Frequency Response of FIR Filters

Lecture #10
Chapter 6
Properties of the Frequency Response

- Relationship of the Frequency Response to the Difference Equation and Impulse Response

Difference Equation $\Leftrightarrow$ Impulse Response

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \Leftrightarrow h[n] = \sum_{k=0}^{M} b_k \delta[n-k]$$

Difference Equation $\Leftrightarrow$ Frequency Response

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \Leftrightarrow y[n] = (\sum_{k=0}^{M} b_k e^{-j\omega k}) A e^{j\phi} e^{j\omega n} = H(e^{j\omega}) A e^{j\phi} e^{j\omega n}$$

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] \Leftrightarrow H(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-j\omega k} = \sum_{k=0}^{M} h[k] e^{-j\omega k}$$

Time Domain $\Leftrightarrow$ Frequency Domain

Go between the difference equation, impulse response and the frequency response by knowing the $b_k$'s
Example

\[ h[n] = -\delta[n] + 3\delta[n-1] - \delta[n-2] \]
\[ \{b_k\} = \{-1, 3, -1\} \]
\[ y[n] = -x[n] + 3x[n-1] - x[n-2] \]
\[ H(e^{j\hat{\omega}}) = -1 + 3e^{-j\hat{\omega}} - e^{-j2\hat{\omega}} \]
Periodicity of the Frequency Response

• The Frequency Response is a periodic function of $2\pi$

$$H(e^{j(\hat{\omega}+2\pi)}) = \sum_{k=0}^{M} b_k e^{-j(\hat{\omega}+2\pi)k}$$

$$= \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} e^{-j2\pi k}$$

$$= H(e^{j\hat{\omega}})$$

since $e^{-j2\pi k} = 1$ when $k = 1$

Therefore, we always express $H(e^{j\hat{\omega}})$ over one period, e.g., $-\pi < \hat{\omega} < \pi$
Conjugate Symmetry

• If the filter coefficients are real (i.e., $b_k = b_k^*$), then the frequency response has conjugate symmetry and

$$H(e^{-j\hat{\omega}}) = H^*(e^{j\hat{\omega}})$$

• As a result,
  – In polar form, the magnitude is an even function and the phase is an odd function
  – In Cartesian form, the real part is an even function and the imaginary part is an odd function

• Therefore, we only have to show the frequency for one half of a period, (e.g., between 0 and $\pi$)
Proof of Conjugate Symmetry

\[ H^\ast(e^{j\hat{\omega}}) = \left( \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} \right) \]

\[ = \sum_{k=0}^{M} b_k^\ast e^{+j\hat{\omega}k} \]

\[ = \sum_{k=0}^{M} b_k e^{-j(\hat{\omega})k} \]

\[ = H(e^{-j\hat{\omega}}) \]
Proof of Conjugate Symmetry

\[ H(e^{j\hat{\omega}}) = |H(e^{j\hat{\omega}})|e^{j\angle H(e^{j\hat{\omega}})} \]

\[ H(e^{-j\hat{\omega}}) = |H(e^{-j\hat{\omega}})|e^{j\angle H(e^{-j\hat{\omega}})} \]

\[ = |H(e^{j\hat{\omega}})|e^{-j\angle H(e^{j\hat{\omega}})} \]

\[ = |H(e^{j\hat{\omega}})|e^{j\angle H(e^{j\hat{\omega}})} \]

\[ |H(e^{-j\hat{\omega}})| = |H(e^{j\hat{\omega}})| \text{ even function} \]

\[ \angle H(e^{-j\hat{\omega}}) = -\angle H(e^{j\hat{\omega}}) \text{ odd function} \]
Proof of Conjugate Symmetry

\[ H(e^{j\hat{\omega}}) = \Re \{ H(e^{j\hat{\omega}}) \} + j \Im \{ H(e^{j\hat{\omega}}) \} \]

\[ H(e^{-j\hat{\omega}}) = \Re \{ H(e^{-j\hat{\omega}}) \} + j \Im \{ H(e^{-j\hat{\omega}}) \} \]

