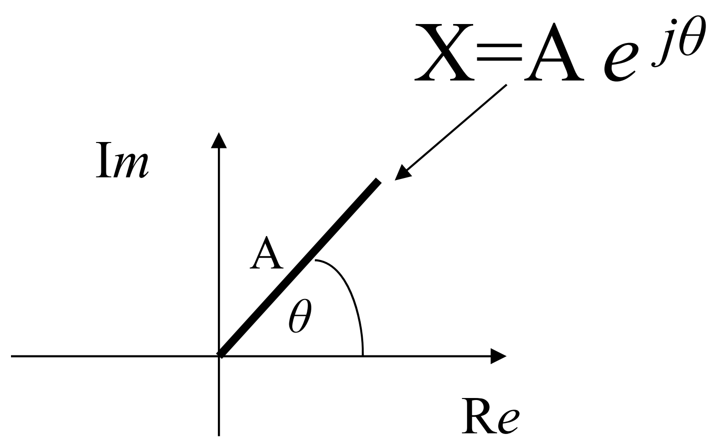
*t=1 seconds*

*arg(z(t))=2π x1+π/4 = 9π/4 = π/4*



# Graphing a phasor

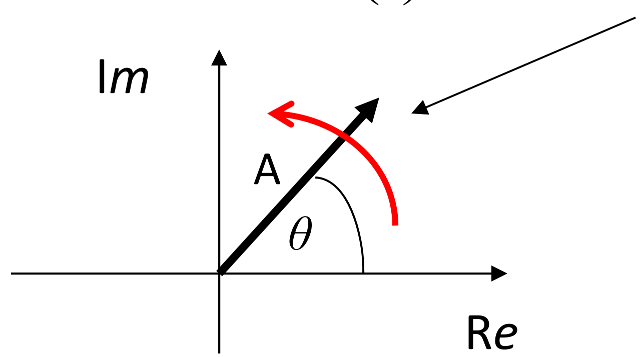
- **X**=A $e^{j\theta}$ can be graphed in the complex plane with magnitude A and angle $\theta$:



# Graphing a Complex Signal in terms of its phasors

- Since a complex signal, $z(t)$, is a phasor multiplying a complex exponential signal $e^{j\omega t}$, then a complex signal can be viewed as a phasor rotating in time:

$$z(t) = Ae^{j(\omega t+\theta)} = \mathbf{X}e^{j\omega t}$$



# Voltage and Current Relationship for a Capacitor

Let's assume that $\mathbf{I}$ is a sinusoidal signal and is represented by a complex exponential signal that has zero phase angle, $\mathbf{I} = Ie^{j\omega t}$

Let's calculate the voltage across a capacitor and resistor using the impedance for each.

–For a capacitor $\mathbf{Z}_C = \dfrac{1}{j\omega C} = \dfrac{1}{\omega C\angle\frac{\pi}{2}} = \dfrac{1}{\omega C}\angle -\dfrac{\pi}{2} \Rightarrow \dfrac{1}{\omega C}e^{-j\frac{\pi}{2}}$

$\mathbf{V} = \mathbf{I}\mathbf{Z}_C = Ie^{j\omega t}\dfrac{1}{\omega C}e^{-j\frac{\pi}{2}} = \dfrac{I}{\omega C}e^{j\omega t - \frac{\pi}{2}};$

Since the phase angle of the voltage is $-\dfrac{\pi}{2}$; we say the voltage across a capacitor ***lags*** the current through it by $\dfrac{\pi}{2}$ or $90^\circ$.

# Voltage and Current Relationship for a Resistor and Inductor

– For a resistor $\mathbf{Z}_R = R = R\angle 0 = Re^{j0}$

$\mathbf{V} = \mathbf{IZ}_R = Ie^{j\omega t}Re^{j0} = IRe^{j\omega t+0} = IRe^{j\omega t}$;

Since the phase angle of the voltage is 0; we say the voltage across a resistor is ***in phase*** with the current through it.