\[ = \Re \{ H(e^{j\hat{\omega}})^* \} + j \Im \{ H(e^{j\hat{\omega}})^* \} \]

\[ = \Re \{ H(e^{j\hat{\omega}}) \} - j \Im \{ H(e^{j\hat{\omega}}) \} \]

\[ \Re \{ H(e^{-j\hat{\omega}}) \} = \Re \{ H(e^{j\hat{\omega}}) \} \text{ even function} \]

\[ \Im \{ H(e^{-j\hat{\omega}}) \} = -\Im \{ H(e^{j\hat{\omega}}) \} \text{ odd function} \]
Graphical Representation of the Frequency Response

- The frequency response varies with frequency
- By choosing the coefficients of the difference equation, the shape of the frequency response vs frequency can be developed.
- Examples are:
  - filters which only pass low frequencies
  - filters which only pass high frequencies
  - filters which only alter the phase
- Therefore, we usually plot the amplitude and phase of the frequency response vs. frequency
  - This is sometimes called the Bode Plot
Delay System

• A simple FIR filter: \( y[n] = x[n-n_0] \)
• Therefore, from the difference equation, \( k = n_0 \)
  and \( b_{n_0} = 1 \) and the Frequency Response becomes:
  \[
  H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}n_0}
  \]

\[
\angle H(e^{j\hat{\omega}}) = -\hat{\omega}n_0
\]
First-Difference System
High Pass Filter

\[ y[n] = x[n] - x[n-1] \]
\[
H(e^{j\hat{\omega}}) = 1 - e^{-j\hat{\omega}} = e^{-j\hat{\omega}/2} (e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2})
\]
\[
= 2je^{-j\hat{\omega}/2} \sin(\hat{\omega} / 2)
\]
\[
= 2e^{j\pi/2} e^{-j\hat{\omega}/2} \sin(\hat{\omega} / 2)
\]
\[
= 2e^{-j(\hat{\omega}-\pi)/2} \sin(\hat{\omega} / 2)
\]
\[
|H(e^{j\hat{\omega}})| = 2\left|\sin \frac{\hat{\omega}}{2}\right|
\]
\[
\angle H(e^{j\hat{\omega}}) = -(\hat{\omega} - \pi) / 2
\]
Simple Low Pass Filter

\[ y[n] = x[n] + 2x[n-1] + x[n-2] \]

\[ H(e^{j\hat{\omega}}) = 1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} \]

\[ = e^{-j\hat{\omega}}(e^{j\hat{\omega}} + 2 + e^{-j\hat{\omega}}) \]

\[ = (2 + 2\cos \hat{\omega})e^{-j\hat{\omega}} \]

\[ |H(e^{j\hat{\omega}})| = (2 + 2\cos \hat{\omega}) \]

\[ \angle H(e^{j\hat{\omega}}) = -\hat{\omega} \]
Cascaded LTI Systems

• Recall that two systems cascaded together, then the overall impulse response is the convolution of the two individual impulse responses.

• It turns out the the frequency response of a cascaded system is the product of the individual frequency responses.
Proof of the Frequency Response of Cascaded Systems

\[ y_1[n] = x_2[n] \]
\[ y_2[n] = y[n] \]
\[ h_1[n] \otimes h_2[n] \]

\[ y_1[n] = H_1(e^{j\omega})e^{j\omega n} \]
\[ y[n] = y_2[n] = H_2(e^{j\omega})y_1[n] = H_2(e^{j\omega})H_1(e^{j\omega})e^{j\omega n} \]
\[ H_T(e^{j\omega}) = H_2(e^{j\omega})H_1(e^{j\omega}) \]

Therefore, these processes are related
\[ h_1[n] \otimes h_2[n] \Leftrightarrow H_2(e^{j\omega})H_1(e^{j\omega}) \]
Running-Average Filtering