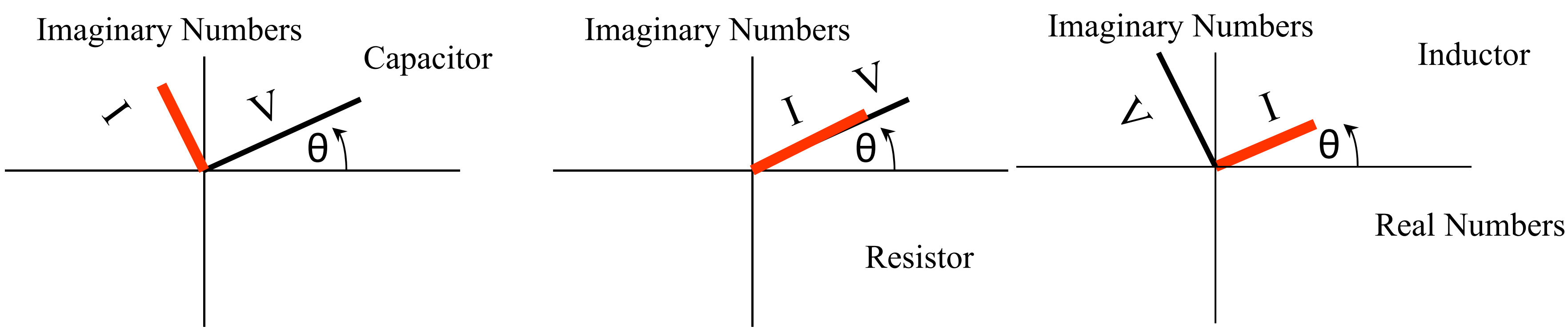
– For an inductor $\mathbf{Z}_L = j\omega L = \omega L\angle\frac{\pi}{2} = \omega Le^{j\frac{\pi}{2}}$

$\mathbf{V} = \mathbf{IZ}_L = Ie^{j\omega t}\omega Le^{j\frac{\pi}{2}} = I\omega Le^{j\omega t+\frac{\pi}{2}}$;

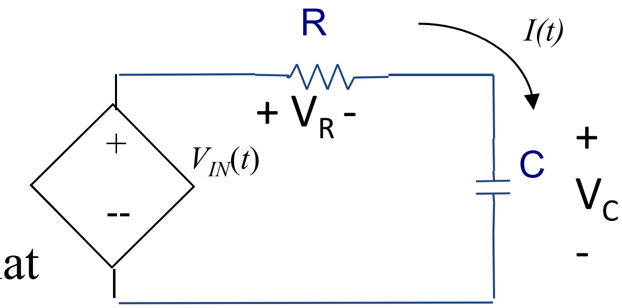
Since the phase angle of the voltage is $\frac{\pi}{2}$; we say the voltage across an inductor ***leads*** the current through it by $\frac{\pi}{2}$ or 90 degrees.

# Sinusoidal Steady State Continued

- For a capacitor, $Z_C = \dfrac{1}{j\omega C} \Rightarrow \dfrac{1}{\omega C} \angle -\dfrac{\pi}{2}.$
- For a resistor, $Z_R = R \Rightarrow R\angle 0$
- For an inductor, $Z_L = j\omega L \Rightarrow \omega L \angle \dfrac{\pi}{2}$



# Let’s Revisit the Series RC circuit




$$V_{IN}(t) = IR + \frac{1}{C}\int I\,dt$$

To solve this differential equation, let's assume that

$V_{IN}(t) = V\cos\omega t;$ since this is a linear system then $I(t)$ must

also have the form $A\cos(\omega t + \theta)$ where A and $\theta$ are unknowns.

Substituting for $I$ and $V_{IN}(t)$ we get

$$V\cos\omega t = A\cos(\omega t + \theta)R + \frac{1}{\omega C}A\sin(\omega t + \theta)$$

To solve this we have to use trigomentic idenities such as

$\cos\ (a\ +\ b)\ =\ \cos\ a\ \cos\ b\ -\ \sin\ a\ \sin\ b$

$\sin\ (a\ +\ b)\ =\ \sin\ a\ \cos\ b\ +\ \cos\ a\ \sin\ b$

You try it!!!

# Let's Revisit the Series RC circuit

$$V\cos\omega t = A\cos(\omega t+\theta)R + \frac{1}{\omega C}A\sin(\omega t+\theta)$$

$$V\cos\omega t = AR(\cos\ \omega t\ \cos\ \theta\ -\ \sin\ \omega t\ \sin\ \theta) + \frac{1}{\omega C}A(\sin\ \omega t\ \cos\ \theta\ +\ \cos\ \omega t\ \sin\ \theta)$$

$$V\cos\omega t = AR\cos\ \omega t\ \cos\ \theta + \frac{1}{\omega C}A\cos\ \omega t\ \sin\ \theta - AR\sin\ \omega t\ \sin\ \theta + \frac{1}{\omega C}A(\sin\ \omega t\ \cos\ \theta)$$

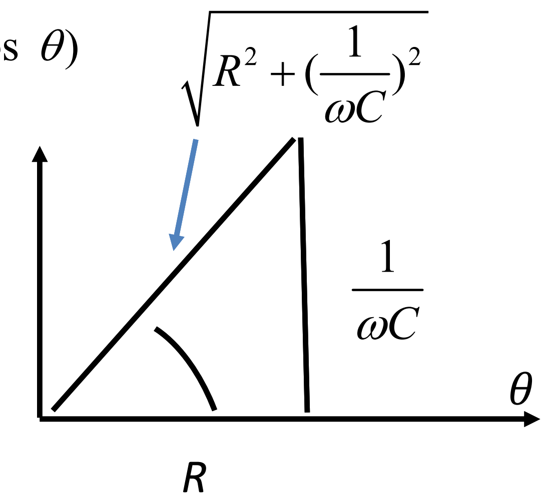
$$V\cos\omega t = A\cos\ \omega t(R\ \cos\ \theta + \frac{1}{\omega C}\ \sin\ \theta) - A\sin\ \omega t(R\ \sin\ \theta - \frac{1}{\omega C}\ \cos\ \theta)$$

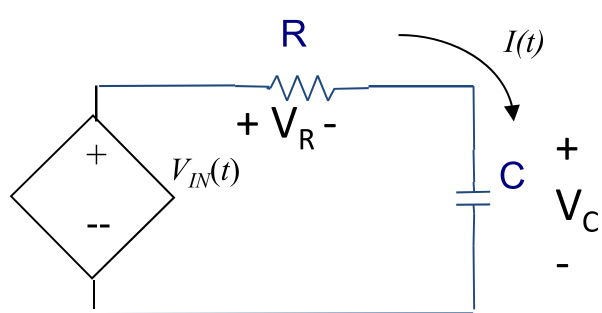
$$V = A(R\ \cos\ \theta + \frac{1}{\omega C}\sin\ \theta);(R\ \sin\ \theta - \frac{1}{\omega C}\ \cos\ \theta) = 0$$

$$A = \frac{V}{(R\ \cos\ \theta + \frac{1}{\omega C}\sin\ \theta)};R\ \sin\ \theta = \frac{1}{\omega C}\ \cos\ \theta$$

$$\theta = \tan^{-1}(\frac{1}{\omega RC});A = \frac{V}{\sqrt{R^2+(\frac{1}{\omega C})^2}}$$

$$\text{since } \cos\ \theta = \frac{R}{\sqrt{R^2+(\frac{1}{\omega C})^2}};\sin\ \theta = \frac{\frac{1}{\omega C}}{\sqrt{R^2+(\frac{1}{\omega C})^2}}$$



# Let's Revisit the Series RC circuit

Now let's use complex exponentials.

$V_{IN}(t) = IR + \frac{1}{C}\int I\,dt$

To solve this differential equation, let's assume that

$V_{IN}(t) = Ve^{j\omega t}$; since this is a linear system then $I(t)$ must also have the form $Ae^{j(\omega t+\theta)}$ where A and $\theta$ are unknowns.

Substituting for $I$ and $V_{IN}(t)$ we get

$$Ve^{j\omega t} = Ae^{j(\omega t+\theta)}R + \frac{1}{C}\frac{Ae^{j(\omega t+\theta)}}{j\omega}$$

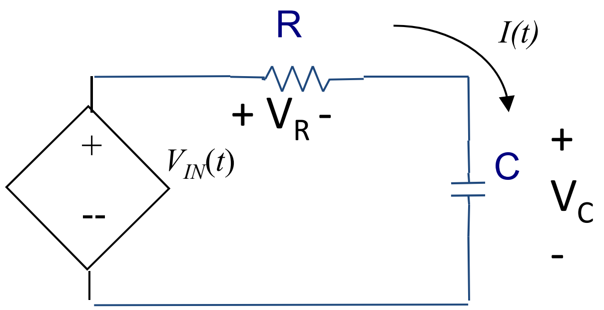
$$Ve^{j\omega t} = Ae^{j\theta}e^{j\omega t}R - j\frac{1}{\omega C}Ae^{j\theta}e^{j\omega t}$$

$$V = Ae^{j\theta}R - j\frac{1}{\omega C}Ae^{j\theta}$$

$$V = (R - j\frac{1}{\omega C})Ae^{j\theta}$$

$$Ae^{j\theta} = \frac{V}{R - j\frac{1}{\omega C}} = \frac{V}{\sqrt{R^2 + (\frac{1}{\omega C})^2}\,e^{j-\tan^{-1}(\frac{1}{\omega RC})}} = \frac{V}{\sqrt{R^2 + (\frac{1}{\omega C})^2}}e^{j\tan^{-1}(\frac{1}{\omega RC})}$$

$$\therefore A = \frac{V}{\sqrt{R^2 + (\frac{1}{\omega C})^2}}; \theta = \tan^{-1}(\frac{1}{\omega RC})$$