- A simple LTI system defined as the L-point running average

\[ y[n] = \frac{1}{L} \sum_{k=0}^{L-1} x[n-k] \]

\[ = \frac{1}{L} (x[n] + x[n-1] + \cdots + x[n-(L-1)]) \]

- The frequency response is then

\[ H(e^{j\omega}) = \frac{1}{L} \sum_{k=0}^{L-1} e^{-j\omega k} \]
The Frequency Response of the Running Average Filter

\[ H(e^{j\hat{\omega}}) = \frac{1}{L} \sum_{k=0}^{L-1} e^{-j\hat{\omega}k} \]

Using the formula for the partial sums of a geometric series

\[ \sum_{k=0}^{L-1} \alpha^k = \frac{1 - \alpha^L}{1 - \alpha} \]

\[ H(e^{j\hat{\omega}}) = \frac{1}{L} \sum_{k=0}^{L-1} e^{-j\hat{\omega}k} = \left( \frac{1}{L} \right) \left( 1 - e^{-j\hat{\omega}L} \right) \left( \frac{1 - e^{-j\hat{\omega}}}{1 - e^{-j\hat{\omega}}} \right) \]

\[ = \left( \frac{1}{L} \right) \left( \frac{e^{-j\hat{\omega}L/2} - e^{j\hat{\omega}L/2}}{e^{-j\hat{\omega}/2} - e^{j\hat{\omega}/2}} \right) \]

\[ = \left( \frac{1}{L} \right) \left( \frac{\sin \hat{\omega}L/2}{\sin \hat{\omega}/2} \right) e^{-j\hat{\omega}(L-1)/2} \]

Therefore,

\[ H(e^{j\hat{\omega}}) = D_L(e^{j\hat{\omega}}) e^{-j\hat{\omega}(L-1)/2} \]

\[ D_L(e^{j\hat{\omega}}) = \frac{\sin \hat{\omega}L/2}{L \sin \hat{\omega}/2} \iff \text{Dirichlet Function} \]
Plot of the Dirichlet Function

What happens when \( \hat{\omega} = 0 \)?

\[
D_L(e^{j0}) = \frac{\sin 0}{L \sin 0} = \frac{0}{0}
\]

Using L'Hopital's rule

\[
\lim_{\hat{\omega}\to0} D_L(e^{j\hat{\omega}}) = \lim_{\hat{\omega}\to0} \frac{\sin \hat{\omega}L/2}{L \sin \hat{\omega}/2} = \frac{\lim_{\hat{\omega}\to0} \frac{d}{d\hat{\omega}} \sin \hat{\omega}L/2}{L \lim_{\hat{\omega}\to0} \frac{d}{d\hat{\omega}} \sin \hat{\omega}/2} = \frac{\lim_{\hat{\omega}\to0} \cos \hat{\omega}L/2}{L \lim_{\hat{\omega}\to0} \cos \hat{\omega}/2} = \frac{(L/2)}{(L/2)} = 1
\]

**Properties of** \( D_L(e^{j\hat{\omega}}) \):

- Even Function and periodic in \( 2\pi \)
- Maximum at 0
- Has zeroes at integer multiples of \( 2\pi / L \)

**Low Pass Filter**
Smoothing an Image

• See Figures 6-11 through 6-15 for example of the application of the running average filter.
Reconstruction of a Continuous-time signal

• Recall:
  – The sampling theorem suggests that a process exists for reconstructing a continuous-time signal from its samples.
  – If we know the sampling rate and know its spectrum then we can reconstruct the continuous-time signal by scaling the principal alias of the discrete-time signal to the frequency of the continuous signal.
  – The principal alias will always be in the range between $0 \sim \pi$ if the sampling rate is greater than the Nyquist rate.
Continued

• If continuous-time signal has a frequency of $\omega$, then the discrete-time signal will have a principal alias of

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$

• So we can use this equation to determine the frequency of the continuous-time signal from the principal alias:

$$\omega = \hat{\omega} f_s = \frac{\hat{\omega}}{T_s}$$

• Note that the principal alias must be less than $p$ if the Nyquist rate is used

$$\hat{\omega} = \omega_{\text{MAX}} T_s = 2\pi f_{\text{MAX}} T_s = \frac{2\pi f_{\text{MAX}}}{f_s} = \frac{2\pi f_{\text{MAX}}}{f_s (\geq 2 f_{\text{MAX}})} \leq \pi$$