Therefore the current is

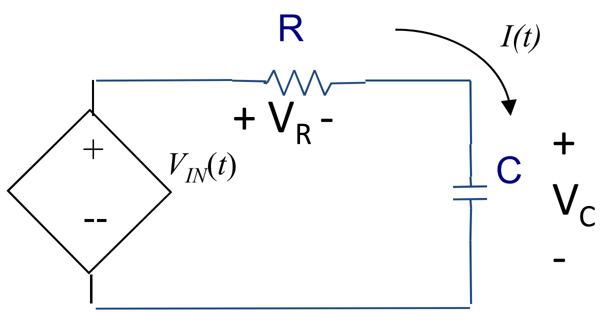
$$I(t) = \frac{V}{\sqrt{R^2 + (\frac{1}{\omega C})^2}}\cos(\omega t + \tan^{-1}(\frac{1}{\omega RC}))$$

and the voltage is

$V(t) = V\cos(\omega t)$

since we represented it as complex exponental $V(t) = Ve^{j\omega t}$.

# Let's Revisit the Series RC circuit




Now let's use phasors and impedances.

Recall the phasor $Ae^{j(\omega t+\theta)}$ is $Ae^{j\theta} = A\angle\theta$

$$V_{IN}(t) = IR + \frac{1}{C}\int Idt$$

To solve this differential equation, let's assume that

$V_{IN}(t) = Ve^{j\omega t} \Rightarrow \mathbf{V} = V\angle 0; \;\; I(t) = Ae^{j(\omega t+\theta)} \Rightarrow \mathbf{I} = A\angle\theta$ where A and $\theta$ are unknowns.

Using impedances and KVL we get

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_R + \mathbf{I}\mathbf{Z}_C = (\mathbf{Z}_R + \mathbf{Z}_C)\mathbf{I} = (R + \frac{1}{j\omega C})\mathbf{I} = (R - j\frac{1}{\omega C})\mathbf{I}$$

$$\mathbf{I} = \frac{\mathbf{V}}{(R - j\frac{1}{\omega C})} = \frac{\mathbf{V}}{\sqrt{R^2 + (\frac{1}{\omega C})^2}\angle -\tan^{-1}(\frac{1}{\omega C})} = \frac{V}{\sqrt{R^2 + (\frac{1}{\omega C})^2}}\angle \tan^{-1}(\frac{1}{\omega RC})$$

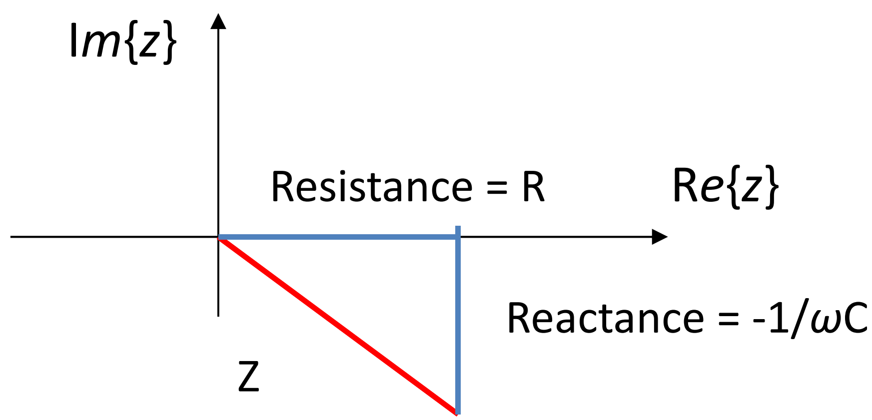
From this we get the same answer for the current but a lot less work.

$$I(t) = \frac{V}{\sqrt{R^2 + (\frac{1}{\omega C})^2}}\cos(\omega t + \tan^{-1}(\frac{1}{\omega RC}))$$

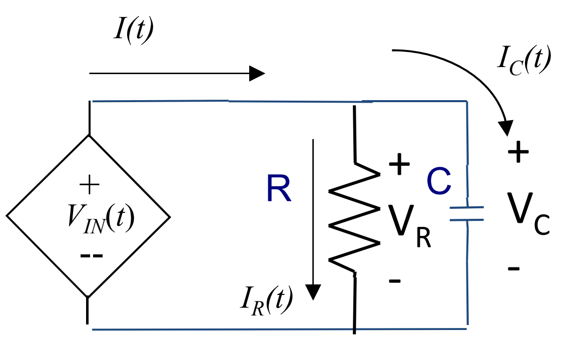
# Let's look at the Impedance of this example

$$Z = (R - j\frac{1}{\omega C})$$

In general, the real part of the impedance is resistive part of the impedance.
However, the imaginery part of the impedance is call the reactance and represents the contribution of the capacitor (inductor).
In this case, the reactance is inversely proportional to the frequency of the sinusoid.
We can plot it in the complex plan as:



# Parallel




Note that R and C are in parallel and $V_R = V_C$.

Finally $I(t) = I_R(t) + I_C(t)$

Using impedances and phasors

$$\mathbf{I}_R = \frac{\mathbf{V}}{R}$$

$$\mathbf{I}_C = \frac{\mathbf{V}}{\frac{1}{j\omega C}} = j\omega C\mathbf{V}$$

$$\mathbf{I} = \mathbf{I}_R + \mathbf{I}_C = \frac{\mathbf{V}}{R} + j\omega C\mathbf{V} = (\frac{1}{R} + j\omega C)\mathbf{V} = (\frac{1+j\omega RC}{R})\mathbf{V}$$

$$= \frac{1}{R}\sqrt{(1+(\omega RC)^2}\angle\tan^{-1}(\omega RC)\mathbf{V}$$

We could have gotten this from calculating the impedance. Since R and C are in parallel we have using the reciprocal rule:

$$Z = \frac{1}{\frac{1}{R} + \frac{1}{\frac{1}{j\omega C}}} = \frac{1}{\frac{1}{R} + j\omega C} = \frac{R}{1+j\omega RC}$$

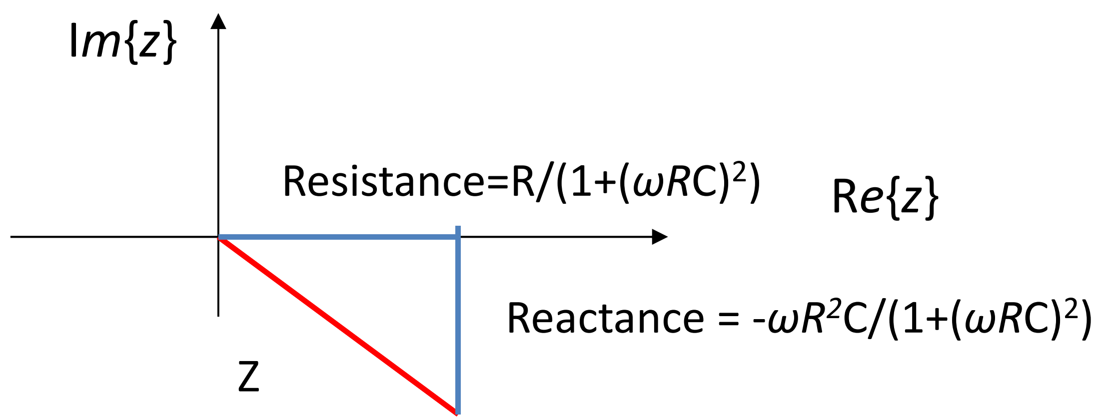
$$\mathbf{I} = \frac{\mathbf{V}}{Z} = \frac{\mathbf{V}}{\frac{R}{1+j\omega RC}} = (\frac{1+j\omega RC}{R})\mathbf{V}$$

$$= \frac{1}{R}\sqrt{(1+(\omega RC)^2}\angle\tan^{-1}(\omega RC)\mathbf{V}$$

# Impedances

$$Z = \frac{1}{\frac{1}{R} + \frac{1}{\frac{1}{j\omega C}}} = \frac{1}{\frac{1}{R} + j\omega C} = \frac{R}{1 + j\omega RC}$$

$$= \frac{R}{1 + j\omega RC} \times \frac{1 - j\omega RC}{1 - j\omega RC} = \frac{R - j\omega R^2 C}{1 + j\omega RC - j\omega RC + (\omega RC)^2} = \frac{R - j\omega R^2 C}{1 + (\omega RC)^2}$$