• And the reconstructed continuous-time frequency must be

$$\omega = 2\pi f = \hat{\omega} f_s \Rightarrow f = \frac{\hat{\omega} f_s}{2\pi} \leq \frac{f_{\text{MAX}}}{2\pi} = \frac{f_s}{2}$$
Low Pass Filter

- Since we are within the Nyquist rate, the principal alias is $< \pi$
- Best reconstruction is Low Pass Filter or what the text calls: Ideal Bandlimited Interpolation
Reconstruction of a Continuous-time signal in terms of the Frequency Response

\[ x(t) = Xe^{j\omega t} \quad \iff \text{Continuous-Time Signal} \]
\[ x[n] = Xe^{j\omega nT_s} = Xe^{j\hat{\omega} n} \quad \iff \text{Sampled Continuous-Time Signal} \]
\[ \hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} \]

\[ y[n] = H(e^{j\hat{\omega}}) Xe^{j\hat{\omega} n} \iff \text{Applying the a filter to recover the signal} \]
\[ = H(e^{j\omega T_s}) Xe^{j\omega T_s n} \]
\[ y(t) = H(e^{j\omega T_s}) Xe^{j\omega t} \iff \text{Ideal D-to-C conversion} \]
\[ \iff \text{only good for } -\pi/T_s < \omega < \pi/T_s \]
\[ \iff \text{since for } -\pi < \hat{\omega} < \pi \text{ to obtain the principal alias} \]
Example

\[ y[n] = \frac{1}{11} \sum_{k=0}^{10} x[n-k] \quad \Leftarrow \text{11-point filter} \]

\[ H(e^{j\hat{\omega}}) = \frac{\sin \hat{\omega} \{1/2\}}{11 \sin \hat{\omega}/2} e^{-j\hat{\omega}5} \quad \Leftarrow \text{Its Frequency Response} \]

\[ x(t) = \cos(2\pi(25)t) + \cos(2\pi(250)t) \quad \Leftarrow \text{Analog Signal } f_s = 1000 \]

\[ H(e^{j\hat{\omega}}) = H(e^{j\omega/1000}) = H(e^{j2\pi/1000}) \]

\[ H(e^{j2\pi(25)/1000}) = \frac{\sin(2\pi(25)/1000 \times 11/2)}{11 \sin(2\pi(25)/1000 / 2)} e^{-j2\pi(25)/1000 \times 5} \]

\[ = \frac{\sin(\pi(25)(11)/1000)}{11 \sin(\pi(25)/1000)} e^{-j\pi(25)/1000} \]

\[ = 0.8811 e^{-j\pi/4} \quad \Leftarrow \text{FR at } f = 25 \]

\[ H(e^{j2\pi(250)/1000}) = \frac{\sin(2\pi(250)/1000 \times 11/2)}{11 \sin(2\pi(250)/1000 / 2)} e^{-j2\pi(250)/1000 \times 5} \]

\[ = \frac{\sin(\pi(250)(11)/1000)}{11 \sin(\pi(250)/1000)} e^{-j\pi(250)/100} \]

\[ = 0.0909 e^{-j(2\pi + \pi/2)} = 0.0909 e^{-j\pi/2} \quad \Leftarrow \text{FR at } f = 250 \]

\[ y(t) = .8811 \cos(2\pi(25)t - \pi/4) + 0.0909 \cos(2\pi(250)t - \pi/2) \quad \Leftarrow \text{Reconstructed Signal} \]
Homework

• Exercises:
  – 6.2-6.6

• Problems:
  – 6.14 Use Matlab to plot the Frequency Response; show your code
  – 6.15
  – 6.17, 6.19, Use Matlab to plot the Frequency Response; show your code
  – 6.20, 6.21