# Homework

1. Simplify the following complex numbers

$2a)\ 2e^{j\pi/4} + 2e^{j\pi/2}$

$2b)\ e^{j5\pi/4} + e^{j\pi/4}$

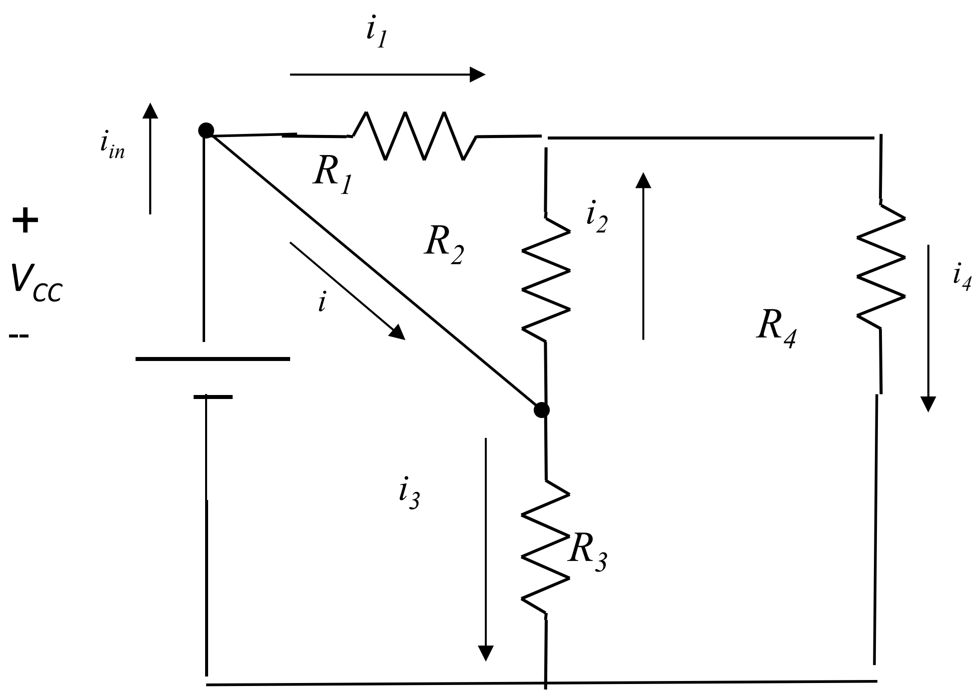
$2c)\ (3+4j)^5$

$2d)\ \Re e\{\frac{\sqrt{2}e^{j\pi/2}}{-j}\}$

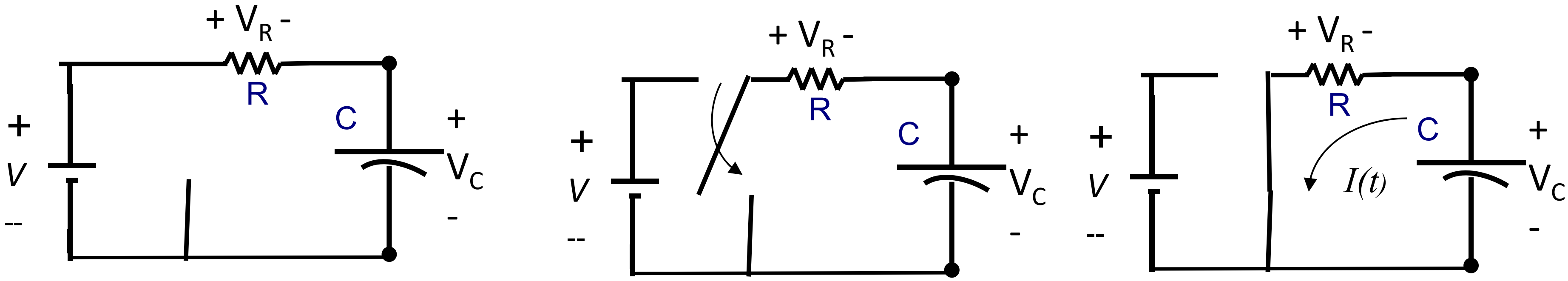
# Homework

2. Calculate the current labeled, *i*.

$R_1 = 5\Omega$, $R_2 = 5\Omega$, $R_3 = 5\Omega$, $R_4 = 2.5\Omega$, $V_{cc} = 5v$



# Homework



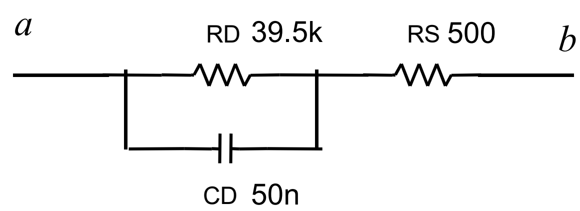

3. a. Given that R=1kOhm, and C=50μF calculate the time constant.

b. How would you increase the time constant of the circuit by a factor of five?

c. Given that the voltage V1=10V, calculate the voltage of the capacitor at 1,2,3,4, and 5 time constants.

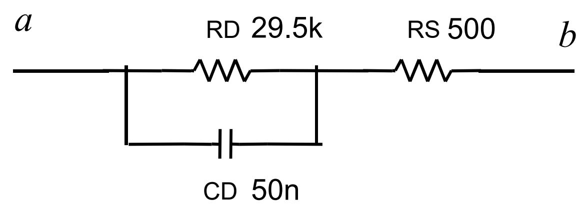
d. At time = 0s, the switch closes and stays closed for 2 time constants (see figure on the left).
At that point the switch is opened for 2 time constants (see center figure).
The switch is then switched as shown in the right figure for a total of 2 time constants before it is opened again. Plot the voltage across the capacitor, the charge on the capacitor as a function of time from time = 0 until 2 time constants after the circuits opened for the final time.

# Homework




4. The circuit shown is an equivalent circuit of an electrode where RD and CD are the resistance and capacitance associated with the interface of the electrode and the body and RS is the resistance of the device itself. Find the impedance $Z_{ab}(j\omega)$ as function of $\omega$.

# solution




Note the impedance from a to b is RS is series with the parallel combination of RD and CD.

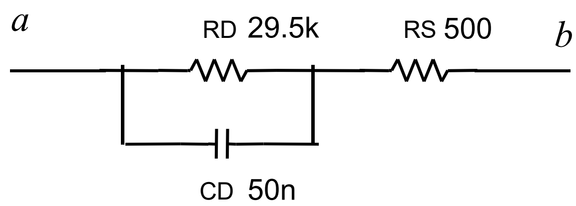
$$Zab(\omega) = R_S + R_D \,||\, C_D$$

$$Zab(\omega) = R_S + \frac{R_D \times \frac{1}{j\omega C_D}}{R_D + \frac{1}{j\omega C_D}} = R_S + \frac{R_D}{1 + j\omega R_D C_D}$$

$$Zab(\omega) = \frac{R_S(1 + j\omega R_D C_D) + R_D}{1 + j\omega R_D C_D} = \frac{R_S + R_D + j\omega R_D R_S C_D}{1 + j\omega R_D C_D}$$

$$= \frac{\sqrt{(R_S + R_D)^2 + (\omega R_D R_S C_D)^2}\angle \tan^{-1}(\frac{\omega R_D R_S C_D}{R_S + R_D})}{\sqrt{1 + (\omega R_D C_D)^2}\angle \tan^{-1}(\omega R_D C_D)} = \frac{\sqrt{(R_S + R_D)^2 + (\omega R_D R_S C_D)^2}}{\sqrt{1 + (\omega R_D C_D)^2}}\angle \tan^{-1}(\frac{\omega R_D R_S C_D}{R_S + R_D}) - \tan^{-1}(\omega R_D C_D)$$

# solution




Three points: $\omega$=0;$\omega \to \infty$; and some other point.

$$Zab(0) = \frac{R_S + R_D + j0R_DR_SC_D}{1 + j0R_DC_D} = R_S + R_D = 39.5k + 500 = 40k$$

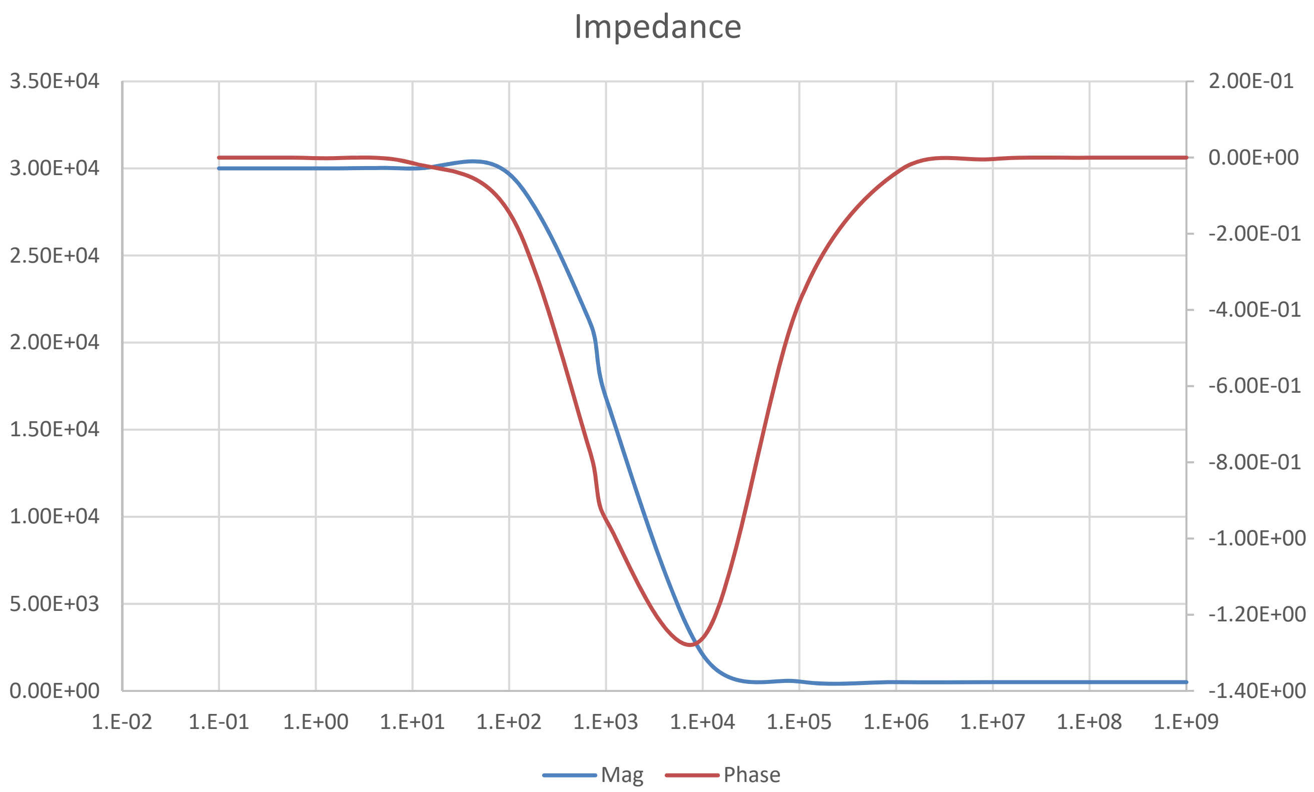
$$Zab(\omega \to \infty) = \frac{R_S + R_D + j\omega R_DR_SC_D}{1 + j\omega R_DC_D}|_{\omega\to\infty} \to \frac{j\omega R_DR_SC_D}{j\omega R_DC_D} \to R_S = 500$$

$$Zab(\omega = \frac{1}{R_DC_D} = 670rad = 107.9Hz) = \frac{R_S + R_D + j\omega R_DR_SC_D}{1 + j\omega R_DC_D}|_{\omega\to=\frac{1}{R_DC_D}} = \frac{R_S + R_D + j\frac{1}{R_DC_D}R_DR_SC_D}{1 + j\frac{1}{R_DC_D}R_DC_D}$$

$$= \frac{R_S + R_D + jR_S}{1 + j} = \frac{\sqrt{(R_S + R_D)^2 + (R_S)^2}\angle \tan^{-1}(\frac{R_S}{R_S + R_D})}{\sqrt{2}\angle\frac{\pi}{4}}$$

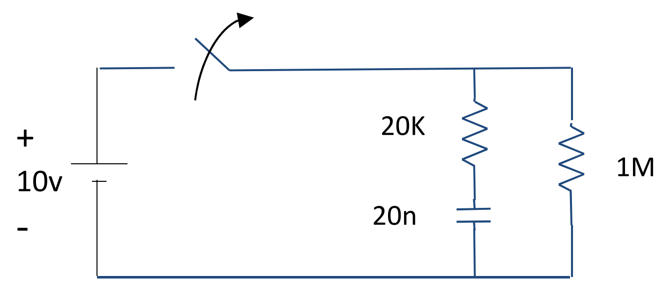
$$= \frac{\sqrt{(R_S + R_D)^2 + (R_S)^2}}{\sqrt{2}}\angle\{\tan^{-1}(\frac{R_S}{R_S + R_D}) - \frac{\pi}{4}\} = 40k\angle -0.77$$

# Solution



Impedance
3.50E+04
3.00E+04
2.50E+04
2.00E+04
1.50E+04
1.00E+04
5.00E+03
0.00E+00
2.00E-01
0.00E+00
-2.00E-01
-4.00E-01
-6.00E-01
-8.00E-01
-1.00E+00
-1.20E+00
-1.40E+00
1.E-02
1.E-01
1.E+00
1.E+01
1.E+02
1.E+03
1.E+04
1.E+05
1.E+06
1.E+07
1.E+08
1.E+09
Mag
Phase

# Homework




5. HONORS STUDENTS ADD THE FOLLOWING

   Assume the switch has been closed for a very long time. Once the switch is opened, how long will it take for the capacitor voltage to discharge to 5volts. What will the voltage be after the switch is closed again 2 msec later?

# Homework

6. HONORS STUDENTS ADD THE FOLLOWING

Plot the voltage phasors for an R-C series circuit where R=5k ohms, C=500 nF, and the source voltage Vin=10V is a sinusoidal signal at f=100 k Hz. What is the phase angle between the voltage of the capacitance $V_C$ and $V_R$ of the circuit? Which one is ahead?